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# Disjoint paths of bounded length in large generalized cycles

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**Abstract**

A generalized $p$-cycle is a digraph whose set of vertices is partitioned in $p$ parts that can be ordered in such a way that a vertex is adjacent only to vertices in the next part. The families of $BGC(p,d,d^p)$ and $KGC(p,d,d^{p+k}+d^p)$ are the largest known $p$-cycles for their degree and diameter.

In this paper we present a lower bound for the fault-diameter of a generalized cycle. Then we calculate the wide-diameter and the fault-diameter of the families mentioned above, by constructing disjoint paths between any pair of vertices. We conclude that the values of these parameters for $BGC(p,d,d^p)$ and $KGC(p,d,d^{p+k}+d^p)$ exceed the lower bound at most in one unit. 

*Keywords:* Fault-tolerant networks; Fault-diameter; Generalized cycles; Iterated line digraphs

## 1. Introduction

Interconnection networks are usually modeled by graphs. The switching elements or processors are represented by the vertices. The communication links are represented by edges (if they are bidirectional) or arcs (if they are unidirectional). Some concepts in Graph Theory appear to be specially useful in order to analyze the efficiency and the reliability of an interconnection network modeled by a graph. For instance, the diameter is a measure of the transmission delay and the connectivity and the fault-diameter are related to the fault tolerance of the network.

Some other parameters, such as the $s$-wide-distance and the $s$-wide-diameter [9–11,15], have been considered in relation to the existence of a number of dis-

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joint paths of bounded length between the vertices of the graph. The $s$-wide-diameter is closely related to the $s$-fault-diameter. In some cases, these two parameters have the same value and, in general, the $s$-wide-diameter is an upper bound for the $s$-fault-diameter. The definitions of these concepts are given in Section 2.

The existence of disjoint paths of bounded length in de Bruijn and Kautz digraphs have been studied in [3,14]. As a conclusion, its $s$-wide-diameter and $s$-fault-diameter appear to be almost optimal. These parameters were studied in [17] for the bipartite digraphs $BD(d, d^{D-3} + d^{D-1})$ with similar results. These families of digraphs were proposed as solutions to the $(d, D)$-digraph problem, which consists in finding digraphs with order as large as possible for fixed values of the maximum out-degree $d$ and the diameter $D$. Actually, de Bruijn and Kautz digraphs are the best-known general solutions to this problem and the digraphs $BD(d, d^{D-3}+d^{D-1})$ [5] are the largest known bipartite digraphs for general values of the degree $d$ and the diameter $D$. All these digraphs are iterated line digraphs [5,6]. Actually, the disjoint paths of bounded length between the vertices are constructed using this fact. Besides, iterated line digraphs have in general very good properties in relation to the connectivity and the fault-diameter [4,16].

A generalized $p$-cycle is a digraph whose set of vertices is partitioned in $p$ parts that can be cyclically ordered in such a way that any vertex is adjacent only to vertices in the next part.

In this paper, we study the $s$-wide-diameter and the $s$-fault-diameter of two families of generalized cycles that have been recently introduced in [8]: the digraphs $BGC(p, d, d^{D-p+1})$ and $KGC(p, d, d^{D-p+1} + d^{D-2p+1})$. These digraphs, which are iterated line digraphs, are the largest known generalized cycles for general values of their number of parts $p$, degree $d$ and diameter $D$. If we take $p = 1$, we find the de Bruijn and Kautz digraphs in these families. The bipartite digraphs $BD(d, d^{D-3}+d^{D-1})$ appear when $p = 2$.

In the next section we present the most relevant definitions and the notation we are going to use in the following. In Section 3, we will calculate a lower bound for the $s$-fault-diameter of a generalized $p$-cycle. In Sections 4 and 5, we determine the $s$-wide-diameter and the $s$-fault-diameter of the generalized cycles $BGC(p, d, d^{D-p+1})$ and $KGC(p, d, d^{D-p+1} + d^{D-2p+1})$. To do this, we prove that there exist in these digraphs $d$ disjoint paths with length at most $D + 2$ between any pair of different vertices. The $s$-fault-diameter of these digraphs exceed the above mentioned lower bound in at most one unit. These results generalize the previous results about the $s$-wide-diameter and the $s$-fault-diameter of the de Bruijn and Kautz digraphs and the bipartite digraphs $BD(d, d^{D-3} + d^{D-1})$.

## 2. Definitions and notation

We are concerned only in *directed graphs*, called *digraphs* for short. See [2] for the definitions of the concepts about digraphs that are not defined here.

We define next two parameters, the wide-diameter and the fault-diameter, related to the vulnerability of the network. These parameters and the notation we use here were first introduced in [9,10]

The $(s-1)$-*vertex-fault-diameter*, $D_s(G)$, of a digraph $G$ is the maximum of the diameters of the digraphs obtained by removing at most $s-1$ vertices from $G$. The $(s-1)$-*arc-fault-diameter*, $D'_s(G)$, is defined analogously.

Let $x$ and $y$ be two vertices of a digraph $G$. Two paths from $x$ to $y$ are said to be *vertex-disjoint* or *disjoint* if they do not have any internal vertex in common. A *container* from a vertex $x$ to another vertex $y$ is a set $C(x,y)$ of disjoint paths from $x$ to $y$. The *width* $w(C(x,y))$ of a container $C(x,y)$ is the number of disjoint paths that it contains, and its *length*, $l(C(c,y))$, is the maximum length of its paths. For an integer $s$, $0 \leqslant s \leqslant \kappa(G)$, the $s$-*wide-distance* from $x$ to $y$, $d_s(x,y)$, is the minimum length of all containers of width $s$ from $x$ to $y$. Finally, the $s$-*wide-diameter* of the digraph $G$, $d_s(G)$, is the maximum $s$-wide-distance among all pairs of different vertices in $G$.

