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Connectivity and fault-tolerance of hyperdigraphs ☆

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Abstract

Directed hypergraphs are used to model networks whose nodes are connected by directed buses. We study in this paper two parameters related to the fault-tolerance of directed bus networks: the connectivity and the fault-diameter of directed hypergraphs. Some bounds are given for those parameters. As a consequence, we obtain that de Bruijn-Kautz directed hypergraphs and, more generally, iterated line directed hypergraphs provide models for highly fault-tolerant directed bus networks.

Keywords: Directed hypergraphs; Bus interconnection networks; Connectivity; Fault-tolerant interconnection networks

1. Introduction

A bus network consists in a set of processors and a set of buses providing communication channels between subsets of processors. Bus networks are represented by hypergraphs. If the buses are unidirectional, directed hypergraphs, called *hyperdigraphs* for short, are considered. See [2,3,8] for more details about this modelization.

Some basic requirements related to the design of bus interconnection networks lead to the search of families of hyperdigraphs with a good relation between their order and diameter for any given values of the maximum vertex-degree and bus-size. De Bruijn and Kautz hyperdigraphs [3] and, more generally, iterated line hyperdigraphs [1] are some of such families.

There exist two basic parameters that are generally considered for graphs and hypergraphs in relation to the fault-tolerance of interconnection networks [12]: the connectivity and the fault-diameter. The connectivity is the minimum number of vertices or edges that have to be deleted in order to disconnect the graph or hypergraph. The fault

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diameter is the maximum diameter of the subgraphs obtained by removing a given number of vertices or edges. In this paper, we study these parameters for directed hypergraphs. We present some bounds on those parameters. Some interesting properties of de Bruijn and Kautz hyperdigraphs and iterated line hyperdigraphs are derived from those bounds. The results we present here generalize some previous results about the connectivity and the fault-diameter of digraphs [9,5,14,6].

The main definitions and the notation that will be used in the following are given in Section 2. Some basic results about the connectivity of hyperdigraphs are presented in Section 3. Some properties about the vertex-connectivity and the vertex-fault-diameter of hyperdigraphs are derived from properties of these parameters for digraphs in Section 4. Previous results about digraphs cannot be directly applied to study the connectivity and fault-diameter when one considers the deletion of hyperarcs. This case is studied in Sections 5 and 6 by introducing some new parameters related to the bipartite representation of hyperdigraphs. Finally, some results about the connectivity and the fault-diameter of de Bruijn and Kautz hyperdigraphs and iterated line hyperdigraphs are given in Section 7.

2. Preliminaries

A *directed hypergraph*, also called *hyperdigraph* for short, H is a pair $(\mathscr{V}(H), \mathscr{E}(H))$, where $\mathscr{V}(H)$ is a non-empty set of *vertices* and $\mathscr{E}(H)$ is a set of ordered pairs of non-empty subsets of $\mathscr{V}(H)$, called *hyperarcs*. The *order* of H, denoted by $n(H)$, is the number of vertices, $n(H)=|\mathscr{V}(H)|$, and $m(H)$ will denote the number of hyperarcs. If $E=(E^-,E^+)$ is an hyperarc, we say that E^- and E^+ are, respectively, the *in-set* and *out-set* of E. The cardinalities of the in-set and the out-set of a hyperarc $E=(E^-,E^+)$ are, respectively, the *in-size*, $s^-(E)=|E^-|$, and the *out-size*, $s^+(E)=|E^+|$, of E. The *in-degree*, $d^-(v)$, of a vertex v is the number of hyperarcs containing v in the out-set, and the *out-degree*, $d^+(v)$, is the number of hyperarcs such that v is in their in-sets. The *minimum out-size* of a hyperdigraph H is defined by

$$s^+(H)=\min\{|E^+|\colon E\in\mathscr{E}(H)\}.$$

The *minimum in-size*, and the *maximum in-* and *out-size* are defined analogously. Similarly, the *minimum out-degree* of H is

$$d^+(H)=\min\{d^+(v)\colon v\in\mathscr{V}(H)\}.$$

Equally, one can define the *maximum in-degree*, $d^-(H)$. The *minimum degree* of H is $d(H)=\min\{d^+(H),d^-(H)\}$. The *maximum in-* and *out-degree* and the *maximum degree* are defined analogously. A hyperdigraph is *d-regular* if $d^-(v)=d^+(v)=d$ for any vertex $v\in\mathscr{V}(H)$. Similarly, a hyperdigraph is *s-uniform* if the out-size and the in-size of all its hyperarcs are equal to s.

A *path* of *length* k from a vertex u to a vertex v in H is an alternating sequence of vertices and hyperarcs $u=v_0,E_1,v_1,E_2,v_2,\ldots,E_k,v_k=v$ such that $v_i\in E^-_{i+1}$,

$(i=0,\ldots,k-1)$ and $v_i \in E_i^+$, $(i=1,\ldots,k)$. The *distance* from u to v is the length of a shortest path from u to v. The *diameter* of H, $D(H)$, is the maximum distance between every pair of vertices of H.

A hyperdigraph is *connected* if there exists at least one path between any pair of vertices. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph H is the minimum number of vertices that have to be removed from H to obtain a non-connected or trivial hyperdigraph (i.e., with only one vertex). The *hyperarc-connectivity*, $\lambda(H)$, is defined similarly.

