Average Connectivity and Average Edge-connectivity in Graphs

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joint work with
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Basic Definitions

- The **connectivity** of a graph $G$, written $\kappa(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected.
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- The **edge-connectivity** of a graph $G$, written $\kappa'(G)$, is the minimum size of an edge set $F$ such that $G - F$ is disconnected.
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The connectivity and the edge-connectivity of a graph measure the difficulty of breaking the graph apart. However, since these values are based on a worst-case situation, it does not reflect the “global (edge) connectedness” of the graph.

Figure: Two Graphs $G_1$ and $G_2$ with $\kappa = \kappa' = 1$

Basic Definitions

The **average connectivity** of a graph $G$ with $n$ vertices, written $\overline{\kappa}(G)$, is $\frac{\sum_{u,v \in V(G)} \kappa(u,v)}{\binom{n}{2}}$, where $\kappa(u, v)$ is the minimum number of vertices whose deletion makes $v$ unreachable from $u$.

The **average edge-connectivity** of a graph $G$ with $n$ vertices, written $\overline{\kappa}'(G)$, is $\frac{\sum_{u,v \in V(G)} \kappa'(u,v)}{\binom{n}{2}}$, where $\kappa'(u, v)$ is the minimum number of edges whose deletion makes $v$ unreachable from $u$.

![Graphs](image)

**Figure:** Two Graphs with $\overline{\kappa}(G_1) = \overline{\kappa}'(G_1) = \frac{27}{7}$ and $\overline{\kappa}(G_2) = \overline{\kappa}'(G_2) = \frac{12}{7}$
In 2002, Beineke, Oellermann and Pippert introduced the average connectivity and found several properties of it.

**Theorem (Dankelmann and Oellermann 2003)**

If $G$ has average degree $\overline{d}$ and $n$ vertices, then \[ \frac{\overline{d}^2}{n-1} \leq \kappa(G) \leq \overline{d}. \]
Average Connectivity and Matching Number

In 2002, Beineke, Oellermann and Pippert introduced the average connectivity and found several properties of it.

**Theorem (Dankelmann and Oellermann 2003)**

If $G$ has average degree $\bar{d}$ and $n$ vertices, then $\frac{-d^2}{n-1} \leq \kappa(G) \leq \bar{d}$.

We prove a bound on the average connectivity in terms of the matching number.
Average Connectivity and Matching Number

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We prove a bound on the average connectivity in terms of the matching number.

**Theorem (Kim and O 2013)**

For a connected graph $G$, $\kappa(G) \leq 2\alpha'(G)$, and this is sharp. Furthermore, if $G$ is connected and bipartite, then $\kappa(G) \leq \left(\frac{9}{8} - \frac{3n-4}{8n^2-8n}\right) \alpha'(G)$, and this is sharp.
Proof (Average Connectivity and Matching Number)

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For a connected graph $G$, $\bar{\kappa}(G) \leq 2\alpha'(G)$. This is sharp only for complete graphs with an odd number of vertices.

- If $G$ has a perfect matching or is a complete graph, then we are done. Assume not.
- Let $M$ be a maximum matching in $G$ and let $S = V(G) - M$. 
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- If $G$ has a perfect matching or is a complete graph, then we are done. Assume not.
- Let $M$ be a maximum matching in $G$ and let $S = V(G) - M$.
- For $v v' \in M$, put $v$ and $v'$ into $T$, $T'$ and $R$ as follows: If neither $v$ nor $v'$ has a neighbor in $S$, then put both in $T$. If $v'$ has a neighbor in $S$ and $v$ does not, then put $v$ in $T$ and $v'$ in $T'$.
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Average Edge-connectivity in Regular Graphs
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Consider three cases to obtain upper bounds on $\kappa(u, v)$ depending on the possible locations of distinct vertices $u$ and $v$. 
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Consider three cases to obtain upper bounds on $\kappa(u, v)$ depending on the possible locations of distinct vertices $u$ and $v$.

- **Case 1:** $u \in S$. If $P$ and $P'$ are distinct internally disjoint $u, v$-paths, then both of them must visit $V(M) - T$ immediately after $u$. $\kappa(u, v) \leq 2m - t$. 
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- **Case 2:** $u, v \in T'$. $\kappa(u, v) \leq n - 1 = 2m + s - 1$.
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- **Case 2:** $u, v \in T'$. $\kappa(u, v) \leq n - 1 = 2m + s - 1$.
- **Case 3:** $u \in R \cup T$. For the vertex after $u$ on a $u, v$-path, at most one vertex of $S$ is available. Thus, $\kappa(u, v) \leq 2m$. 