In general, the following relations hold between the wide-diameter and the fault-diameters: $d_s(G) \geqslant D_s(G)$ and $d_s(G) \geqslant D'_s(G)$. Also there exist some relations between these two parameters, the connectivities and the diameter. From the definition, $D_1(G)$ and $D'_1(G)$ coincide with the diameter of $G$. If $\kappa = \kappa(G)$ there is a container of width $\kappa$ between every pair of distinct nodes. Clearly, $D_s(G) \leqslant D_{s+1}(G)$ and $D'_s(G) \leqslant D'_{s+1}(G)$. In particular, since $D(G) = \infty$ if $G$ is not strongly connected, the vertex-connectivity $\kappa = \kappa(G)$ and the arc-connectivity $\lambda = \lambda(G)$ are, respectively, the minimum values of $s$ satisfying $D_{s+1}(G) = \infty$ and $D'_{s+1}(G) = \infty$.

We recall here the definition and some properties of line digraphs. See, for example, [6] for proofs and more information.

In the *line digraph* $LG$ of a digraph $G$ each vertex represents an arc of $G$, that is, $V(LG) = \{uv \mid (u,v) \in A(G)\}$. A vertex $uv$ is adjacent to a vertex $wz$ if $v = w$, that is, whenever the arc $(u,v)$ of $G$ is adjacent to the arc $(w,z)$. The maximum and minimum out- and in-degrees of $LG$ are equal to those of $G$. Therefore, if $G$ is $d$-regular with order $n$, then $LG$ is $d$-regular and has order $dn$. Besides, if $G$ is a strongly connected digraph different from a directed cycle, the diameter of $LG$ is the diameter of $G$ plus one.

The iteration of the line digraph operation is a good method to obtain large digraphs with fixed degree and diameter. If $G$ is $d$-regular with $d > 1$, has diameter $D$ and order $n$, then $L^kG$ is $d$-regular, has diameter $D + k$ and order $d^k n$, that is, the order increases in an asymptotically optimal way in relation to the diameter.

The set of vertices of the iterated line digraph $L^kG$ can be considered as the set of all paths of length $k$ in $G$, that is, the set of the sequences of vertices of $G$ with length $k+1$, $x_0x_1 \ldots x_k$, where $(x_i, x_{i+1})$ is an arc of $G$. A vertex $\boldsymbol{x} = x_0x_1 \ldots x_k$ in $L^kG$ is adjacent to the vertices $\boldsymbol{y} = x_1 \ldots x_k x_{k+1}$ for all $x_{k+1}$ adjacent from $x_k$. A path of length $h$ in $L^kG$ can be written as a sequence of $k+h+1$ vertices of $G$. The vertices of this path are the subsequences of $k+1$ consecutive vertices of $G$.

The purpose of this paper is to calculate the $s$-wide-diameter and the $s$-fault-diameters of the digraphs in two families of generalized cycles, $BGC(p,d,d^{D-p+1})$ and $KGC(p,d,d^{D-p+1}+d^{D-2p+1})$, that were introduced in [8]. We define next these digraphs and present some basic properties. See [8] for proofs and more information.

A *generalized p-cycle* is a digraph $G$ such that its set of vertices can be partitioned in $p$ parts, $V(G)=\bigcup_{\alpha\in Z_p} V_\alpha$ in such a way that the vertices in the partite set $V_\alpha$ are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\boldsymbol{Z}_p$. Observe that, for instance, a bipartite digraph is a generalized 2-cycle.

The *conjunction* of a directed cycle of length $p$ with a digraph $G=(V,A)$, $C_p\otimes G$, has set of vertices $\boldsymbol{Z}_p\times V$ and a vertex $(\alpha,x)$ is adjacent to the vertices $(\alpha+1,y)$ for any $y$ adjacent from $x$ in the digraph $G$. Observe that $C_p\otimes G$ is a generalized $p$-cycle for any digraph $G$. The line digraph of $C_p\otimes G$ is isomorphic to $C_p\otimes LG$. It is not difficult to see that the mapping $(\alpha,x)(\alpha+1,y)\mapsto(\alpha+1,xy)$ defines a digraph isomorphism between $L(C_p\otimes G)$ and $C_p\otimes LG$ [8].

## 3. Bounds on the fault-diameters of generalized cycles

If $G$ is a digraph with $D_s(G)=D'$, there exist at least $s+1$ paths (not necessarily disjoint) of length at most $D'$ between any pair of non-adjacent vertices of $G$.

Let $G$ be a generalized $p$-cycle with maximum out-degree $d$ and order $n$. Let $r$ such that $D_s(G)=D'$, with $D'-(p-1)=pm+r$ and $0\leqslant r\leqslant p-1$. Then, if $x\in V_{\alpha-r}$ and $y\in V_\alpha$ are two non-adjacent vertices of $G$, there must exist $s+1$ paths of length at most $pm+r$ from $x$ to $y$. There are at most $d^r(1+d^p+d^{2p}+\cdots+d^{pm})$ paths of length less or equal than $pm+r$ from a vertex in $V_{\alpha-r}$ to the vertices in $V_\alpha$. Therefore,

$$n=\sum_{\alpha\in Z_p}|V_\alpha|\leqslant\begin{cases} p(1+\lfloor(d^p+d^{2p}+\cdots+d^{pm})/(s+1)\rfloor) & \text{if } r=0,\\ p(d+\lfloor(d^{p+1}+d^{2p+1}+\cdots+d^{pm+1})/(s+1)\rfloor) & \text{if } r=1,\\ p(\lfloor d^r(1+d^p+\cdots+d^{pm})/(s+1)\rfloor) & \text{if } r\neq 0,1.\end{cases}$$

Then, $D_s(G)=D'=p(m+1)+r+1\geqslant \ell_K^r$, where

$$\ell_D^r=\begin{cases}\lceil\log_d((\tfrac{n}{p}-1)(s+1)(d^p-1)+d^p)\rceil-1 & \text{if } r=0,\\ \lceil\log_d((\tfrac{n}{p}-d)(s+1)(d^p-1)+d^{p+1})\rceil-1 & \text{if } r=1,\\ \lceil\log_d(\tfrac{n}{p}(s+1)(d^p-1)+d^r)\rceil-1 & \text{if } r\neq 0,1.\end{cases}$$

Therefore, if $G$ is a generalized $p$-cycle with maximum out-degree $d$ and order $n$, then, $D_s(G)\geqslant\min_{0\leqslant r\leqslant p-1}\ell_D^r=\ell_D^1$. We have found a lower bound for the $(s-1)$-vertex-fault-diameter.