The *vertex-fault-diameter*, $D_w(H)$, of a hyperdigraph H, is the maximum diameter of the hyperdigraphs obtained when w arbitrary vertices are removed from H. The *hyperarc-fault-diameter*, $D'_w(H)$, is defined similarly.

The vertices of the *dual hyperdigraph*, H^*, of a hyperdigraph H coincide with the hyperarcs of H, that is, $\mathscr{V}(H^*)=\mathscr{E}(H)$, and its hyperarcs are in one-to-one correspondence with the vertices of H. For every vertex v of H, there is an hyperarc $V=(V^-,V^+)$ of H^* such that, for any $E\in\mathscr{V}(H^*)=\mathscr{E}(H)$, $E\in V^-$, if and only if, $v\in E^+$, and $E\in V^+$, if and only if, $v\in E^-$.

The *underlying digraph* of a hyperdigraph H is the digraph $\hat{H}=(\mathscr{V}(\hat{H}),\mathscr{A}(\hat{H}))$, where $\mathscr{V}(\hat{H})=\mathscr{V}(H)$ and $(u,v)\in\mathscr{A}(\hat{H})$, if and only if, there exists $E\in\mathscr{E}(H)$ such that $u\in E^-$ and $v\in E^+$. That is, there is an arc from a vertex u to a vertex v in $\hat{H}$, if and only if, there is a hyperarc joining u to v in H. Therefore, paths in $\hat{H}$ and H are in one-to-one correspondence and, hence, $D(\hat{H})=D(H)$ and $\kappa(\hat{H})=\kappa(H)$. If there are more than one hyperarc joining two vertices in H, then $\hat{H}$ will have multiple arcs between these two vertices. Therefore, $\hat{H}$ is, in general, a multidigraph.

The *bipartite representation* of a hyperdigraph H is a bipartite digraph $R=R(H)=(V(R),A(R))$ with vertices $V(R)=V_0(R)\cup V_1(R)$, where $V_0(R)=\mathscr{V}(H)$ and $V_1(R)=\mathscr{E}(H)$, and arcs $A(R)=\{(u,E)|u\in V_0,E\in V_1,u\in E^-\}\cup\{(F,v)|v\in V_0,F\in V_1,v\in F^+\}$. Observe that, if u,v are two vertices of H, a path of length h from u to v in H corresponds to a path of length $2h$ in $R(H)$ and then, $d_R(u,v)=2d_H(u,v)$.

The *line hyperdigraph* of $H=(\mathscr{V}(H),\mathscr{E}(H))$ is defined in [1] as the hyperdigraph $LH=(\mathscr{V}(LH),\mathscr{E}(LH))$, where

$$\mathscr{V}(LH)=\bigcup_{E\in\mathscr{E}(H)}\{(uEv)\colon u\in E^-,v\in E^+\}$$

and

$$\mathscr{E}(LH)=\bigcup_{v\in\mathscr{V}(H)}\{(EvF)\colon v\in E^+,v\in F^-\},$$

with $(EvF)^-=\{(wEv):w\in E^-\}$ and $(EvF)^+=\{(vFw):w\in F^+\}$. Note that if H is a digraph, LH coincides with the line digraph of H. Besides, the underlying digraph of LH coincides with the line digraph $L\hat{H}$. The iteration of the line hyperdigraph technique provides a method to find hyperdigraphs with a large number of vertices for their values of the degree, bus size and diameter. In fact, de Bruijn and Kautz hyperdigraphs are iterated line hyperdigraphs [1]. We refer to [1] for other properties of the line hyperdigraph technique.

3. Basic results on connectivity

We say that a hyperdigraph H is *simple* if its underlying digraph $\hat{H}$ has no parallel arcs. That is, a hyperdigraph H is simple, if and only if, there does not exist any pair of hyperarcs E_1, E_2 of H with $E_1^- \cap E_2^- \neq \emptyset$ and $E_1^+ \cap E_2^+ \neq \emptyset$.

Proposition 3.1. *Let H be a hyperdigraph. Then, its line digraph LH is a simple hyperdigraph.*

Proof. Let E, F be hyperarcs of LH such that $E^- \cap F^- \neq \emptyset$. Suppose that $E=(E_1 v_1 F_1)$ and $F=(E_2 v_2 F_2)$. If $E^- \cap F^- \neq \emptyset$, then $E_1=E_2$ and $v_1=v_2$. Then, $E^+=\{(v_1 F_1 w_i)\colon w_i \in F_1^+\}$ and $F^+=\{(v_1 F_2 z_i)\colon z_i \in F_2^+\}$. If $E \neq F$, it must be $F_1 \neq F_2$ and then, $E^+ \cap F^+=\emptyset$. □

Let H be a hyperdigraph with minimum degree d and minimum size s. We denote by $\hat{d} = d(\hat{H})$ the minimum degree of the underlying digraph $\hat{H}$. Let κ and λ be, respectively, the vertex and hyperarc-connectivities of H and let $\hat{\kappa}$ and $\hat{\lambda}$ be the vertex and arc-connectivities of the underlying digraph $\hat{H}$.

It is clear that $\kappa = \hat{\kappa}$ and, from the properties of the connectivities of digraphs, $\hat{\kappa} \leqslant \hat{\lambda} \leqslant \hat{d}$. On the other hand, it is obvious that $\lambda \leqslant d$.

