Partial Line Directed Hypergraphs

Daniela Ferrero
Department of Mathematics, Southwest Texas State University, San Marcos, Texas 78666

Carles Padró
Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Campus Nord, Edifici C3, C. Jordi Girona 1-3, 08034 Barcelona, Spain

The partial line digraph technique was introduced in [7] in order to construct digraphs with a minimum diameter, maximum connectivity, and good expandability. To find a new method to construct directed hypergraphs with a minimum diameter, we present in this paper an adaptation of that technique to directed hypergraphs. Directed hypergraphs are used as models for interconnection networks whose vertices are linked by directed buses. The connectivity and expandability of partial line directed hypergraphs are studied. Besides, we prove a conjecture by J-C. Bermond and F. Ergincan about the characterization of line directed hypergraphs. © 2002 Wiley Periodicals, Inc.

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1. INTRODUCTION

Graphs and hypergraphs are used as interconnection network models. While point-to-point networks are modeled by graphs, hypergraphs are used as models for bus networks. If the buses are unidirectional, directed hypergraphs, called hyperdigraphs for short, are considered. See [3, 9] for more details about this modeling technique.

Since the number of buses that a vertex can be connected to and the number of vertices that can be communicated by a bus are both limited, it is not possible to interconnect every pair of vertices in a bus network with a large number of vertices. Therefore, several buses should be used, in general, to connect two vertices. The diameter of the hypergraph that models the network measures the transmission delay in the communications. Therefore, it is interesting to find families of hypergraphs, directed or not, with a minimum diameter for all fixed values of the maximum vertex degree and bus size.

We present here a new method to construct directed hypergraphs with a minimum diameter. This method is an adaptation of the partial line digraph technique [7] to hyperdigraphs and can be seen as a generalization of the line hyperdigraph technique [2].

The iteration of the line hyperdigraph technique provides hyperdigraphs with a large order for fixed values of the diameter, vertex degree, and bus size. Besides, iterated line hyperdigraphs have good properties in relation to the fault-tolerance of bus networks [6]. In particular, de Bruijn and Kautz hyperdigraphs [1] are iterated line hyperdigraphs.

We introduce in this paper the partial line hyperdigraph technique, which makes it possible to construct hyperdigraphs with a minimum diameter for any number of vertices and fixed values of the maximum vertex degree and bus size. We study the connectivity and the expandability of partial line hyperdigraphs. The expandability is related to the capability of a network to increase its number of processors without loss of performance [7]. The relation between line and partial line hyperdigraphs is also studied. As a consequence, we obtain that partial line hyperdigraphs constructed from iterated line hyperdigraphs, such as Kautz hyperdigraphs, have a minimum diameter, maximum connectivity, and good expandability, which are all very good properties in relation to the design of directed bus networks.

Some definitions and notation are given in the next section. The partial line hyperdigraph technique is presented in Section 3. Some basic properties of partial line hyperdigraphs are given in this section. The connectivity and the expandability of partial line hyperdigraphs are studied in Sections 4 and 5, respectively. In Section 6, we apply the partial line hyperdigraph technique to Kautz hyperdigraphs to obtain a new family of hyperdigraphs with a minimum diameter. Finally, in Section 7,

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Correspondence to: C. Padró; e-mail: matcpl@mat.upc.es

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we prove a conjecture by J.-C. Bermond and F. Ergincan in relation to the characterization of line hyperdigraphs.

2. PRELIMINARIES

A *directed hypergraph* H, also called a *hyperdigraph* for short, is a pair $(\mathscr{V}(H), \mathscr{E}(H))$, where $\mathscr{V}(H)$ is a nonempty set of *vertices* and $\mathscr{E}(H)$ is a set of ordered pairs of nonempty subsets of $\mathscr{V}(H)$, called *hyperarcs*. The *order* of H, denoted by $n(H)$, is the number of vertices, $n(H) = |\mathscr{V}(H)|$, and $m(H)$ will denote the number of hyperarcs. If $E = (E^-, E^+)$ is a hyperarc, we say that E^- and E^+ are, respectively, the *in-set* and *out-set* of E. The cardinalities of the in-set and the out-set of a hyperarc $E = (E^-, E^+)$ are, respectively, the *in-size*, $s^-(E) = |E^-|$, and the *out-size*, $s^+(E) = |E^+|$, of E. The *in-degree*, $d^-(v)$, of a vertex v is the number of hyperarcs containing v in the out-set, and the *out-degree*, $d^+(v)$, is the number of hyperarcs such that v is in their in-sets. The *maximum out-size* of a hyperdigraph H is defined by

$$s^+(H) = \max\{|E^+| : E \in \mathscr{E}(H)\}.$$

The *maximum in-size* and the *minimum in-* and *out-size* are defined analogously. Similarly, the *maximum out-degree* of H is

$$d^+(H) = \max\{d^+(v) : v \in \mathscr{V}(H)\}.$$

Equally, one can define the *maximum in-degree* and the *minimum in-* and *out-degree*. A hyperdigraph is *d-regular* if $d^-(v) = d^+(v) = d$ for any vertex $v \in \mathscr{V}(H)$. Similarly, a hyperdigraph is *s-uniform* if the out-size and the in-size of all its hyperarcs are equal to s.