**Proposition 3.1.** *Let $G$ be a generalized $p$-cycle with maximum out-degree $d$ and order $n$. Then, for any $s = 2,\dots,d$,*

$$D_s(G) \geqslant D_{\min}(s,p,d,n) = \left\lceil \log_d \left( \left( \frac{n}{p} - d \right)(s+1)(d^p - 1) + d^{p-1} \right) \right\rceil - 1.$$

If $G$ is a digraph with $D'_s(G) = D'$, there must exist at least $s+1$ paths of length at most $D'$ between any pair of different vertices. Reasoning in the same way as in the vertex case, we can obtain a lower bound for the $(s-1)$-arc-fault-diameter of a generalized cycle.

**Proposition 3.2.** *Let $G$ be a generalized $p$-cycle with maximum out-degree $d$ and order $n$. Then, for any $s = 2,\dots,d$,*

$$D'_s(G) \geqslant D'_{\min}(s,p,d,n) = \left\lceil \log_d \left( \left( \frac{n}{p} - 1 \right)(s+1)(d^p - 1) + d^p \right) \right\rceil - 1.$$

Since $d_s(G) \geqslant D_s(G)$ and $d_s(G) \geqslant D'_s(G)$, the bounds obtained are also bounds for the $s$-wide-diameter.

## 4. Fault-diameters of $BGC(p, d, d^{k+1})$

The values of the vertex-fault-diameter and the arc-fault-diameter of the generalized cycles $BGC(p,d,d^{k+1})$, $p \geqslant 2$, are given in this section.

The generalized cycle $BGC(p,d,d^{D-p+1}) = BGC(p,d,d^{k+1})$ is, by definition, equal to $C_p \otimes B(d,k+1)$, where $B(d,k+1)$ is the de Bruijn digraph with degree $d$ and diameter $k+1$. The de Bruijn digraph is an iterated line digraph, $B(d,k+1) = L^k K_d^*$, where $K_d^*$ is the complete digraph with a loop on each vertex. Therefore, $BGC(p,d,d^{k+1})$ is also an iterated line digraph,

$$BGC(p,d,d^{k+1}) = C_p \otimes L^k K_d^* = L^k(C_p \otimes K_d^*) = L^k BGC(p,d,d).$$

The set of vertices of the digraph $BGC(p,d,d)$ is $\mathbf{Z}_p \times \mathbf{Z}_d$ and a vertex $(\alpha,x)$ is adjacent to $(\alpha+1,y)$ for any $y \in \mathbf{Z}_d$. This digraph is $d$-regular and has diameter $p$. The vertices of $BGC(p,d,d^{k+1})$, which is $d$-regular and has diameter $p+k$, can be seen as sequences of vertices of $BGC(p,d,d)$ $(\alpha,y_0)(\alpha+1,y_1)\dots(\alpha+k,y_k)$, where $\alpha \in \mathbf{Z}_p$ and $y_i \in \mathbf{Z}_d$, $i = 0,1,\dots,k$. Observe that $BGC(p,d,d^p)$ which has diameter $2p-1$ and order $pd^p$, is isomorphic to the *directed butterfly* $B_d(p)$ [1].

Particularly, we are going to prove, in a constructive way, that there exist a container of width $d$ and length at most $p+k+2$ between any pair of different vertices of $BGC(p,d,d^{k+1})$. Besides, only one path of this container have length $p+k+2$ and the other paths have length at most $p+k+1$.

First of all, we present these containers in the smallest digraph in this family: the generalized cycle $BGC(p,d,d) = C_p \otimes K_d^*$. Next, we show how to construct containers of bounded length between any pair of adjacent vertices of $BGC(p,d,d^{k+1})$. Finally, taking into account that these digraphs are iterated line digraphs, we prove that containers between any pair of different vertices can be found from containers between vertices in a smaller digraph in the same family.

**Proposition 4.1.** *Let $x$ and $y$ be two (not necessarily different) vertices of the generalized cycle $BGC(p,d,d) = C_p \otimes K_d^+$, $p \geqslant 2$. There exist in this digraph a container of width $d$ from $x$ to $y$ with length at most $p+1$.*

**Proof.** We can suppose that $x$ is in the partite set $V_0 = \{0\} \times \mathbf{Z}_d$ and that $y \in V_h = \{h\} \times \mathbf{Z}_d$, $1 \leqslant h \leqslant p$ (of course, $V_p = V_0$). If $p \geqslant 3$ and $h \neq 1,2$, there exist many different ways to find a container with width $d$ and length $h$ from $x$ to $y$. Any path of these containers is in the form $x\alpha_1^i \ldots \alpha_{h-1}^i y$, $1 \leqslant i \leqslant d$, where $\alpha_r^i \in V_r$ and $\alpha_r^i \neq \alpha_r^j$ if $i \neq j$. If $p \geqslant 2$ and $h = 1$, we have a path of length 1, the arc $(x,y)$, and we can take $d-1$ disjoint paths with length $p+1$ in the form $x\alpha_1^i \ldots \alpha_{p-1}^i \alpha_p^i y$, $1 \leqslant i \leqslant d-1$, where $\alpha_r^i \in V_r$, $\alpha_r^i \neq \alpha_r^j$ if $i \neq j$, $\alpha_1^i \neq y$ and $\alpha_p^i \neq x$. Finally, if $h = 2$ there are exactly $d$ paths of length 2 from $x$ to $y$, which are disjoint. □