If the hyperdigraph H is s-uniform, we have that $\hat{d} \leqslant ds$. If, besides, H is simple, $\hat{d} = ds$. Then, in the uniform case, $\kappa = \hat{\kappa} \leqslant \hat{\lambda} \leqslant \hat{d} \leqslant ds$. Another relation between the connectivities of a hyperdigraph is given in next proposition.

Proposition 3.2. *Let $H = (\mathscr{V}(H), \mathscr{E}(H))$ be an s-uniform hyperdigraph with vertex-connectivity κ, hyperarc-connectivity λ and order $n > 2\lambda s$. Then, $\kappa \leqslant \lambda s$.*

Proof. Let $\mathscr{F} = \{E_1, \ldots, E_\lambda\} \subset \mathscr{E}(H)$ be a disconnecting set of H. Let us consider $\mathscr{F}^- = E_1^- \cup \cdots \cup E_\lambda^-$ and $\mathscr{F}^+ = E_1^+ \cup \cdots \cup E_\lambda^+$. Since the order of H is $n > 2\lambda s$, there is a vertex $z \notin \mathscr{F}^- \cup \mathscr{F}^+$. Let us consider two vertices $x, y \in \mathscr{V}(H)$ such that there is no path from x to y in $H - \mathscr{F}$. Then, all paths from x to z contain an internal vertex in $\mathscr{F}^+$ or all paths from z to y pass through $\mathscr{F}^-$. If not, we could find a path from x to y in $H - \mathscr{F}$. Therefore, one of the hyperdigraphs $H - \mathscr{F}^-$ or $H - \mathscr{F}^+$ is not connected and, hence, $\kappa(H) \leqslant \lambda s$. □

We say that a hyperdigraph H is *maximally connected* if $\kappa = \hat{d}$ and $\lambda = d$. Note that, if $\kappa = ds$ and Proposition 3.2 holds, $\lambda = d$. In this case, a hyperdigraph is maximally connected if and only if $\kappa = ds$.

4. Fault-tolerance under deletion of vertices

The vertex-connectivity of a hyperdigraph H coincides with the vertex-connectivity of its underlying digraph $\hat{H}$, that is, $\kappa = \kappa(H) = \hat{\kappa} = \kappa(\hat{H})$. The same occurs with the w-vertex-fault-diameter: $D_w(H) = D_w(\hat{H})$ for any $w = 1, \ldots, \hat{d} - 1$, where $\hat{d}$ is the minimum degree of $\hat{H}$. Therefore, the numerous known results about this parameter for digraphs, can be applied for hyperdigraphs just by considering the underlying digraph.

Next results are obtained by considering the results about vertex-connectivity of digraphs given in [5]. These results are based on the parameter ℓ_π, that was introduced in the same paper. We recall here its definition.

Definition 4.1. Let G be a digraph with minimum degree $d \geqslant 2$ and diameter D. Let π be an integer such that $0 \leqslant \pi \leqslant d - 2$. We define $\ell_\pi = \ell_\pi(G)$ as the maximum integer, with $1 \leqslant \ell_\pi \leqslant D$, such that for any pair of vertices $x, y \in V(G)$,

- if $d(x, y) < \ell_\pi$, there is only one shortest path from x to y and there are at most π paths from x to y with length $d(x, y) + 1$.
- if $d(x, y) = \ell_\pi$, there is only one shortest path from x to y.

Observe that, in any case, $\ell_\pi(G) \geqslant 0$. If $\pi \geqslant 1$ or $\pi = 0$ and G is loopless, then $\ell_\pi(G) \geqslant 1$.

Proposition 4.1. *Let H be a simple hyperdigraph with diameter D and vertex-connectivity κ. Let $\hat{d} \geqslant 2$ be the minimum degree of the underlying digraph $\hat{H}$ and consider $\ell_\pi = \ell_\pi(\hat{H})$, where $0 \leqslant \pi \leqslant \hat{d} - 2$. Then, $\kappa \geqslant \hat{d} - \pi$ if $D \leqslant 2\ell_\pi - 1$.*

Some interesting corollaries about the vertex-connectivity of iterated line hyperdigraphs are deduced from this proposition. The following one is proved by taking into account that $\widehat{L^k H} = L^k \hat{H}$ and that $\ell_\pi(L^k \hat{H}) = \ell_\pi(\hat{H}) + k$ whenever H is a simple digraph, $\hat{H}$ is not a cycle and $\ell_\pi(\hat{H}) \geqslant 1$ [5].

Corollary 4.2. *Let H be a simple hyperdigraph with diameter D. Let $\hat{d} \geqslant 2$ be the minimum degree of the underlying digraph $\hat{H}$ and consider $\ell_\pi = \ell_\pi(\hat{H})$, where $0 \leqslant \pi \leqslant \hat{d} - 2$ and $\ell_\pi(\hat{H}) \geqslant 1$. Then, $\kappa(L^k H) \geqslant \hat{d} - \pi$ if $k \geqslant D - 2\ell_\pi + 1$.*

The particular case $\pi = 0$ is specially interesting.