A *path of length k* from a vertex u to a vertex v in H is an alternating sequence of vertices and hyperarcs $u = v_0, E_1, v_1, E_2, v_2, \ldots, E_k, v_k = v$ such that $v_i \in E^-_{i+1}, (i = 0, \ldots, k-1)$ and $v_i \in E^+_i, (i = 1, \ldots, k)$. The *distance* from u to v is the length of a shortest path from u to v. The *diameter* of $H, D(H)$, is the maximum distance between every pair of vertices of H.

A hyperdigraph is *connected* if there exists at least one path between any pair of vertices. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph H is the minimum number of vertices that have to be removed from H to obtain a disconnected or trivial hyperdigraph (i.e., one having a single vertex). The *hyperarc-connectivity*, $\lambda(H)$, is defined similarly.

The vertices of the *dual hyperdigraph*, H^*, of a hyperdigraph H coincide with the hyperarcs of H, that is, $\mathscr{V}(H^*) = \mathscr{E}(H)$, and its hyperarcs are in one-to-one correspondence with the vertices of H. For every vertex v of H, there is a hyperarc $V = (V^-, V^+)$ of H^* such that, for any $E \in \mathscr{V}(H^*) = \mathscr{E}(H), E \in V^-$ if and only if $v \in E^+$, and $E \in V^+$ if and only if $v \in E^-$.

The *underlying digraph* of a hyperdigraph H is the digraph $\hat{H} = (\mathscr{V}(\hat{H}), \mathscr{A}(\hat{H}))$, where $\mathscr{V}(\hat{H}) = \mathscr{V}(H)$ and $(u, v) \in \mathscr{A}(\hat{H})$ if and only if there exists $E \in \mathscr{E}(H)$ such that $u \in E^-$ and $v \in E^+$, that is, there is an arc from a vertex u to a vertex v in $\hat{H}$ if and only if there is a hyperarc joining u to v in H. Therefore, paths in $\hat{H}$ and H are in one-to-one correspondence and, hence, $D(\hat{H}) = D(H)$ and $\kappa(\hat{H}) = \kappa(H)$. If there is more than one hyperarc joining two vertices in H, then $\hat{H}$ will have multiple arcs between these two vertices. Therefore, $\hat{H}$ is, in general, a multidigraph.

The *line hyperdigraph* of $H = (\mathscr{V}(H), \mathscr{E}(H))$ is defined in [2] as the hyperdigraph $LH = (\mathscr{V}(LH), \mathscr{E}(LH))$, where

$$\mathscr{V}(LH) = \bigcup_{E \in \mathscr{E}(H)} \{(uEv) : u \in E^-, v \in E^+\}$$

and

$$\mathscr{E}(LH) = \bigcup_{v \in \mathscr{V}(H)} \{(EvF) : v \in E^+, v \in F^-\},$$

with $(EvF)^- = \{(wEv) : w \in E^-\}$ and $(EvF)^+ = \{(vFw) : w \in F^+\}$. The next two propositions were proved by Bermond and Ergincan in [2].

Proposition 2.1. *Let H be a hyperdigraph. The underlying digraph of LH coincides with the line digraph $L\hat{H}$. Besides, if H is a digraph, LH coincides with the line digraph of H.*

Proof. Observe that there is a one-to-one correspondence between the arcs of the underlying digraph $\hat{H}$ and the set $\mathscr{V}(LH)$ of vertices of the line hyperdigraph LH, that is, there exists a one-to-one correspondence between the sets of vertices of the digraphs $L\hat{H}$ and $\widehat{LH}$. It is easy to check that this mapping is, in fact, a digraph isomorphism. In the particular case that H is a digraph, we have that $\hat{H} = H$ and, hence, the line hyperdigraph of H coincides with the line digraph of H. ∎

Proposition 2.2. *Let H be a hyperdigraph. The dual of the line hyperdigraph of H is isomorphic to the line hyperdigraph of the dual hyperdigraph of H, that is, $(LH)^* \cong LH^*$.*

Proof. Observe that

$$\mathscr{V}((LH)^*) = \mathscr{E}(LH) = \bigcup_{v \in \mathscr{V}(H)} \{(EvF) : v \in E^+, v \in F^-\}$$

and

$$\mathscr{E}((LH)^*) = \mathscr{V}(LH) = \bigcup_{E \in \mathscr{E}(H)} \{(uEv) : u \in E^-, v \in E^+\}.$$