Let $\boldsymbol{x}$, $\boldsymbol{y}$ be a pair of adjacent vertices of $BGC(p,d,d^{k+1})$. Since this digraph is the iterated line digraph $L^k(C_p \otimes K_d^*)$, we can put $\boldsymbol{x} = x_0 x_1 \ldots x_k$ and $\boldsymbol{y} = x_1 \ldots x_k x_{k+1}$, where $x_r$, $0 \leqslant r \leqslant k+1$, is a vertex of $C_p \otimes K_d^*$. Besides, we can suppose that $x_r = (r,j) \in \mathbf{Z}_p \times \mathbf{Z}_d$, that is, that $x_r$ is a vertex in the partite set $V_r$ of the generalized cycle $C_p \otimes K_d^*$. We want to construct $d$ disjoint paths with length at most $p+k+2$ from $\boldsymbol{x}$ to $\boldsymbol{y}$. The first of these paths is the arc $(\boldsymbol{x},\boldsymbol{y}) = x_0 x_1 \ldots x_k x_{k+1}$. The other paths are going to be constructed from disjoint paths from $x_k$ to $x_1$ in $C_p \otimes K_d^*$ and will have the form $x_0 x_1 \ldots x_k a_{k+1} a_{k+2} \ldots a_{k+r} x_1 \ldots x_k x_{k+1}$ , with $r \leqslant p+1$. Since these paths cannot contain the arc $(\boldsymbol{x},\boldsymbol{y})$, we must take $a_{k+1} \neq x_{k+1}$ and $a_{k+r} \neq x_0$. That is, we have to find a container of width $d-1$ and length at most $p+2$ from $x_k$ to $x_1$ in $C_p \otimes K_d^*$ such that all the paths in it have their first and last arcs, respectively, different from $(x_k, x_{k+1})$ and $(x_0, x_1)$.

If $p \geqslant 3$ and $k \equiv h \pmod{p}$, $1 \leqslant h \leqslant p-2$, then $x_k = x_1$ or $3 \leqslant d(x_k,x_1) = p-h+1 \leqslant p$. In this case, we consider $d-1$ paths from $x_k$ to $x_1$ in the following form: $x_k \alpha_{h+1}^s \alpha_{h+2}^s \cdots \alpha_p^s x_1$, where $1 \leqslant s \leqslant d-1$, $\alpha_{h+1}^s \neq x_{k+1}$, $\alpha_p^s \neq x_0$ and $\alpha_r^s \neq \alpha_r^{s'}$ if $s \neq s'$. These paths are disjoint and have length $p-h+1$.

If $k \equiv p-1 \pmod{p}$, then $d(x_k,x_1) = 2$ and there are exactly $d$ paths of length 2 from $x_k$ to $x_1$. If $x_{k+1} = x_0$, we consider the $d-1$ paths of length 2 that avoid the vertex $x_0$: the paths $x_k \alpha_p^s x_1$, where $1 \leqslant s \leqslant d-1$, $\alpha_p^s \neq x_0$ and $\alpha_p^s \neq \alpha_p^{s'}$ if $s \neq s'$. If $x_{k+1} \neq x_0$, we can take only $d-2$ of these paths: the paths $x_k \alpha_p^s x_1$, where $1 \leqslant s \leqslant d-2$, $\alpha_p^s \neq x_0$, $x_{k+1}$ and $\alpha_p^s \neq \alpha_p^{s'}$ if $s \neq s'$. In this case, we have to consider also a path with length $p+2$: $x_k x_0 \alpha_1^{d-1} \ldots \alpha_{p-1}^{d-1} x_{k+1} x_1$.

If $k \equiv 0 \pmod{p}$, then $d(x_k,x_1) = 1$. In this case, we take $d-1$ paths of length $p+1$: $x_k \alpha_1^s \ldots \alpha_p^s x_1$, where $1 \leqslant s \leqslant d-1$, $\alpha_1^s \neq x_{k+1}$, $\alpha_p^s \neq x_0$ and $\alpha_r^s \neq \alpha_r^{s'}$ if $s \neq s'$.

Then, we can construct $d$ paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ in $BGC(p,d,d^{k+1})$: one path of length 1, the arc

$$A = x_0 x_1 \ldots x_k x_{k+1},$$

$d-1$ or $d-2$ paths with length $D-h+1 \leqslant D+1$,

$$Q_s = x_0 x_1 \ldots x_k \alpha^s_{h+1} \alpha^s_{h+2} \ldots \alpha^s_p x_1 \ldots x_k x_{k+1},$$

where $k \equiv h \pmod p$ and $0 \leqslant h \leqslant p-1$, and, if $k \equiv p-1 \pmod p$, we may need one path with length $D+2$,

$$R = x_0 x_1 \ldots x_k x_0 \alpha^{d-1}_1 \ldots \alpha^{d-1}_{p-1} x_{k+1} x_1 \ldots x_k x_{k+1}.$$

In order to prove that these paths are disjoint, we are going to use the same techniques as in the previous works about disjoint paths in the de Bruijn and Kautz digraphs [14] and the bipartite digraphs $BD(d,d^{D-1}+d^{D-3})$ [17].

**Proposition 4.2.** *If $s \neq t$, the paths $Q_s$ and $Q_t$ are disjoint.*

**Proof.** If $q_{s,j}$ is the $j$th vertex of the path $Q_s$,

$$a_{s,j} = \begin{cases} x_j \ldots, x_k \alpha^s_{h+1} \ldots \alpha^s_{h+j} & j = 1, \ldots, p-h, \\ x_j \ldots x_k \alpha^s_{h+1} \ldots \alpha^s_p x_1 \ldots x_{j-p+h} & j = p-h+1, \ldots, k, \\ \alpha^s_{h+j-k} \ldots \alpha^s_p x_1 \ldots x_{j-p+h} & j = k+1, \ldots, k+p-h, \end{cases}$$

and, of course, we have the analogous expressions for the vertices of the path $Q_t$. We have to prove that $q_{s,j} \neq q_{t,i}$ for any $i,j = 1, \ldots, k+p-h$. By the symmetry of the paths, it suffices to compare $q_{s,j}$ with $q_{t,i}$ when $j \leqslant i$. The case $i = j$ is trivial because $\alpha^s_r \neq \alpha^t_r$ for any $r = h+1, \ldots, p$. Besides, since the paths are in a $p$-cycle, it is only necessary to prove that $q_{s,j} \neq q_{t,i}$ when $i \equiv j \pmod p$ and $i \geqslant j+p$.

Let us suppose that there exist $i,j$, where $i \equiv j \pmod p$ and $i \geqslant j+p$, such that $q_{s,j} = q_{t,i}$.