Corollary 4.3. *Let H be a simple hyperdigraph with diameter D such that its underlying digraph $\hat{H}$ is loopless. Let us consider $\ell_0 = \ell_0(\hat{H}) \geqslant 1$. Then, $\kappa(L^k H) = \hat{d}$ if $k \geqslant D - 2\ell_0 + 1$.*

Since the line hyperdigraph LH is simple for any hyperdigraph H, we can see from the last corollary that, for any hyperdigraph H such that $\hat{H}$ is loopless, the vertex-connectivity of $L^k H$ is maximum if the number of iterations k is large enough. If, besides, H is s-uniform, we have seen that H is maximally connected if and only if $\kappa = ds$. Therefore, in that case, the iterated line hyperdigraph $L^k H$ is maximally connected if k is large enough.

In a similar way, we can apply the results in [6] for fault-diameters of digraphs to find bounds on the vertex-fault-diameter of hyperdigraphs. In particular, from the results for iterated line digraphs, we can see that, if k is large enough, the w-vertex-fault-diameter of an iterated line hyperdigraph L^kH is $D_w(L^kH) \leqslant D(L^kH) + C$, where C is a constant that depends only on w and $\hat{H}$, but does not depend on the number of iterations k.

5. Hyperarc-connectivity

The aim of this section is to present some bounds for the hyperarc-connectivity of any hyperdigraph H. Sufficient conditions for a hyperdigraph to have maximum hyperarc-connectivity are derived.

Let us recall that the bipartite representation of a hyperdigraph H is a bipartite digraph $R = R(H) = (V(R), A(R))$ with set of vertices $V(R) = V_0(R) \cup V_1(R)$, where $V_0(R) = \mathscr{V}(H)$ and $V_1(R) = \mathscr{E}(H)$, and set of arcs

$$A(R) = \{(u,E) \mid u \in V_0, E \in V_1, u \in E^-\} \cup \{(F,v) \mid v \in V_0, F \in V_1, v \in F^+\}.$$

Observe that, if u, v are two vertices of H, a path of length h from u to v in H correspond to a path of length $2h$ in $R(H)$ and, then, $d_R(u,v) = 2d_H(u,v)$. Also the bipartite representation of the line hyperdigraph LH is $R(LH) = L^2R(H)$.

The hyperarc-connectivity $\lambda = \lambda(H)$ of a hyperdigraph H can be expressed in terms of the bipartite representation of H. In effect, λ is the minimum cardinality of all the subsets $\mathscr{F} \subset V_1$ such that there exist two vertices $u, v \in V_0$ such that there is no path from u to v in $R - \mathscr{F}$.

We define next a parameter, similar to the parameter ℓ_π, that will be useful to bound the hyperarc-connectivity. This parameter is defined for bipartite digraphs and will be applied to the bipartite representation of the hypergraph. Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph. Let us consider $d_0^+(R) = \min_{v \in V_0} d^+(v)$, the minimum out-degree of the vertices in V_0, and $d_0^-(R)$, the minimum in-degree of the vertices in V_0. Let us take $d_0 = d_0(R) = \min\{d_0^+, d_0^-\}$.

Definition 5.1. Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph with $d_0(R) \geqslant 2$ and diameter D. Let π be an integer such that $0 \leqslant \pi \leqslant d_0 - 2$. We define $h_\pi = h_\pi(R)$ as the maximum integer, with $1 \leqslant h_\pi \leqslant D$, such that for any pair of vertices x, y, with $x \in V_i$, $y \in V_j$ and $i \neq j$,

- if $d(x,y) < h_\pi$, there is only one shortest path from x to y and there are at most π paths from x to y with length $d(x,y) + 2$;
- if $d(x,y) = h_\pi$, there is only one shortest path from x to y.

Observe that the vertices x and y that appear in the definition of the parameter $h_\pi(R)$ are different, that is, $d(x,y) \geqslant 1$. Then, it is clear that, for any bipartite digraph R, there exists $h_\pi(R)$ and $h_\pi(R) \geqslant 1$.

Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph. Then, the iterated line digraph L^2R is a bipartite digraph and, in a natural way, we can put, for $i = 0, 1$, $V_i(L^2R) = \{x_0x_1x_2 \in V(L^2R) \mid x_0 \in V_i(R)\}$. In this situation, $d_0(L^2R) = d_0(R)$ and we can consider $h_\pi(R)$ and $h_\pi(L^2R)$ for the same values of π.

Proposition 5.1. *Let $R=(V_0(R)\cup V_1(R),A(R))$ be a bipartite digraph different from a cycle. Then, $h_\pi(L^2R)=h_\pi(R)+2$ for any $\pi=1,\dots,d_0-2$. If there are no cycles of length 2 in R, then $h_0(L^2R)=h_0(R)+2$.*