On the other hand,

$$\mathscr{V}(LH^*) = \bigcup_{V \in \mathscr{E}(H^*)} \{(EVF) : E \in V^-, F \in V^+\}$$

and

$$\mathscr{E}(LH^*) = \bigcup_{E \in \mathscr{V}(H^*)} \{(UEV) : E \in U^+, E \in V^-\}.$$

It is obvious that there exist natural one-to-one mappings $\phi : \mathscr{V}((LH)^*) \to \mathscr{V}(LH^*)$ and $\Phi : \mathscr{E}((LH)^*) \to \mathscr{E}(LH^*)$, defining an isomorphism between $(LH)^*$ and LH^*. ∎

3. PARTIAL LINE HYPERDIGRAPHS

The main definitions and some basic properties about the partial line hyperdigraph technique are given in this section.

Let $H = (\mathscr{V}(H), \mathscr{E}(H))$ be a connected hyperdigraph. To define a *partial line hyperdigraph* of H, we consider

- A set $\mathscr{W} \subset \mathscr{V}(LH)$ of vertices of LH such that, for any $v \in \mathscr{V}(H)$, there exists in $\mathscr{W}$ at least one vertex of the form uEv.
- A mapping ϕ, which is defined as follows: For every pair (E, vFw), where vFw is a vertex LH and E is a hyperarc of H such that $v \in E^+$ and there exist a vertex $uEv \in \mathscr{W}$,

$$\phi(E, vFw) = \begin{cases} vFw & \text{if } vFw \in \mathscr{W} \\ v'F'w \in \mathscr{W} & \text{otherwise,} \end{cases}$$

where v' and F' are chosen arbitrarily.

The *partial line hyperdigraph* $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ has the set of vertices $\mathscr{V}(\mathscr{L}H) = \mathscr{W}$ and the set of hyperarcs

$$\mathscr{E}(\mathscr{L}H) = \{\overline{EvF} : EvF \in \mathscr{E}(LH) \text{ such that there exists } uEv \in \mathscr{W}\},$$

where $(\overline{EvF})^- = \{uEv \in \mathscr{W} : u \in E^-\}$ and $(\overline{EvF})^+ = \{\phi(E, vFw) : w \in F^+\}$.

Notice that there exist as many partial line hyperdigraphs of H as different pairs $(\mathscr{W}, \phi)$ with the above properties. In particular, if H has order N, we can construct a partial line hyperdigraph of H with order N' for any $N' = N, N+1, \ldots, |\mathscr{V}(LH)|$. Observe that $\mathscr{L}H = LH$ if $N' = |\mathscr{V}(LH)|$. Besides, in the particular case that H is a digraph, $\mathscr{L}H$ coincides with the partial line digraph defined in [7]. So, the partial line hyperdigraph technique is a generalization of the line hyperdigraph [2] and the partial line digraph techniques [7].

Example. Let $H = (\mathscr{V}(H), \mathscr{E}(H))$ be the hyperdigraph with vertices $\mathscr{V}(H) = \{0, 1, 2, 3, 4, 5\}$ and hyperarcs $\mathscr{E}(H) = \{E_0, E_1, E_2\}$, where

$$\begin{array}{ll} E_0^- = \{0, 3\} & E_0^+ = \{4, 5\} \\ E_1^- = \{1, 4\} & E_1^+ = \{2, 3\} \\ E_2^- = \{2, 5\} & E_2^+ = \{0, 1\}. \end{array}$$

We are going to construct a partial line hyperdigraph of H with a set of vertices

$$\mathscr{W} = \{0E_04, 0E_05, 1E_13, 2E_20, 2E_21, 3E_05, 4E_12, 4E_13, 5E_21\} \subset \mathscr{V}(LH).$$

Observe that $\mathscr{V}(LH) = \mathscr{W} \cup \{1E_12, 3E_04, 5E_20\}$, that is, we have chosen nine vertices in $\mathscr{V}(LH)$, which has 12 elements. At this point, we only have to define a function ϕ to construct a partial line hyperdigraph $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$. According to the definition, the only possible choice for this function is

$$\phi(E_2, 1E_12) = 4E_12, \qquad \phi(E_1, 3E_04) = 0E_04, \qquad \phi(E_0, 5E_20) = 2E_20,$$

and $\phi(E, vFw) = vFw$ for every $vFw \in \mathscr{W}$ and $v \in E^+$. In this case, the partial line hyperdigraph $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ has the set of vertices $\mathscr{V}(\mathscr{L}H) = \mathscr{W}$ and the set of hyperarcs

$$\mathscr{E}(\mathscr{L}H) = \{\overline{E_04E_1}, \overline{E_05E_2}, \overline{E_12E_2}, \overline{E_13E_0}, \overline{E_20E_0}, \overline{E_21E_1}\},$$