If $1 \leqslant j \leqslant p-h$ and $p-h+1 \leqslant i \leqslant k$, we have that

$$q_{s,j} = x_j, \ldots, x_k, \alpha^s_{h+1}, \ldots, \alpha^s_{h+j} = x_i, \ldots, x_k, \alpha^t_{h+1}, \ldots, \alpha^t_p, x_1 \ldots, x_{i-p+h} = q_{t,i}.$$

Let us consider the subsequences formed by the vertices of $C_p \otimes K^*_d$ in the partite set $V_{h+j}$:

$$x_{h+j} x_{h+p+j} \ldots x_{k-p+j} \alpha^s_{h+j} = x_{h+i} \ldots x_{k-p+i} \alpha^t_{h+j} x_{h+j} \ldots x_{i-p-h}.$$

Observe that $h+j \leqslant p$. We consider the *equivalence relation digraph* [14] given by this equality. The arcs of the equivalence relation digraph join a symbol appearing in the first sequence with the symbol that appears in the same place in the second sequence. In this digraph, the vertices $x_r$ have in-degree and out-degree equal to one, $\alpha^s_{h+j}$ have in-degree 0 and out-degree 1 and $\alpha^t_{h+j}$ have in-degree 1 and out-degree 0.

Then, there exists in the equivalence relation digraph a path from $\alpha^s_{h+j}$ to $\alpha^t_{h+j}$. That means that $\alpha^s_{h+j} = \alpha^t_{h+j}$, a contradiction.

If $p-h+1 \leqslant j < i \leqslant k$, we consider $j_0$ such that $j_0 \equiv j \equiv i \pmod p$ and $0 \leqslant j_0 \leqslant p-1$. From the equality $q_{s,j} = q_{t,i}$, we take the subsequences formed by the vertices in the partite set $V_p$ of $C_p \otimes K_d^*$:

$$x_{j+p-j_0} \ldots x_{k-h}\alpha^s_p x_p \ldots x_{j-j_0-\ell} = x_{i+p-j_0} \ldots x_{k-h}\alpha^t_p x_p \ldots x_{i-j_0-\ell}$$

where $\ell = 0$ if $j_0 \geqslant p-h$ and $\ell = p$ otherwise. As before, using the equivalence relation digraph, we obtain that $\alpha^s_p = \alpha^t_p$, a contradiction.

The remaining case, $p-h+1 \leqslant j \leqslant k$ and $k+1 \leqslant i \leqslant k+p-h$, is solved analogously. □

The following propositions are proved in a similar way.

**Proposition 4.3.** *The paths $Q_s$ and $R$ do not contain the arc $A$.*

**Proposition 4.4.** *The path $R$ is disjoint with any path $Q_s$.*

Therefore, we have constructed $d$ disjoint paths between any pair of adjacent vertices of $BGC(p,d,d^{k+1})$: one of length 1, $d-2$ of length at most $p+k+1$ and one of length at most $p+k+2$.

**Theorem 4.5.** *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be any pair of different vertices of the generalized cycle $BGC(p,d,d^{k+1})$, where $p \geqslant 2$. There exist a container from $\boldsymbol{x}$ to $\boldsymbol{y}$ of width $d$ and length less than or equal to $p+k+2 = D+2$ composed by one path with minimum length $d(\boldsymbol{x},\boldsymbol{y})$, $d-2$ with length at most $D+1$ and one of length at most $D+2$.*

**Proof.** We are going to prove this theorem by induction on $k$. If $k=0$, the result is true by Proposition 4.1. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two different vertices of $BGC(p,d,d^{k+1})$, $k \geqslant 1$. We have proved the existence of these paths if $d(\boldsymbol{x},\boldsymbol{y}) = 1$. Let us suppose that $d(\boldsymbol{x},\boldsymbol{y}) \geqslant 2$. Since $BGC(p,d,d^{k+1})$ is isomorphic to the line digraph $LBGC(p,d,d^k)$, we can put $\boldsymbol{x} = x_0x_1$ and $\boldsymbol{y} = y_0y_1$, where $x_0$, $x_1$, $y_0$ and $y_1$ are vertices of $BGC(p,d,d^k)$. Besides, $x_1 \neq y_0$, because $\boldsymbol{x}$ is not adjacent to $\boldsymbol{y}$. Then, in $BGC(p,d,d^k)$ there exist a container from $x_1$ to $y_0$ with width $d$ and length at most $p+k+1$. In this container there are one path of length $d(x_1,y_0)$, $d-2$ of length at most $p+k$, one of length at most $p+k+1$. These paths induce in the line digraph $LBGC(p,d,d^k) = BGC(p,d,d^{k+1})$ a container of width $d$ from $\boldsymbol{x} = x_0x_1$ to $\boldsymbol{y} = y_0y_1$, with one path of minimum length $d(\boldsymbol{x},\boldsymbol{y}) = d(x_1,y_0)+1$, $d-2$ of length at most $p+k+1 = D+1$ and one path of length at most $p+k+2 = D+2$. □

As a corollary, we obtain the values of the $s$-wide-diameter and the $s$-fault-diameters of $BGC(p,d,d^{k+1})$. We can see that these values are almost optimal by comparing them with the lower bounds given in Propositions 3.1 and 3.2. In effect, for any $s=2,\dots,d$,

$$D_{\min}(s,p,d,pd^{k-1}) = D'_{\min}(s,p,d,pd^{k-1}) = p-k+1.$$

**Theorem 4.6.** *Let $G$ be the generalized cycle $BGC(p,d,d^{k+1})$, where $p \geqslant 2$. Then*
- $d_s(G)=D'_s(G)=p+k+1=D+1$ *if* $2 \leqslant s \leqslant d-1$ *or* $s=d$ *and* $0 \leqslant k \leqslant p-2$.
- $d_d(G)=D'_d(G)=p+k+2=D+2$ *if* $k \geqslant p-1$.
- $d_s(G)=D_s(G)=p+k+1=D+1$ *if* $2 \leqslant s \leqslant d-1$ *or* $s=d$ *and* $0 \leqslant k \leqslant p-1$.
- $d_d(G)=D_d(G)=p+k+2=D+2$ *if* $k \geqslant p$.