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$$\kappa(G) \leq \frac{(2m-t)[\binom{s}{2} + s(n-s)] + (2m+s-1)\binom{t'}{2} + 2m[\binom{n}{2} - \binom{s}{2} - s(n-s) - \binom{t'}{2}]}{\binom{n}{2}}$$
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$$\leq \frac{(2m-t)[\binom{s}{2}+st]+(2m+s-1)\binom{t'}{2}+2m[\binom{n}{2}-\binom{s}{2}-st-\binom{t'}{2}]}{\binom{n}{2}}$$
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= 2m + \frac{(s-1)\binom{t'}{2}-t\binom{s}{2}-t^2s}{\binom{n}{2}} \leq 2m - t \frac{s^2+t-1}{n(n-1)} \leq 2m.
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\overline{\kappa}(G) \leq \frac{(2m-t)[\binom{s}{2}+s(n-s)]+(2m+s-1)\binom{t'}{2}+2m[\binom{n}{2}-(\binom{s}{2})-s(n-s)-(\binom{t'}{2})]}{\binom{n}{2}}
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To have equality in the last inequality, $t = 0$ or 1.
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$$\leq \frac{(2m-t)[\binom{s}{2}+st]+(2m+s-1)(t')\binom{n}{2}+2m[\binom{n}{2}-(\binom{s}{2}-st-(t')\binom{t'}{2}]}{\binom{n}{2}}$$

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$t = 0$ requires $s = 1$. $G$ is the complete graph with $n$ vertices.
Proof (Average Connectivity and Matching Number)

**Theorem (Kim and O 2013)**

If $G$ is connected and bipartite, then

$$\bar{\kappa}(G) \leq \left( \frac{9}{8} - \frac{3n-4}{8n^2-8n} \right) \alpha'(G).$$

This is sharp only for $K_{q,3q-2}$ for a positive integer $q$. 
Definitions
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Average Edge-connectivity in Regular Graphs

Average Edge-connectivity and Matching Number

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Average Edge-connectivity and Average Connectivity

The above graphs show that there can be a huge gap between average edge-connectivity and average connectivity.
Average Edge-connectivity and Average Connectivity

Question 1. What is the largest gap between the average edge-connectivity and the average connectivity in an $n$-vertex connected graph?

$,\eta:\text{odd}$.
Question 1. What is the largest gap between the average edge-connectivity and the average connectivity in an \( n \)-vertex connected graph?

Question 2. What is the largest ratio of the average edge-connectivity and the average connectivity in an \( n \)-vertex connected graph?
An extremal problem: What is the smallest average edge-connectivity of an $n$-vertex connected $r$-regular graph?
Average Edge-connectivity in Cubic Graphs

We found the best lower bound for the first nontrivial case $r = 3$. 

Theorem (Kim and O 2013) If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

$$\kappa'(G) \geq \left(\frac{n}{2}\right) + \frac{7n + 58}{4}.$$ 

Equality holds only for graphs in the following family. If a graph $G$ has a cut-edge, then we get components after we delete all cut-edges of $G$. We define an $i$-balloon to be such a component incident to $i$ cut-edges. Let $B_1 = P_3 + K_2$ and let $B_1' = K_4 - e$. 

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If a graph $G$ has a cut-edge, then we get components after we delete all cut-edges of $G$. We define an $i$-balloon to be such a component incident to $i$ cut-edges. Let $B_1 = P_3 + K_2$ and let $B'_1 = K_4 - e$. 
Sketch of Proof

**Theorem (Kim and O 2013)**

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then 
\[
\left(\frac{n}{2}\right)\kappa'(G) \geq \binom{n}{2} + \frac{7n+58}{4}.
\]
Equality holds only for graphs in a special family.
Theorem (Kim and O 2013)

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

$$\binom{n}{2}\kappa'(G) \geq \binom{n}{2} + \frac{7n+58}{4}. \text{ Equality holds only for graphs in a special family.}$$

Sketch of proof: Consider a minimal counterexample $G$.

$k'(G) = 1$: If not, then $k'(G)\binom{n}{2} \geq 2\binom{n}{2} \geq \binom{n}{2} + \frac{7n+58}{4}$.

Every 1-balloon of $G$ is $B_1$: If not, then there exists an 1-balloon $D_1$ of $G$ such that $D_1 \neq B_1$. 
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Equality holds only for graphs in a special family.

**Sketch of proof:** Consider a minimal counterexample $G$.

1. $\kappa'(G) = 1$: If not, then $\kappa'(G)\left(\frac{n}{2}\right) \geq 2\left(\frac{n}{2}\right) \geq \left(\frac{n}{2}\right) + \frac{7n+58}{4}$.

2. Every 1-balloon of $G$ is $B_1$: If not, then there exists an 1-balloon $D_1$ of $G$ such that $D_1 \neq B_1$. Let $|V(D_1)| = 5 + a$.

Let $G'$ be the graph obtained from $G$ by replacing $D_1$ with $B_1$. 
Definitions

Average Connectivity and Matching Number

Average Edge-connectivity in Regular Graphs

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If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

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If not, then there exists an 1-balloon $D_1$ of $G$ such that $D_1 \neq B_1$. Let $|V(D_1)| = 5 + a$.