Proof. Let us consider $\boldsymbol{x}=x_0x_1x_2\in V_i(L^2R)$ and $\boldsymbol{y}=y_0y_1y_2\in V_j(L^2R)$, where $i\neq j$. If $d(\boldsymbol{x},\boldsymbol{y})\geqslant 3$ and $d(\boldsymbol{x},\boldsymbol{y})\leqslant h_\pi(R)+2$, then $d(x_2,y_0)\leqslant h_\pi(R)$. Therefore, the shortest path from x_2 to y_0 is unique and so is the shortest path from $\boldsymbol{x}$ to $\boldsymbol{y}$. If $d(\boldsymbol{x},\boldsymbol{y})\geqslant 3$ and $d(\boldsymbol{x},\boldsymbol{y})<h_\pi(R)+2$, then $d(x_2,y_0)<h_\pi(R)$ and there are at most π paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ with length $d(\boldsymbol{x},\boldsymbol{y})+2$. If $d(\boldsymbol{x},\boldsymbol{y})=1$, then the vertices x_1x_2 and y_0y_1 of LR are equal. Since in LR there is at most one cycle of length 2 on the vertex x_1x_2, in L^2R there is at most one path with length $3=d(\boldsymbol{x},\boldsymbol{y})+2$ from $\boldsymbol{x}$ to $\boldsymbol{y}$. If R has no cycles of length 2, there is no path of length 3 from $\boldsymbol{x}$ to $\boldsymbol{y}$. Therefore, $h_\pi(L^2R)\geqslant h_\pi(R)+2$ if $\pi\geqslant 1$ or $\pi=0$ and R has no cycles of length 2. Since R is not a cycle, it is not difficult to see that $h_\pi(L^2R)\leqslant h_\pi(R)+2$. □

Proposition 5.2. *Let $R=(V_0\cup V_1,A)$ be a bipartite digraph and let us consider $h_\pi=h_\pi(R)$, where $0\leqslant\pi\leqslant d_0-2$. Let us consider a vertex $x\in V_0$, a subset $\mathscr{F}\subset V_1$, with $|\mathscr{F}|\leqslant d_0-\pi-1$, and a vertex $y\in\mathscr{F}$. Then,*

- *There exists a vertex $x_1\in V_0$ and a path xy_1x_1 such that $y_1\notin\mathscr{F}$ and $d(x_1,y)\geqslant\min\{d(x,y)+2,h_\pi\}$ and $d(x_1,y')\geqslant\min\{d(x,y'),h_\pi\}$ for any $y'\in\mathscr{F}$.*
- *There exists a vertex $x_{-1}\in V_0$ and a path $x_{-1}y_{-1}x$ such that $y_{-1}\notin\mathscr{F}$ and $d(y,x_{-1})\geqslant\min\{d(y,x)+2,h_\pi\}$ and $d(y',x_{-1})\geqslant\min\{d(y',x),h_\pi\}$ for any $y'\in\mathscr{F}$.*

Proof. We are going to prove the first statement. The second one is proved analogously. Let $\Gamma^+(x)$ be the set of vertices that are adjacent from x and let us consider the set $v(x\to\mathscr{F})\subset\Gamma^+(x)$ defined by: $z\in v(x\to\mathscr{F})$ if and only if there exists $y'\in\mathscr{F}$ such that $d(x,y')\leqslant h_\pi$ and (x,z) is the first arc of the shortest path from x to y'. Since $|v(x\to\mathscr{F})|\leqslant d_0-\pi-1$, there exists a vertex $y_1\in\Gamma^+(x)-v(x\to\mathscr{F})$ such that the first vertex of any path from x to y with length $d(x,y)+2$ is different from y_1. Let x_1 be any vertex in $\Gamma^+(y_1)$. It is not difficult to prove that this vertex satisfies the required conditions. □

Theorem 5.3. *Let H be a hyperdigraph with minimum degree d, diameter D and hyperarc-connectivity λ. Let $R=R(H)$ be its bipartite representation and consider $h_\pi=h_\pi(R)$. Then, $\lambda\geqslant d-\pi$ if $D\leqslant h_\pi-1$.*

Proof. We are going to prove that, if $D\leqslant h_\pi-1$, for any set of vertices of the bipartite representation $\mathscr{F}\subset V_1=\mathscr{E}(H)$, with $|\mathscr{F}|\leqslant d-\pi-1$, and for any pair of vertices $u,v\in V_0$, there exists a path from u to v in $R-\mathscr{F}$. Effectively, from Proposition 5.2, we can find in R a path $uE_1u_1E_2u_2,\dots,E_mu_m$ such that $E_i\notin\mathscr{F}$ and $d_R(u_m,\mathscr{F})\geqslant h_\pi$. Equally, we can find a path $v_{-n}E_{-n},\dots,v_{-2}E_{-2}v_{-1}E_{-1}v$ such that $E_{-i}\notin\mathscr{F}$ and $d_R(v_{-n},\mathscr{F})\geqslant h_\pi$. Then, a shortest path from u_m to v_{-n}, of length at most $2D<2h_\pi$, will avoid $\mathscr{F}$. □

The following corollary is a direct consequence of Theorem 5.3 and Proposition 5.1.

Corollary 5.4. *Let H be a hyperdigraph with minimum degree d, and diameter D. Let $R = R(H)$ be its bipartite representation and consider $h_\pi = h_\pi(R)$. Then,*
- $\lambda(L^k H) \geqslant d - \pi$ *if* $k \geqslant D - h_\pi + 1$.
- *If R has no cycles of length 2, then* $\lambda(L^k H) = d$ *if* $k \geqslant D - h_0 + 1$.

6. Hyperarc-fault-diameter

The *hyperarc-fault-diameter*, $D'_w(H)$, of a hyperdigraph H, which is defined as the maximum diameter of the hyperdigraphs obtained from H by removing at most w hyperarcs.