**Proof.** It is obvious from Propositions 3.1 and 3.2 and Theorem 4.5 that $d_s(G)=D_s(G)=D'_s(G)=p+k+1=D+1$ if $2 \leqslant s \leqslant d-1$. The minimum value of $k$ for which we need a path with length $D+2$ (the path $R$) in order to construct the $d$ disjoint paths is $k=p-1$. Then, $D'_d(G)=D_{d-1}(G)=D+1$ if $0 \leqslant k \leqslant p-2$. Since in the digraph $BGC(p,d,d^{p-1})$ there are containers of width $d$ and length at most $p+k-1$ between any pair of different vertices, in the digraph $BGC(p,d,d^p)=LBGC(p,d,d^{p-1})$ we can find a container of width $d$ and length at most $p+k$ between any pair of non-adjacent vertices. Therefore, $D_d(G)=D+1$ if $k=p-1$. Finally, let us consider in $BGC(p,d,d^p)$ the vertices $\boldsymbol{x}=x_0x_1\dots x_{p-1}$ and $\boldsymbol{y}=x_1\dots x_{p-1}x_p$, where $x_0=(0,0)\in \boldsymbol{Z}_p\times\boldsymbol{Z}_d$ and $x_p=(0,d-1)$. If we remove from $BGC(p,d,d^p)$ the $d-1$ arcs $\boldsymbol{e}_i=x_0x_1\dots x_{p-1}\alpha^i_p$, where $\alpha^i_p=(0,i)$, $1 \leqslant i \leqslant d-1$, the distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ in the resulting digraph will be equal to $2p+1=D+2$. Using the line digraph technique, it is not difficult to find, for any $k>p-1$, $d-1$ vertices or arcs to be removed from $BGC(p,d,d^{k-1})$ in order to obtain a digraph with diameter $p+k+2$. Therefore, $D'_d(G)=D+2$ if $k \geqslant p-1$ and $D_d(G)=p+k+2=D+2$ if $k \geqslant p$. $\square$

## 5. Fault-diameters of $KGC(p,d,d^{p+k}+d^k)$

The goal of this section is to determine the $s$-wide-diameter and the $s$-fault-diameters of the generalized cycles $KGC(p,d,d^{D-p+1}+d^{D-2p-1})$. As we did in Section 4, we are going to find $d$ disjoint paths with length at most $D+2$ between any pair of different vertices of these digraphs.

The set of vertices of the generalized cycle $KGC(p,d,n)$ [8] is $\boldsymbol{Z}_p\times\boldsymbol{Z}_n$. If $0 \leqslant \alpha \leqslant p-2$, the vertex $(\alpha,x)$ is adjacent to $(\alpha+1,dx+t)$ for any $t=0,1,\dots,d-1$. The vertex $(p-1,x)$ is adjacent to $(0,-dx-(d-t))$ for any $t=0,1,\dots,d-1$. The digraph $KGC(p,d,d^p+1)$ is $d$-regular and has diameter $2p-1$. The generalized cycle $KGC(p,d,d^{D-p+1}+d^{D-2p+1})=KGC(p,d,d^{p-k}+d^k)$ is isomorphic to the iterated line digraph $L^kKGC(p,d,d^p+1)$. Then, it is $d$-regular and has diameter $D=2p+k-1$. These generalized cycles have large order (optimal if $2p-1 \leqslant 3p-1$) for their degree and diameter. Besides, if $p=2$, the digraphs $KGC(p,d,d^{D-p+1}+d^{D-2p+1})$ are equal to the bipartite digraphs $BD(d,d^{D-1}+d^{D-3})$ constructed by Fiol and Yebra [5]. If $p=1$, we obtain the Kautz digraphs.

We are going to proceed in the same way as in Section 4. First, we find the disjoint paths between vertices in the digraph $KGC(p,d,d^p+1)$, which is the smallest digraph in this family. Next, we construct disjoint paths of bounded length between any pair of adjacent vertices of $KGC(p,d,d^{p+k}+d^k)$. Finally, taking into account that these digraphs are iterated line digraphs, we prove the existence of disjoint paths between any pair of different vertices.

We recall now some properties of the generalized cycle $KGC(p,d,d^p+1)$. See [8] for proofs and more information. If $x$ and $y$ are two different vertices in the same partite set of $KGC(p,d,d^p+1)$, then $d(x,y)=p$ and there is only one shortest path form $x$ to $y$. Besides, there is no cycle of length $p$ in this digraph. For any vertex $x$ there are exactly $1+d+d^2+\cdots+d^p$ vertices $y$ such that $d(x,y)\leqslant p$. That is, if $d(x,y)\leqslant p$ there is only one path from $x$ to $y$ with length at most $p$.

**Proposition 5.1.** *Let $x$, $y$ be any pair of vertices of $KGC(p,d,d^p+1))$. There exist a container of width $d$ from $x$ to $y$ with length at most $2p=D+1$.*

**Proof.** We can suppose that $x\in V_0$ and $y\in V_h$, where $1\leqslant h\leqslant p$. Let $\Gamma^+(x)=\{z_1, z_2,\dots,z_d\}$ be the set of the vertices that are adjacent from $x$ and $\Gamma^-(y)=\{v_1,v_2,\dots,v_d\}$ be the set of vertices that are adjacent to $y$.

If $h=1$ and $(x,y)$ is not an arc, since there is a unique path of length $p$ from any $z_i$ to $y$, we have exactly $d$ paths of length $p+1$ from $x$ to $y$: the paths $xz_i\dots y$, $1\leqslant i\leqslant d$. Using the properties of $KGC(p,d,d^p+1))$, it is not difficult to see that these paths are disjoint. If $(x,y)$ is an arc, we can suppose that $z_1=y$. In this case, we have a path of length $l$, the arc $(x,y)$, and $d-1$ paths of length $p+1$: the paths $xz_i\dots y$, $2\leqslant i\leqslant d$. As before, these paths are disjoint.