where

$$\begin{array}{ll} (\overline{E_04E_1})^- = \{0E_04\} & (\overline{E_04E_1})^+ = \{4E_12, 4E_13\} \\ (\overline{E_05E_2})^- = \{0E_05, 3E_05\} & (\overline{E_05E_2})^+ = \{2E_20, 5E_21\} \\ (\overline{E_12E_2})^- = \{4E_12\} & (\overline{E_12E_2})^+ = \{2E_20, 2E_21\} \\ (\overline{E_13E_0})^- = \{1E_13, 4E_13\} & (\overline{E_13E_0})^+ = \{0E_04, 3E_05\} \\ (\overline{E_20E_0})^- = \{2E_20\} & (\overline{E_20E_0})^+ = \{0E_04, 0E_05\} \\ (\overline{E_21E_1})^- = \{2E_21, 5E_21\} & (\overline{E_21E_1})^+ = \{4E_12, 1E_13\}. \end{array}$$

The next proposition relates the out-degrees and out-sizes of H and $\mathscr{L}H$. As a consequence, the maximum and minimum out-degree and out-size of $\mathscr{L}H$ coincide with those of H. Observe that this does not occur with the in-degrees and in-sizes.

Proposition 3.1. *Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H. For any vertex (uEv) and any hyperarc $(\overline{EvF})$ of $\mathscr{L}H$,*

- $d^+_{\mathscr{L}H}(uEv) = d^+_H(v)$,
- $s^+_{\mathscr{L}H}(\overline{EvF}) = s^+_H(F)$.

We prove in the next proposition that the underlying digraph of $\mathscr{L}H$ is a partial line digraph of the underlying digraph of H.

Proposition 3.2. *Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of a connected hyperdigraph H. Then, there exist a set $\hat{\mathscr{W}}$ of arcs of the underlying digraph $\hat{H}$ and a mapping $\hat{\phi}$ such that*

$$\widehat{\mathscr{L}H} \cong \mathscr{L}\hat{H} = \mathscr{L}(\hat{H}, \hat{\mathscr{W}}, \hat{\phi}).$$

Proof. To simplify the proof, let us suppose that $\hat{H}$ is a simple digraph. The proof can be easily adapted to the case that $\hat{H}$ is a multidigraph. Let $\hat{\mathscr{W}}$ be the set of arcs $(u, v) \in A(\hat{H})$ such that there exists $E \in \mathscr{E}(H)$ with $(uEv) \in \mathscr{W}$. For any pair $((u, v), (v, w))$ of arcs of $\hat{H}$ with $(u, v) \in \hat{\mathscr{W}}$, we define $\hat{\phi}((u, v), (v, w)) = (v', w) \in \hat{\mathscr{W}}$, where $\phi(E, vFw) = v'F'w$ and $(uEv) \in \mathscr{W}$. It is not difficult to check that $\widehat{\mathscr{L}H} \cong \mathscr{L}(\hat{H}, \hat{\mathscr{W}}, \hat{\phi})$. ∎

As a direct consequence of this proposition, some properties of partial line hyperdigraphs can be derived from properties of partial line digraphs. For instance, we obtain in this way the diameter of partial line hyperdigraphs.

Proposition 3.3. *Let H be a hyperdigraph with order N and diameter D. Let $\mathscr{L}H$ be a partial line digraph of H with order $N' > N$. Then, the diameter of $\mathscr{L}H$ is $D(\mathscr{L}H) = D + 1$.*

Proof. This result is proved in [7] for partial line digraphs. Therefore, using Proposition 3.2 and the properties of the underlying digraph of a hyperdigraph,

$$D(\mathscr{L}H) = D(\widehat{\mathscr{L}H}) = D(\mathscr{L}\hat{H}) = D(\hat{H}) + 1 = D(H) + 1.$$

∎

We present next the relation between line and partial line hyperdigraphs. We prove that some partial line hyperdigraphs of iterated line hyperdigraphs are also iterated line hyperdigraphs.

Theorem 3.4. *Let H be a connected hyperdigraph and let LH be the line hyperdigraph of H. Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H. Then, there exists $\mathscr{L}LH = \mathscr{L}(LH, \mathscr{W}_1, \phi_1)$, a partial line hyperdigraph of LH, such that $L\mathscr{L}H \cong \mathscr{L}LH$.*