In the same way as we did for the hyperarc-connectivity, we are going to use the bipartite representation $R(H)$ to study that parameter. In particular, we present a bound on $D'_w(H)$ in terms of $h_0(R)$ and the parameter $M_{0,1}(R)$. The parameter $M_{\pi,r}$ was defined in [6]. We present here its definition for the particular case $\pi = 0$ and $r = 1$.

Definition 6.1. Let G be a loopless digraph $d \geqslant 2$. A $(0,1)$-*double detour* in G is a set of four paths $\{C_1, C'_1, C_2, C'_2\}$ such that
- C_1 and C'_1 are paths from x to f, with lengths s and s', respectively, where $s' \geqslant s$ and $s' \geqslant 1$. C_2 and C'_2 are paths from f to y, with lengths t and t', respectively, where $t' \geqslant t$ and $t' \geqslant 1$. Besides, $\max\{s,t\} \geqslant 1$.
- If $s \neq 0$ and (x, x_1) and (x, x'_1) are, respectively, the first arcs of C_1 and C'_1, then $x'_1 \neq x_1$.
- If $t \neq 0$ and (y_1, y) and (y'_1, y) are, respectively, the last arcs of C_2 and C'_2, then $y'_1 \neq y_1$.

The *length* of a $(0,1)$-double detour is defined to be $s' + t'$. We define $M_{0,1}(G)$ as the minimum length of a $(0,1)$-double detour in G.

The following two propositions are proved in [6].

Proposition 6.1. $M_{0,1}(G) \geqslant 4$ *for any loopless digraph G with minimum degree $d \geqslant 2$.*

Proposition 6.2. *Let G be a loopless digraph with minimum degree $d \geqslant 2$. Then, for any positive integer k, $M_{0,1}(L^k(G)) = M_{0,1}(G) + k$.*

We are going to use the following lemma, which is proved in a similar way as Proposition 5.2.

Lemma 6.3. *Let $R = (V_0 \cup V_1, A)$ be a bipartite digraph without cycles of length 2 and $h_0 = h_0(R)$. Let us consider a vertex $x \in V_0$ and a subset $\mathscr{F} \subset V_1$, with $|\mathscr{F}| \leqslant d_0 - 1$.*

Then,

- *There exists a vertex* $x_1 \in V_0$ *and a path* xy_1x_1 *such that* $y_1 \notin \mathscr{F}$ *and* $d(x_1, y) \geqslant \min\{d(x, y) + 2, h_\pi\}$ *for any* $y \in \mathscr{F}$.
- *There exists a vertex* $x_{-1} \in V_0$ *and a path* $x_{-1}y_{-1}x$ *such that* $y_{-1} \notin \mathscr{F}$ *and* $d(y, x_{-1}) \geqslant \min\{d(y, x) + 2, h_\pi\}$ *for any* $y \in \mathscr{F}$.

Theorem 6.4. *Let* H *be a hyperdigraph with minimum degree* d *and diameter* D *such that its bipartite representation* $R = R(H)$ *has no cycles of length* 2. *Let us consider* $h = h_0(R)$ *and* $M = M_{0,1}(R)$. *Then, if* $D \leqslant h - 1$, *for any* $w = 1, \ldots, d - 1$, *the w-hyperarc-fault-diameter of* H *verifies* $D'_w(H) \leqslant D + C$, *where*

$$C = \max\left\{D - \left\lfloor \frac{M-1}{2} \right\rfloor + 4, 2\left(D - \left\lfloor \frac{h}{2} \right\rfloor\right)\right\}.$$

Proof. Let $\mathscr{F} \subset \mathscr{E}(H) = V_1(R)$ be a set of faulty hyperarcs with $|\mathscr{F}| = w < d$. We are going to prove that, for any pair of vertices $x, y \in \mathscr{V}(H) = V_0(R)$, there exists in H a path from x to y with length at most $D + C$ avoiding the hyperarcs in $\mathscr{F}$. From Lemma 6.3, there exist paths xE_1x_1 and $y_{-1}E_{-1}y$ in R such that $E_1, E_{-1} \notin \mathscr{F}$ and $d_R(x_1, \mathscr{F}), d_R(\mathscr{F}, y_{-1}) \geqslant 3$ (observe that $h \geqslant D + 1 \geqslant 2$). Besides, from the definition of the parameter $M_{0,1}(R)$, we have that $d_R(x_1, F) + d_R(F, y_{-1}) \geqslant M - 4$ for any $F \in \mathscr{F}$. Applying again Lemma 6.3, for any $m, n \geqslant 1$ we can find paths $xE_1x_1 \ldots E_mx_m$ and $y_{-n}E_{-n} \ldots y_{-1}E_{-1}y$ such that, for any $F \in \mathscr{F}$,

$$d_R(x_m, F) \geqslant \min\{d_R(x_1, F) + 2(m-1), h\},$$

$$d_R(F, y_{-n}) \geqslant \min\{d_R(F, y_{-1}) + 2(n-1), h\}.$$

Then, if $m, n \geqslant D - \lfloor h/2 \rfloor$ and $m + n = C$, it is not difficult to see that $d_R(x_m, F) + d_R(F, y_{-n}) > 2D$ for any $F \in \mathscr{F}$. Therefore, any shortest path in R from x_m to y_{-n}, which has length at most $2D$, will avoid $\mathscr{F}$. Hence, we have found a path from x to y in H with length at most $D + m + n = D + C$ avoiding the faulty hyperarcs in $\mathscr{F}$. □

As a consequence of Theorem 6.4, we obtain the following result about the hyperarc-fault-diameter of iterated line hyperdigraphs.

