Routes versus Atlases, and Numbers on Taxis

Jason Starr

6pm November 7, 2018 Room P-131

Presented by the Stony Brook Math Club

Abstract

A famous math anecdote recounts that when the eminent Cambridge mathematician G. H. Hardy visited the Indian math prodigy Srinavasa Ramanujan in hospital, he remarked on the dull number of his taxi, 1729, to which Ramanujan immediately replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." (one cubed plus twelve cubed equals nine cubed plus ten cubed).

What are all ways of writing integers as sums of two cubes in two different ways? This is a "Diophantine problem" of finding rational points on a "cubic surface". The simplest curves on this surface, the famous 27 lines, give many ways to solve this problem using a little arithmetic of complex numbers. One geometric explanation for this is that a cubic surface is "rationally connected": it is an algebraic-geometric space where for every two locations, there is an effective way to find a route between these locations. Finding "routes" is more efficient than compiling an "atlas" of all locations at once, i.e., it is easier to carry a GPS device that can find routes than to carry a satchel full of city maps.

I will describe theorems (and some conjectures) that solve Diophantine problems for rationally connected spaces and for "rationally simply connected spaces" (rationally connected spaces with effective "routes joining any two nearby routes") for several number systems: complex numbers, complex "function fields", finite fields (arising naturally in cryptography), number fields like the system of fractions of whole numbers, and "global function fields" that are intermediate between finite fields and number fields in their algebraic properties. Behind the scenes, the new theorems are inspired by tools from mathematical physics and from the geometry of "Fano manifolds" – complex manifolds that are positively curved in a weak sense.