Proof. The vertices of $L\mathscr{L}H$ are of the form $(uEv)(\overline{EvF})(v'F'w)$, where $uEv \in \mathscr{W}$ and $v'F'w = \phi(E, vFw)$. The vertices of L^2H can be identified with paths of length 2 in H, that is, they are of the form $(uEv)(EvF)(vFw)$ and can be denoted by $uEvFw$. Let us consider

$$\mathscr{W}_1 = \{uEvFw \in \mathscr{V}(L^2H) : uEv \in \mathscr{W}\}.$$

For any pair $(EvF, vFwGx)$, where $vFwGx$ is a vertex of L^2H and EvF is a hyperarc of LH such that there exists $uEvFw \in \mathscr{W}_1$, we define $\phi_1(EvF, vFwGx) = v'F'wGx$, where $v'F'w = \phi(E, vFw)$. It is not difficult to check that the mappings $\Phi : \mathscr{W}_1 \to \mathscr{V}(L\mathscr{L}H)$ and $\Psi : \mathscr{E}(\mathscr{L}LH) \to \mathscr{E}(L\mathscr{L}H)$ defined, respectively, by $\Phi(uEvFw) = (uEv)(\overline{EvF})(v'F'w)$ and $\Psi(EvFwG) = (\overline{EvF})(v'F'w)(\overline{F'wG})$, where $v'F'w = \phi(E, vFw)$, define an isomorphism between $\mathscr{L}(LH, \mathscr{W}_1, \phi_1)$ and $L\mathscr{L}H$. ∎

Corollary 3.5. *Let H be a connected hyperdigraph and let L^kH be an iterated line hyperdigraph of H. Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H. Then, there exists $\mathscr{L}L^kH = \mathscr{L}(L^kH, \mathscr{W}_1, \phi_1)$, a partial line hyperdigraph of L^kH, such that $L^k\mathscr{L}H \cong \mathscr{L}L^kH$.* ∎

Let H be a hyperdigraph with order N and minimum in-degree d. Observe that, for any integer N' with $dN \le N' \le |\mathscr{V}(LH)|$, we can choose $\mathscr{W}$ and ϕ in such a way that $\mathscr{E}(\mathscr{L}H) = \{\overline{EvF} : EvF \in \mathscr{E}(LH)\}$ and $\phi(E, vFw) = v'Fw$. In this situation, the hyperarcs of $\mathscr{L}H$ are in one-to-one correspondence with the hyperarcs of the line hyperdigraph LH. Besides, all vertices in $(\overline{EvF})^+$ are of the form $v'Fw$.

4. CONNECTIVITY

A hyperdigraph H has *no redundant short paths* when there is at most one path of length one or two between every pair of vertices (different or not) of H. Note that under this restriction we can still work with interesting hyperdigraphs, for instance, the generalized de Bruijn and Kautz hyperdigraphs.

Lemma 4.1. *Let H be a hyperdigraph. Then, H has no redundant short paths if and only if $\hat{H}$ has no redundant short paths.* ∎

Theorem 4.2. *Let H be a connected hyperdigraph without redundant short paths. Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H. Let $\hat{d}$ be the minimum degree of the underlying digraph $\widehat{\mathscr{L}H}$. Then,*

$$\kappa(\mathscr{L}H) \ge \min\{\kappa(H), \hat{d}\}.$$

Proof. By Lemma 4.1, $\hat{H}$ has no redundant short paths. Then, from the bound on the connectivity of partial line digraphs given in [7] and Proposition 3.2, we have $\kappa(\mathscr{L}\hat{H}) \ge \min\{\kappa(\hat{H}), \hat{d}\}$. The proof is concluded by taking into account that the vertex-connectivities of a hyperdigraph and its underlying digraph are equal. ∎

To find bounds on the hyperarc-connectivity, we need the following result of [2]:

Lemma 4.3. *Let H be a hyperdigraph with hyperarc-connectivity λ. Then, every vertex v in H is on λ hyperarc-disjoint cycles.* ∎

Theorem 4.4. *Let H be a connected hyperdigraph with order N and minimum degree d. Let $\mathscr{L}H = \mathscr{L}(H, \mathscr{W}, \phi)$, where $|\mathscr{W}| \ge dN$, be a partial line hyperdigraph of H such that its hyperarcs are in one-to-one correspondence with the hyperarcs of the line hyperdigraph LH. Then, the hyperarc-connectivity of $\mathscr{L}H$ satisfies $\lambda(\mathscr{L}H) \ge \lambda(H)$.*

Proof. It is enough to prove that there exist $\lambda = \lambda(H)$ hyperarc-disjoint paths between any pair of different vertices of $\mathscr{L}H$. Let $(uEv), (xFy)$ be two different vertices of $\mathscr{L}H$. We consider two cases:

1. If $v \ne x$, we have λ hyperarc-disjoint paths from v to x in H:
$$P_i = v, E_1^i, v_1^i, E_2^i, v_2^i, \ldots, E_{n_i-1}^i, v_{n_i-1}^i, E_{n_i}^i, x,$$
where $i = 1, \ldots, \lambda$. Each path P_i gives rise to a path from (uEv) to (xFy), $\mathscr{L}P_i$ in $\mathscr{L}H$, defined by
$$\mathscr{L}P_i = (uEv), (\overline{EvE_1^i}), (v'E_1^iv_1^i), (\overline{E_1^iv_1^iE_2^i}), ((v_1^i)'E_2^iv_2^i), \ldots, ((v_{n_i}^i)'E_{n_i}^ix), (\overline{E_{n_i}^ixF}), (xFy).$$
It is not difficult to see that the paths $\mathscr{L}P_i$ are hyperarc-disjoint.
2. If $v = x$, we proceed as before but with hyperarc-disjoint cycles in H. By Lemma 4.3, if the hyperarc-connectivity is λ, each vertex of H is in λ hyperarc-disjoint cycles. In the same way as we do with paths P_i, we can obtain λ paths in $\mathscr{L}H$ from these cycles in H. Again, since the original cycles are hyperarc-disjoint, these new paths are hyperarc-disjoint also.

∎

5. EXPANDABILITY

The concept of expandability is related to the capability of a network to increase its number of processors without loss of performance [7]. Given two hyperdigraphs H and H' with, respectively, N and N' vertices, $N \leq N'$, we define the *index of expandability* of H to $H', e(H, H')$, as the minimum number of hyperarcs that have to be modified in H in order to obtain H' by adding $N' - N$ vertices and some appropriate hyperarcs, if necessary.

If H is a hyperdigraph, $\mathscr{L}_N H$ will denote a partial line hyperdigraph of H with order N. We prove next that any hyperdigraph $\mathscr{L}_N H$ has good expandability to some $\mathscr{L}_{N+1} H$.

Theorem 5.1. *Let H be a connected hyperdigraph with maximum degree d and let $\mathscr{L}_N H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H, where $|\mathscr{V}(H)| \leq N \leq |\mathscr{V}(LH)| - 1$. Then, there exists a partial line hyperdigraph $\mathscr{L}_{N+1} H = \mathscr{L}(H, \mathscr{W}', \phi')$ such that the index of expandability of $\mathscr{L}_N H$ to $\mathscr{L}_{N+1} H$ satisfies*

$$e(\mathscr{L}_N H, \mathscr{L}_{N+1} H) \leq 2d.$$

Proof. The hyperdigraph $\mathscr{L}_{N+1} H$ can be obtained from $\mathscr{L}_N H$ by the following algorithm:

1. Choose a vertex (uEv) of LH which is not in $\mathscr{W}$. Such a vertex exists because $|\mathscr{W}| \leq |\mathscr{V}(LH)| - 1$.
2. Add the vertex (uEv) to $\mathscr{L}_N H$, that is, consider $\mathscr{W}' = \mathscr{W} \cup \{uEv\}$.
3. For every hyperarc of $\mathscr{L}_N H$ denoted by $\overline{FuE}$, replace in their out-sets the vertex $u'E'v = \phi(F, uEv)$ by the vertex $uEv = \phi'(F, uEv)$.
4. For every hyperarc F of H with $v \in F^-$,

- If $\overline{EvF}$ is not a hyperarc of $\mathscr{L}_N H$, define $\phi'(E, vFw)$ for every $w \in F^+$ and add the hyperarc $\overline{EvF}$ to $\mathscr{L}_N H$, where $(\overline{EvF})^- = \{uEv\}$ and $(\overline{EvF})^+ = \{\phi'(E, vFw) : w \in F^+\}$.
- If $\overline{EvF}$ is a hyperarc of $\mathscr{L}_N H$, add the vertex uEv to its in-set.

Observe that the hyperarcs of $\mathscr{L}_N H$ are only modified in steps 3 and 4 of the algorithm. Therefore, at most $d_H^-(u) + d_H^+(v)$ hyperarcs are modified. ∎

The above proof gives an algorithm to expand partial line hyperdigraphs. With a few changes, it can be used to decrease the number of vertices, that is, to construct $\mathscr{L}_N H$ from $\mathscr{L}_{N+1} H$.

If H is 1-uniform, only the out-sets of existing hyperarcs have to be modified in the above algorithm. Therefore, the expandability of partial line *digraphs* satisfies $e(\mathscr{L}_N H, \mathscr{L}_{N+1} H) \leq d$, where d is the maximum degree of H. This result was proved in [7].

In some applications, it could also be useful to compute the number of vertex-to-vertex connections that have to be modified to expand the hyperdigraph. The next proposition is a direct consequence of the results about the expandability of partial line digraphs given in [7].