If $h\geqslant 2$ and $d(x,y)=h$, we can suppose that the unique path of minimum length from $x$ to $y$ has the form $xz_1\dots v_1y$ (where $z_1=v_1$ if $h=2$). Let $\sigma$ be any permutation in $\{2,\dots,d\}$. For any $i=2,\dots,d$, let $w_i\in V_{h-1}$ be any vertex such that $d(z_i,w_i)=h-2$ and consider the path $xz_i\dots w_i\dots v_{\sigma i}y$, which is a path from $x$ to $y$ with length $p+h\leqslant 2p$. Observe that, since $w_i\neq v_{\sigma i}$ if $2\leqslant i\leqslant d$, there is a unique path of length $p$ from $w_i$ to $v_{\sigma i}$. By the properties of the generalized cycle $KGC(p,d,d^p+1))$, these $d$ paths from $x$ to $y$ are disjoint. If $d(x,y)=p+h$ or $h=p$ and $x=y$, we can consider any permutation $\sigma$ in $\{1,2,\dots,d\}$ and construct $d$ paths from $x$ to $y$ with length $2p$: the paths $xz_i\dots w_i\dots v_{\sigma i}y$, where $i=1,\dots,d$ and $w_i$ is a vertex in $V_{h-1}$ such that $d(z_i,w_i)=h-2$. As before, these paths are disjoint. $\square$

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be any pair of adjacent vertices of the generalized cycle $KGC(p,d, d^{p+k}+d^k)$. As we did in Section 4 for $BGC(p,d,d^{k+1})$, we are going to construct a container from $\boldsymbol{x}$ to $\boldsymbol{y}$ of width $d$ and bounded length. Since that digraph is isomorphic to the iterated line digraph $L^kKGC(p,d,d^p+1)$, its vertices can be written as paths of length $k$ in $KGC(p,d,d^p+1)$. Then, we can write $\boldsymbol{x}=x_0x_1\dots x_k$ and $\boldsymbol{y}=x_1\dots x_kx_{k+1}$, where $x_i$, $i=0,1,\dots,k+1$ are vertices of the generalized cycle $KGC(p,d,d^p+1)$. We are going to find $d-1$ disjoint paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ that do not contain the arc $A=(\boldsymbol{x},\boldsymbol{y})$. These paths are going to be to be constructed from disjoint paths from $x_k$ to $x_1$ in $KGC(p,d,d^p+1)$ and will have the form $x_0x_1\dots x_ka_{k+1}a_{k+2}\dots a_{k+r}x_1\dots x_kx_{k+1}$, where $r\leqslant 2p$, $a_{k+1}\neq x_{k+1}$ and $a_{k+r}\neq x_0$. Therefore, we must find in $KGC(p,d,d^p+1)$ a container $C(x_k,x_1)$ with width $d-1$ and length at most $2p+1$ such that their first and last arcs must be, respectively, different from $(x_k,x_{k+1})$ and $(x_0,x_1)$.

**Lemma 5.2.** *Let $x$ and $y$ be two vertices of $KGC(p,d,d^p+1)$ such that $d(x,y)\not\equiv 1 \pmod p$. Let us consider $z\in\Gamma^+(x)$ and $v\in\Gamma^-(y)$. Then, there exist a container of width $d-1$ and length at most $2p$, avoiding the arcs $(x,z)$ and $(v,y)$.*

**Proof.** We can suppose that $x\in V_0$ and $y\in V_h$, where $2\leqslant h\leqslant p$. If $d(x,y)=h$, let $xz_1\dots v_1y$ be the unique shortest path from $x$ to $y$ (if $h=2$, then $z_1=v_1$). We have to distinguish three cases.

*Case* 1: $h=p$ and $x=y$; or $d(x,y)=p+h$; or $d(x,y)=h$, $z=z_1$ and $v=v_1$; or $d(x,y)=h$, $z\neq z_1$ and $v\neq v_1$. By the proof of Proposition 5.1, we can find in this case a set of $d$ disjoint paths from $x$ to $y$ with length at most $2p$ containing a path in the form $xz\dots vy$. The other $d-1$ paths are the paths we are looking for.

*Case* 2: $d(x,y)=h$, $z=z_1$ and $v\neq v_1$. Let us consider a vertex $z_2\in\Gamma^+(x)$, $z_2\neq z$ and a vertex $u\in V_1$ such that $u\neq z_1$ and $d(u,v_1)=h-1$. Then, $u\notin\Gamma^+(x)$ and there is a path with length $p$ from $z_2$ to $u$. Let us consider the following $d-1$ paths with length $p+h$ from $x$ to $y$: the path $xz_2\dots u\dots v_1y$, and the paths $xz_i\dots w_i\dots v_{\sigma i}y$, $3\leqslant i\leqslant d$, constructed as in the proof of Proposition 5.1, where $v_{\sigma i}\neq v$. It is not difficult to prove that these paths are disjoint and do not contain neither the vertex $z$ nor the vertex $v$.

*Case* 3: $d(x,y)=h$, $z\neq z_1$ and $v=v_1$. This case is analogous to Case 2. □

**Lemma 5.3.** *Let $x$ and $y$ be two vertices of $KGC(p,d,d^p+1)$ such that $d(x,y)\equiv 1 \pmod p$. Let us consider $z\in\Gamma^+(x)$ and $v\in\Gamma^-(y)$. Then, there exist a container of width $d-1$ and length at most $2p+1$ from $x$ to $y$ such that all the paths in it have their first and last arcs are, respectively, different from $(x,z)$ and $(v,y)$. Besides, at most one of the paths in the container have length $2p+1$.*

**Proof.** We can suppose that $x\in V_0$ and $y\in V_1$. As we have seen in the proof of Proposition 5.1, there are exactly $d$ paths, which are disjoint, of length at most $p+1$ from $x$ to $y$.

*Case* 1: $d(z,v)=p-1$, or $x=v$ and $y=z$. If $d(z,v)=p-1$, the path $xz\dots vy$ is one of the $d$ disjoint paths from $x$ to $y$ with length at most $p+1$. The other $d-1$ paths are disjoint and avoid the arcs $(x,z)$ and $(v,y)$. If $x=v$ and $y=z$, the two forbidden arcs are equal to $(x, y)$. The other $d-1$ paths with length $p+1$ from $x$ to $y$ are the paths we are looking for.