Corollary 6.5. *Let* H *be a hyperdigraph with minimum degree* d *and diameter* D *such that its bipartite representation* $R = R(H)$ *has no cycles of length* 2. *Let us consider* $h = h_0(R)$ *and* $M = M_{0,1}(R)$. *Then, for any* $k \geqslant D - h + 1$ *and for any* $w = 1, \ldots, d - 1$, *the w-hyperarc-fault-diameter of the iterated line hyperdigraph* L^kH *verifies* $D'_w(L^kH) \leqslant D(L^kH) + C$, *where*

$$C = \max\left\{D - \left\lfloor \frac{M-1}{2} \right\rfloor + 4, 2\left(D - \left\lfloor \frac{h}{2} \right\rfloor\right)\right\}.$$

Proof. Apply Theorem 6.4 by taking into account that $R(L^k H) = L^{2k} R(H)$ and that $h_0(L^{2k} R) = h_0(R) + 2k$ (Proposition 5.1) and $M_{0,1}(L^{2k} R) = M_{0,1}(R) + 2k$ (Proposition 6.2). □

7. Fault-tolerance of de Bruijn and Kautz hyperdigraphs

We apply next the results in the above sections in order to study the connectivities and fault-diameters of de Bruijn and Kautz hyperdigraphs and iterated line hyperdigraphs.

We recall here the definition and some basic properties of de Bruijn and Kautz hyperdigraphs. See [1,3] for proofs and more information about those families. Let n, d, s, m be integers such that $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The *generalized de Bruijn hyperdigraph* $H_1 = GB(d, n, s, m)$ and the *generalized Kautz hyperdigraph* $H_2 = GK(d, n, s, m)$ have set of vertices $\mathscr{V}(H_i) = \boldsymbol{Z}_n$ and set of hyperarcs $\mathscr{E}(H_i) = \boldsymbol{Z}_m$. The incidences in the generalized de Bruijn hyperdigraph H_1 are given by

- $u \in E^-$ if and only if $E \equiv du + \alpha \pmod{m}$, where $0 \leqslant \alpha \leqslant d - 1$
- $v \in E^+$ if and only if $u \equiv sE + \beta \pmod{n}$, where $0 \leqslant \beta \leqslant s - 1$.

The incidences of the generalized Kautz hyperdigraph H_2 are defined by

- $u \in E^-$ if and only if $E \equiv du + \alpha \pmod{m}$, where $0 \leqslant \alpha \leqslant d - 1$
- $v \in E^+$ if and only if $u \equiv -sE - \beta \pmod{n}$, where $1 \leqslant \beta \leqslant s$.

The out-degree of any vertex of H_i is equal to d and all hyperarcs have out-size s. H_i is d-regular and s-uniform if $dn = sm$. The underlying digraph of the generalized de Bruijn hyperdigraph $H_1 = GB(d, n, s, m)$ is the *generalized de Bruijn digraph* or *Reddy–Pradhan–Kuhl digraph* [15,10] with degree ds and order n, that is, $\widehat{H_1} \cong GB(ds, n)$. Equally, the underlying digraph of the generalized Kautz hyperdigraph $H_2 = GK(d, n, s, m)$ is *generalized Kautz digraph* or *Imase–Itoh digraph* [11] with degree ds and order n, that is, $\widehat{H_2} \cong GK(ds, n)$. Therefore, the diameter of H_i is minimum or almost minimum. The line hyperdigraph of a generalized de Bruijn or Kautz hyperdigraph is another hyperdigraph in the same family: $LGB(d, n, s, m) \cong GB(d, dsn, s, dsm)$ and $LGK(d, n, s, m) \cong GK(d, dsn, s, dsm)$.

If we take $n = (ds)^D$ and $m = d^2(ds)^{D-1}$, where $D \geqslant 2$, we obtain the *de Bruijn hyperdigraph*

$$H = HB(d, s, D) = GB(d, (ds)^D, s, d^2(ds)^{D-1}),$$

whose underlying digraph is $\hat{H} \cong B(ds, D)$, the *de Bruijn digraph* [4,7] with degree ds and diameter D. Similarly, by considering $n = (ds)^D + (ds)^{D-1}$ and $m = d^2((ds)^{D-1} + (ds)^{D-2})$, where $D \geqslant 2$, we obtain the *Kautz hyperdigraph*

$$H = HK(d, s, D) = GK(d, (ds)^D + (ds)^{D-1}, s, d^2((ds)^{D-1} + (ds)^{D-2})),$$

whose underlying digraph is $\hat{H} \cong K(ds, D)$, the *Kautz digraph* [13,7] with degree ds and diameter D. De Bruijn and Kautz hyperdigraphs, $HB(d, s, D)$ and $HK(d, s, D)$, are d-regular and s-uniform, and their order is very close to the Moore-like bound for their degree d, size s and diameter D. Observe that the Bruijn and Kautz hyperdigraphs are iterated line hyperdigraphs: for instance, $HK(d,s,D) = L^{D-2}GK(d,(ds)^2 + ds, s, d^2(ds+1)) = L^{D-2}HK(d,s,2)$.