Let $G'$ be the graph obtained from $G$ by replacing $D_1$ with $B_1$.

Then $\kappa'(G')\left(\frac{n-a}{2}\right) \geq \left(\frac{n-1}{2}\right) + \frac{7(n-1)+58}{4}$.
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Let $G'$ be the graph obtained from $G$ by replacing $D_1$ with $B_1$.

Then $k'(G')\binom{n-a}{2} \geq \binom{n-1}{2} + \frac{7(n-1)+58}{4}$.

$k'(G)\binom{n}{2} = k'(G')\binom{n-a}{2} - k'(B_1)\binom{5}{2} - 5(n - a - 5) + k'(D_1)\binom{5+a}{2} + (5 + a)(n - a - 5)$
Sketch of Proof

**Theorem (Kim and O 2013)**

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

$$\frac{n}{2} \kappa'(G) \geq \binom{n}{2} + \frac{7n + 58}{4}.$$  Equality holds only for graphs in a special family.

**Sketch of proof:** Consider a minimal counterexample $G$.

1. **$\kappa'(G) = 1$:** If not, then $\kappa'(G)(\frac{n}{2}) \geq 2(\frac{n}{2}) \geq (\frac{n}{2}) + \frac{7n + 58}{4}.$

2. **Every 1-balloon of $G$ is $B_1$:** If not, then there exists an 1-balloon $D_1$ of $G$ such that $D_1 \neq B_1$. Let $|V(D_1)| = 5 + a$.

3. Let $G'$ be the graph obtained from $G$ by replacing $D_1$ with $B_1$.

Then $\kappa'(G')(\frac{n-a}{2}) \geq \left(\frac{n-1}{2}\right) + \frac{7(n-1) + 58}{4}$.

$$\kappa'(G) \left(\frac{n}{2}\right) = \kappa'(G')(\frac{n-a}{2}) - \kappa'(B_1) \left(\frac{5}{2}\right) - 5(n - a - 5) + \kappa'(D_1) \left(\frac{5+a}{2}\right) + (5 + a)(n - a - 5) \geq \left(\frac{n-a}{2}\right) + \frac{7(n-a) + 58}{4} - 26 - 5(n - a - 5) + 2 \left(\frac{5+a}{2}\right) + (5 + a)(n - a - 5)$$
Average Connectivity and Matching Number

Sketch of Proof

**Theorem (Kim and O 2013)**

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

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Then $\kappa'(G')(\binom{n-a}{2}) \geq (n-1)\binom{5}{2} + \frac{7(n-1)+58}{4}$.

$\kappa'(G)\binom{n}{2} = \kappa'(G')(\binom{n-a}{2}) - \kappa'(B_1)\binom{5}{2} - 5(n-a-5) + \kappa'(D_1)\binom{5+a}{2}$

$+ (5+a)(n-a-5) \geq \binom{n-a}{2} + \frac{7(n-a)+58}{4} - 26 - 5(n-a-5) + 2\binom{5+a}{2} + (5+a)(n-a-5) > \binom{n}{2} + \frac{7n+58}{4}n.$
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If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then 
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\[\kappa'(G) = 1:\]

Every 1-balloon of $G$ is $B_1$. 
#### Definitions

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**Sketch of Proof**

**Theorem (Kim and O 2013)**

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

$$\left(\frac{n}{2}\right)\kappa'(G) \leq \left(\frac{n}{2}\right) + \frac{7n+58}{4}.$$  

Equality holds only for graphs in a special family.

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**Sketch of proof:** Consider a minimal counterexample $G$.

$\kappa'(G) = 1$:

Every 1-balloon of $G$ is $B_1$.

Every 2-balloon of $G$ is $B'_1$.
Sketch of Proof

**Theorem (Kim and O 2013)**

If $G$ is a connected cubic graph with $n$ vertices, other than $K_4$, then

$$\binom{n}{2} \kappa'(G) \leq \binom{n}{2} + \frac{7n + 58}{4}.$$  

Equality holds only for graphs in a special family.

**Sketch of proof:** Consider a minimal counterexample $G$.

$k'(G) = 1$:

Every 1-balloon of $G$ is $B_1$.

Every 2-balloon of $G$ is $B'_1$.

There are no $i$-balloons in $G$ for $i \geq 3$. 
Questions

Question 3. What is the best upper bound for $\bar{\kappa}'(G)$ in an $n$-vertex connected $r$-regular graphs for $r \geq 4$?
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Suppose that $r$ is odd. Let $B_r = P_3 + \frac{r-1}{2} K_2$ and $B'_r = K_{r+1} - e$. For odd $r$, we guess that the graph obtained from the graph in the special family by replacing $B_1$ and $B'_1$ with $B_r$ and $B'_r$ are the extremal graphs.
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Suppose that $r = 4$. 

![Graph Diagram]
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Suppose that $r = 4$. 
Thank you

Thank You : )