Ore-type conditions for the existence of an even \([a, b]\)-factor

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In this report, we consider \(G\) as a finite, simple, and undirected graph. Let \(\delta(G)\) be the minimum degree of \(G\), and let \(\sigma_2(G) = \min\{d(u) + d(v) \mid uv \notin E(G)\}\). For positive integers \(a\) and \(b\) with \(a \leq b\), an even \([a, b]\)-factor of a graph \(G\) is a spanning subgraph \(H\) such that \(d_H(v)\) is even and \(a \leq d_H(v) \leq b\) for all \(v \in V(G)\). A graph \(G\) is \(k\)-edge-connected if for \(S \subseteq E(G)\) with \(|S| < k\), \(G - S\) is connected. The edge-connectivity of \(G\), denoted \(\kappa'(G)\), is the maximum \(k\) such that \(G\) is \(k\)-edge-connected. A graph \(G\) is \(k\)-vertex-connected if \(|V(G)| \geq k + 1\) and for \(S \subseteq V(G)\) with \(|S| < k\), \(G - S\) is connected. The vertex-connectivity of \(G\), denoted \(\kappa(G)\), is the maximum \(k\) such that \(G\) is \(k\)-vertex-connected.

In 2005, Matsuda [2] proved an Ore-type condition for the existence of an even \([2, b]\)-factor.

**Theorem 1** ([2]). Let \(b \geq 2\) be an even integer and let \(G\) be a 2-edge-connected graph of order \(n \geq b + 3\). If \(\sigma_2(G) \geq \frac{4n}{2b+6}\), then \(G\) has an even \([2, b]\)-factor.

For \(a \geq 4\), he proposed the following conjecture.

**Conjecture 2** ([2]). Let \(2 \leq a \leq b\) be even integers and let \(G\) be a 2-edge-connected graph of order \(n \geq 2a + b + \frac{a^2-3a}{b} - 2\). If \(\delta(G) \geq a\) and \(\sigma_2(G) \geq \frac{2an}{a+b}\), then \(G\) has an even \([a, b]\)-factor.

However, this conjecture is not true (See the figure below).

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**Example 3.** Let \(a\) and \(b\) be even integers at least 4 with \(b \geq \frac{a^2-3a+a\sqrt{(a-3)(a+1)}}{2}\). Let \(L_0\) be the trivial graph on \(a - 1\) vertices, and let \(V(L_0) = \{y_1, \ldots, y_{(a-1)}\}\). For \(1 \leq i \leq a\), let \(L_i\) be a copy of the complete graph on \(a + 2\) vertices and let \(V(L_i) = \{x_{i1}, \ldots, x_{i(a+2)}\}\). Let \(t\) be a positive integer such that \((a + 2) \leq a^2 - a + b + \frac{a^2-3a}{b} - 1 \leq t \leq -a^2 - 2a + b + \frac{b}{n} + 2\). Let \(L_{a+1}\) be a copy of the complete graph on \(t\) vertices and let \(V(L_{a+1}) = \{x_{(a+1)1}, \ldots, x_{(a+1)t}\}\). Let \(L\) be the graph obtained from \(L_0, \ldots, L_{a+1}\) by adding edges between \(y_j\) and \(x_{ij}\) for all \(i \in \{1, \ldots, a + 1\}\) and for all \(j \in \{1, \ldots, a - 1\}\).
Cho, Hyun, O, and Park [1] modified some conditions in Conjecture 2 and proved sharp conditions for the existence of an even \([a,b]\)-factor

**Theorem 4** ([1]). Let \(4 \leq a \leq b\) be even integers. If \(G\) is a graph with \(n\) vertices such that (i) \(\kappa(G) \geq a\), (ii) \(n \geq 2a + b + \frac{a^2 - 3a}{6} - 2\), and (iii) \(\delta(G) \geq \frac{an}{a+b}\), then \(G\) contains an even \([a,b]\)-factor.

We were wondering whether if we replace Condition (iii) in Theorem 4 by \(\sigma_2(G) \geq \frac{2an}{a+b}\), then \(G\) contains an even \([a,b]\)-factor. During the workshop, we tried to prove by contradiction when \(a = 4\) by using Lovász’s parity \((g,f)\)-factor theory. Assume that a graph \(G\) does not have an even \([a,b]\)-factor. Then there exist disjoint \(S, T \subseteq V(G)\) such that

\[
\delta(S,T) = q(S,T) - b|S| + 4|T| - \sum_{v \in T} d_{G-S}(v) \geq 2,
\]

where \(q(S,T)\) is the number of the components \(C\) of \(G - (S \cup T)\) such that there are odd number of edges between \(C\) and \(T\). When \(|T| \geq b + 1\) or \(|T| \leq 5\), we are done. The remaining cases to prove are \(6 \leq |T| \leq b\).

**References**