Proposition 5.2. *Let H be a connected hyperdigraph with maximum degree d and maximum size s. Let $\mathscr{L}_N H = \mathscr{L}(H, \mathscr{W}, \phi)$ be a partial line hyperdigraph of H, where $|\mathscr{V}(H)| \leq N \leq |\mathscr{V}(LH)| - 1$. Then, there exists a partial line hyperdigraph $\mathscr{L}_{N+1} H = \mathscr{L}(H, \mathscr{W}', \phi')$ such that the number of connections that have to be modified to transform $\mathscr{L}_N H$ to $\mathscr{L}_{N+1} H$ is at most ds.* ∎

6. PARTIAL LINE HYPERDIGRAPHS WITH MINIMUM DIAMETER

A Moore-like bound for the order of a hyperdigraph with diameter D, maximum out-degree d, and maximum out-size s was given in [4]:

$$N \leq 1 + (ds) + (ds)^2 + \cdots + (ds)^D = \frac{(ds)^{D+1} - 1}{ds - 1}.$$

A lower bound for the diameter of a hyperdigraph with order N, maximum out-degree d, and maximum out-size s can be easily deduced:

$$D \geq D_{\min}(d, s, N) = (\log_{ds}(N(ds - 1) + 1)) - 1.$$

Let H be a hyperdigraph with order N, maximum out-degree d, maximum out-size s, and minimum diameter $D = D_{\min}(d, s, N)$. The partial line hyperdigraphs $\mathscr{L}H$ have maximum out-degree d, maximum out-size s, order N', where $N < N' \leq Nds$, and diameter $D(\mathscr{L}H) = D+1$. Therefore, the diameter of $\mathscr{L}H$ exceeds by at most one the lower bound, that is, $D(\mathscr{L}H) \leq D_{\min}(d, s, N') + 1$.

If, in particular, we apply the partial line hyperdigraph technique to Kautz hyperdigraphs, a generalization of Kautz digraphs that was introduced in [1], we obtain a new family of hyperdigraphs with a minimum or almost minimum diameter. Besides, these hyperdigraphs have other interesting properties (connectivity, expandability) that make them suitable to be considered as models for bus interconnection networks.

We recall here the definition and some basic properties of Kautz hyperdigraphs. See [1, 2] for proofs and more information about that family. Let N, d, s, m be integers such that $dN \equiv 0 (\bmod m)$ and $sm \equiv 0 (\bmod N)$. The sets of vertices and hyperarcs of the *generalized Kautz hyperdigraph* $H = GK(d, N, s, m)$ are, respectively, $\mathscr{V}(H) = \mathbf{Z}_N$ and $\mathscr{E}(H) = \mathbf{Z}_m$. The incidences are given by

- $u \in E^-$ if and only if $E \equiv du + \alpha (\bmod m)$, where $0 \leq \alpha \leq d - 1$
- $v \in E^+$ if and only if $u \equiv -sE - \beta (\bmod N)$, where $1 \leq \beta \leq s$.

The out-degree of any vertex of H is equal to d and all hyperarcs have out-size s. H is d-regular and s-uniform if $dN = sm$. The underlying digraph of the generalized Kautz hyperdigraph $H = GK(d, N, s, m)$ is a *generalized Kautz digraph* or a *Imase–Itoh digraph* [11] with degree ds and order N, that is, $\hat{H} \cong GK(ds, N)$. Therefore, the diameter of H exceeds the lower bound by at most one. The line hyperdigraph of a generalized Kautz hyperdigraph is another generalized Kautz hyperdigraph: $LGK(d, N, s, m) \cong GK(d, dsN, s, dsm)$.

In particular, if we take $N = (ds)^D + (ds)^{D-1}$ and $m = d^2((ds)^{D-1} + (ds)^{D-2})$, where $D \geq 2$, we obtain the *Kautz hyperdigraph*

$$H = HK(d, s, D) = GK(d, (ds)^D + (ds)^{D-1}, s, d^2((ds)^{D-1} + (ds)^{D-2})),$$

whose underlying digraph is $\hat{H} \cong K(ds, D)$, the *Kautz digraph* [8, 12] with degree ds and diameter D. Kautz hyperdigraphs $HK(d, s, D)$ are d-regular and s-uniform, and their order is very close to the Moore-like bound for their degree d, size s, and diameter D. Besides, they are maximally connected and have a small fault-diameter [6]. Finally, observe that Kautz hyperdigraphs $HK(d, s, D)$ are iterated line hyperdigraphs: $HK(d, s, D) = L^{D-2}GK(d, (ds)^2 + ds, s, d^2(ds + 1)) = L^{D-2}HK(d, s, 2)$.