The vertex-connectivity of $GB(d,n,s,m)$ and $GK(d,n,s,m)$ can be directly derived from the results about the vertex-connectivity of their underlying digraphs [9,5].

Theorem 7.1. *Let us consider positive integers d,n,s,m, with $dn \equiv_m 0$ and $sm \equiv_n 0$. Let H be the generalized de Bruijn hyperdigraph $H = GB(d,n,s,m)$. Then, $\kappa(H) = ds - 1$ if $D(H) \geqslant 3$.*

Theorem 7.2. *Let us consider positive integers d,n,s,m, with $dn \equiv_m 0$ and $sm \equiv_n 0$. Let H be the generalized Kautz hyperdigraph $H = GK(d,n,s,m)$. Then, $\kappa(H) \geqslant ds - 1$* if *$D(H) \geqslant 3$. Besides, if $D(H) \geqslant 5$,*

$$\kappa(H) = \begin{cases} ds & \text{if } n \text{ is a multiple of } ds+1 \text{ and } \gcd(ds,n) \neq 1, \\ ds-1 & \text{otherwise.} \end{cases}$$

If $dn = sm$, the hyperdigraphs $H_1 = GB(d,n,s,m)$ and $H_2 = GK(d,n,s,m)$ are s-uniform. In this case, we can find their hyperarc-connectivity because $\kappa(H_i) \leqslant \lambda(H_i)s$. Therefore, if $D(H_i) \geqslant 3$, we have that $\lambda(H_i)s \geqslant ds - 1$ and, hence, $\lambda(H_i) = d$ if $s \geqslant 2$. In particular, if $D, s \geqslant 2$, the hyperdigraphs $HB(d,s,D)$ and $HK(d,s,D)$ have hyperarc-connectivity $\lambda = d$.

In order to find the hyperarc-connectivity of generalized de Bruijn and Kautz hyperdigraphs, $H_1 = GB(d,n,s,m)$ and $H_2 = GK(d,n,s,m)$, we observe that $h_1(H_i) \geqslant 2\lfloor \log_{ds} n \rfloor$. Therefore, if $D(H_i) \geqslant 3$, we can apply Theorem 5.3 and obtain $\lambda(H_i) \geqslant d - 1$.

Since the vertex-fault-diameters of the de Bruijn and Kautz digraphs are equal to those of their underlying digraphs, we can apply the results in [14] about the fault-diameters of de Bruijn and Kautz digraphs. Therefore, $D_w(HB(d,s,D)) \leqslant D + 2$ for any $w = 1, \ldots, ds - 2$ and $D_w(HK(d,s,D)) \leqslant D + 2$ for any $w = 1, \ldots, ds - 1$.

We can apply Corollary 6.5 in order to find the hyperarc-fault-diameter of Kautz hyperdigraphs by taking into account that $HK(d,s,D) = L^{D-2}HK(d,s,2)$. Therefore, $D'_w(HK(d,s,D)) \leqslant D + 2$ for any $w = 1, \ldots, ds - 1$.

References

[1] J.C. Bermond, F. Ergincan, Directed line hypergraphs, private communication.

[2] J.C. Bermond, F. Ergincan, Bus interconnection networks, Discrete Appl. Math. 68 (1996) 1–15.

[3] J.C. Bermond, R.W. Dawes, F. Ergincan, De Bruijn and Kautz Bus Networks, Networks 30 (1997) 205–218.

[4] N.G. de Bruijn, A combinatorial problem, Koninklijke Nederlandse Academie van Wetenschappen Proc. A49 (1946) 758–764.

[5] J. Fàbrega, M.A. Fiol, Maximally connected digraphs, J. Graph Theory 13-6 (1999) 657–688.

[6] D. Ferrero, C. Padró, New bounds on the diameter vulnerability of iterated line digraphs, Discrete Math., to appear.

[7] M.A. Fiol, J.L.A. Yebra, I. Alegre, Line-digraph Iterations and the (d,k) Problem, IEEE Trans. Comput. C-33 (1984) 400–403.

[8] G. Gallo, G. Longo, S. Pallotino, Directed hypergraphs and applications, Discrete Appl. Math. 42 (1993) 177–201.
[9] N. Homobono, C. Peyrat, Connectivity of Imase and Itoh digraphs, IEEE Trans. Comput. C-37 (1988) 1459–1461.
[10] M. Imase, M. Itoh, Design to Minimize Diameter in Building-Block Network, IEEE Trans. Comput. C-30 (1981) 439–442.
[11] M. Imase, M. Itoh, A design for directed graphs with minimum diameter, IEEE Trans. Comput. C-32 (1983) 782–784.
[12] M. Imase, T. Soneoka, K. Okada, Fault-tolerant processor interconnection networks, Systems Comput. Jpn. 17 (1986) 21–30.
[13] W.H. Kautz, Bounds on directed (d,k) graphs, Theory of cellular logic networks and machines, AFCRL-68-0668 Final Report, 1968, pp. 20–28.
[14] C. Padró, P. Morillo, Diameter-vulnerability of iterated line digraphs, Discrete Math. 149 (1996) 189–204.
[15] S.M. Reddy, D.K. Pradham, J. Kuhl, Directed graphs with minimum diameter and maximum connectivity, Technical Report, School of Engineering Oakland University, Rochester, MI 1980.