*Case* 2: $d(z,v)=2p-1$, $x\neq v$ and $y\neq z$. In this case, there are two different paths from $x$ to $y$ with length $p+1$ containing one of the forbidden arcs: the paths $xz\dots v'y$ and $xz'\dots vy$. Then, there are $d-2$ paths from $x$ to $y$ with length at most $p+1$ avoiding the arcs $(x,z)$ and $(v,y)$. Let $w\in V_0$, $w\neq v$, be a vertex such that $d(z',w)=p-1$. Then, $w\notin\Gamma^-(y)$ and there exists a path of length $p$ from $w$ to $v'$. The path $xz'\dots w\dots v'y$, which has length $2p+1$, is disjoint with the above $d-2$ paths and do not contain neither $z$ nor $v$.

*Case* 3: $(x,y)$ is an arc, $x=v$ and $y\neq z$. Then, the $d$ paths from $x$ to $y$ with length at most $p+1$ are: the arc $(x,y)$, the path $xz\dots v'y$ and $d-2$ paths with length $p+1$ that do not contain any of the forbidden arcs. Let $w\in V_1$, $w\neq z$, be a vertex such that $d(w,v')=p-1$. Then, $w\notin\Gamma^+(x)$ and there is a path with length $p$ from $y$ to $w$. The path $xy\dots w\dots v'y$, which has length $2p+1$, is disjoint with the other $d-2$ paths, its first arc is different from $(x,z)$ and its last arc is different $(v,y)$.

*Case* 4: $(x,y)$ is an arc, $x\neq v$ and $y=z$. Analogously to Case 3, we have $d-2$ paths with length $p+1$ that do not contain any of the forbidden arcs. Let $z'$ be the vertex such that $xz'\dots vy$ is a path of length $p+1$. Let $w\in V_0$, $w\neq v$, be a vertex such that $d(z',w)=p-1$. Then, $w\notin\Gamma^-(y)$ and there is a path with length $p$ from $w$ to $x$. The path $xz'\dots w\dots xy$, which has length $2p+1$, and the above $d-2$ paths are the paths we are looking for. □

Let $\boldsymbol{x}=x_0x_1\dots x_k$ and $\boldsymbol{y}=x_1\dots x_kx_{k+1}$ be any pair of adjacent vertices of the generalized cycle $KGC(p,d,d^{p+k}+d^k)$. Let $h$ be the integer such that $h\equiv k \pmod p$ and $1\leqslant h\leqslant p$. By Lemmas 5.2 and 5.3, there exist a container $C(x_k,x_1)$ in $KGC(p,d,d^p+1)$ with width $d-1$ and length at most $2p+1$ such that every path in the container has the first and the last arcs are, respectively, different from $(x_k,x_{k+1})$ and $(x_0,x_1)$. Using these paths, we construct $d-1$ paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ with length at most $2p+k+1=D+2$. By doing that, we have obtained $d$ paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ that will be proved to be disjoint. The first of these paths is the arc

$$A=(\boldsymbol{x},\boldsymbol{y})=x_0x_1\dots x_kx_{k+1}.$$

There can be one path with length $k+p-h+1\leqslant D-p+1$,

$$P=x_0x_1\dots x_k\alpha^1_{h+1}\alpha^1_{h+2}\dots\alpha^1_px_1\dots x_kx_{k+1},$$

$d-1$, $d-2$ or $d-3$ paths with length $k+2p-h+1\leqslant D+1$,

$$Q_s=x_0x_1\dots x_k\alpha^s_{h+1}\alpha^s_{h+2}\dots\alpha^s_p\dots\alpha^s_{2p}x_1\dots x_kx_{k+1},$$

and, if $h=p$, we may need one path with length $D+2$,

$$R=x_0x_1\dots x_k\alpha^{d-1}_1\dots\alpha^{d-1}_p\dots\alpha^{d-1}_{2p}x_1\dots x_kx_{k+1}.$$

It can be proved that these paths are disjoint by using the same techniques as in Section 4. The proof of the following theorem is the same as in Theorem 4.5

**Theorem 5.4.** *Let* $\mathbf{x}$, $\mathbf{y}$ *be any pair of different vertices of the generalized cycle* $KGC(p,d,d^{p+k}+d^k)$, *where* $p \geqslant 2$. *There exists a container from* $\mathbf{x}$ *to* $\mathbf{y}$ *with width* $d$ *and length less than or equal to* $2p+k+1 = D+2$. *Moreover, in such container there is one path with minimum length* $d(\mathbf{x},\mathbf{y})$, *and there are* $d-2$ *paths with length at most* $D+1$ *and one of length at most* $D+2$.

As a corollary, we obtain the value of the $s$-wide-diameter $d_s(G)$, and the $s$-fault-diameters $D_s(G)$ and $D'_s(G)$ being $G = KGC(p,d,d^{p+k}+d^k)$. This theorem is proved in the same way as Theorem 4.6.

**Theorem 5.5.** *Let* $G$ *be the generalized cycle* $KGC(p,d,d^{p+k}+d^k)$, *where* $p \geqslant 2$. *Then*

- $d_s(G) = D'_s(G) = 2p+k = D+1$ *if* $2 \leqslant s \leqslant d-1$ *or* $s=d$ *and* $0 \leqslant k \leqslant p-1$.
- $d_d(G) = D'_d(G) = 2p+k+1 = D+2$ *if* $k \geqslant p$.
- $d_s(G) = D_s(G) = 2p+k = D+1$ *if* $2 \leqslant s \leqslant d-1$ *or* $s=d$ *and* $0 \leqslant k \leqslant p$.
- $d_d(G) = D_d(G) = 2p+k+1 = D+2$ *if* $k \geqslant p+1$.

We can see that these values are almost optimal by comparing them with the lower bounds given in Propositions 3.1 and 3.2. In effect, for any $s = 2,\ldots,d$,

$$D_{\min}(s,p,d,p(d^{p+k}+d^k)) = D'_{\min}(s,p,d,p(d^{p+k}+d^k)) = 2p+k.$$

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