A new family of hyperdigraphs with a minimum or almost minimum diameter is obtained by applying the partial line hyperdigraph technique to Kautz hyperdigraphs. For any values of $d, s \geq 1$ and $D \geq 2$ and for any integer N with $(ds)^D + (ds)^{D-1} \leq N \leq (ds)^{D+1} + (ds)^D$, let us consider a partial line hyperdigraph $\mathscr{L}_N HK(d, s, D) = \mathscr{L}(HK(d, s, D), \mathscr{W}_N, \phi_N)$. The hyperdigraph $\mathscr{L}_N HK(d, s, D)$ has maximum out-degree d, maximum out-size s, diameter $D + 1$ and, of course, order N. Therefore, if

$$\frac{(ds)^{D+1} - 1}{ds - 1} < N \leq (ds)^{D+1} + (ds)^D,$$

the hyperdigraph $\mathscr{L}_N HK(d, s, D)$ has a minimum diameter. Otherwise, the diameter exceeds the lower bound by one.

Since Kautz hyperdigraphs are maximally connected, the vertex-connectivity of $\mathscr{L}_N HK(d, s, D)$ is equal to the minimum degree of its underlying digraph, and the hyperarc-connectivity is equal to d if $N \geq d((ds)^D + (ds)^{D-1})$.

Another interesting property of the hyperdigraphs $\mathscr{L}_N HK(d, s, D)$ is derived from the results about the expandability of partial line hyperdigraphs given in Section 5. Effectively, modifying at most $2d$ hyperarcs, we can add a vertex to any hyperdigraph $\mathscr{L}_N HK(d, s, D)$ to obtain another hyperdigraph $\mathscr{L}_{N+1} HK(d, s, D)$ with a minimum or almost minimum diameter.

If $N = (ds)^k n$, the set $\mathscr{W}_N$ and the mapping ϕ_N can be chosen so that the partial line hyperdigraph $\mathscr{L}_N HK(d, s, D) = \mathscr{L}(HK(d, s, D), \mathscr{W}_N, \phi_N)$ is an iterated line hyperdigraph. In effect, from Corollary 3.5,

$$L^k \mathscr{L}_n HK(d, s, D - k) \cong \mathscr{L}_N L^k HK(d, s, D - k) \cong \mathscr{L}_N HK(d, s, D),$$

for an adequate choice of $\mathscr{W}_N$ and ϕ_N.

7. ON A CONJECTURE BY BERMOND AND ERGINCAN

A conjecture by Bermond and Ergincan [2] about the characterization of line hyperdigraphs is proved to be true in this section.

Theorem 7.1. *Let H be a hyperdigraph. Then, H is a line hyperdigraph if and only if its underlying digraph, $\hat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are both line digraphs, that is, there exists a hyperdigraph H_1 such that $H \cong LH_1$ if and only if there exist two digraphs G and G^* such that $\hat{H} \cong LG$ and $\widehat{H^*} \cong LG^*$.*

Proof. Since $(LH)^*$ is isomorphic to LH^* and $\widehat{LH}$ is isomorphic to $L\hat{H}$ [2], if $H \cong LH_1$, then $\hat{H} \cong L\widehat{H_1}$ and $\widehat{H^*} \cong L\widehat{H_1^*}$.

If $\hat{H}$ is a line digraph, its vertices can be labeled with ordered pairs of vertices of some other digraph, let us say G_1, where $\hat{H} = LG_1$. Moreover, we can assure that any two vertices u_0u_1, v_0v_1 are adjacent in $\hat{H}$ if and only if $u_1 = v_0$.

Analogously, if $\widehat{H^*}$ is a line digraph, the set of vertices of $\widehat{H^*}$ can be labeled with ordered pairs of vertices of some other digraph, let us say G_2, such that $\widehat{H^*} = LG_2$. Besides, two vertices of $\widehat{H^*}$, E_iF_i and E_jF_j, are adjacent if and only if $F_i = E_j$. Then, by the definition of $\widehat{H^*}$, there exists a vertex v of H belonging to the out-set of the hyperarc labeled E_iF_i, $(E_iF_i)^+$ and to the in-set of the hyperarc with label E_jF_j, $(E_jF_j)^-$, if and only if $F_i = E_j$.

Now, we modify the labeling for the vertices of H, introducing the labeling for its hyperarc set, that is, if a vertex labeled with u_0u_1 is in $(E_jF_j)^-$, we relabel it with $u_0E_ju_1$, and if the vertex with label v_0v_1 is in $(E_jF_j)^+$, we relabel it with $v_0F_jv_1$. Note that all the hyperarcs are relabeled since there exists at least one vertex in their in- or out-sets. Moreover, the new labels are unique, because if a vertex belongs to $(E_pF_p)^+$ and to $(E_qF_q)^-$, then $E_q = F_p$, because $\widehat{H^*}$ is a line digraph.

Since we have defined a labeling in H with the line hyperdigraph conditions, H is a line hyperdigraph. ∎

The next corollary is a direct consequence of Theorem 7.1:

Corollary 7.2. *Let H be a hyperdigraph and k be a positive integer. Then, H is a k-iterated line hyperdigraph if and only if its underlying digraph, $\hat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are both k-iterated line digraphs.*

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