Introduction to interconnection network models.

Daniela Ferrero
Universitat Politècnica de Catalunya, Espanya *
Universidad de la República, Uruguay †

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Abstract

This work contains the most important results of the doctoral thesis "Graphs and Hypergraphs as Interconnection Network Models" by Daniela Ferrero supervised by Carles Padró. This thesis was done at the Department of Applied Mathematics and Telematics of the Politechnical University of Catalonia.

We can divide this work into three parts. In the first one, some results about the fault-tolerance of known models based on digraphs are given (poin-to-point networks). The second part is devoted to the study of hyperdigraphs based models (bus networks). This is a new area, so before some equivalent results about the fault-tolerance, we need to prove other topological properties. Finally, a little part concerning with random sequences useful in stream cipher applications is presented.

1 Introduction

From some years ago, interconnection networks are becoming a very useful tool for a wide range of problems of very different nature. Mainly, this is due to the availability of technological possibilities to manage networks with a great number of nodes and a high quantity of connections between them.

To deal with interconnection networks different classifications can be stated. For instance, the objective of economical saving lead us to distinguish between LANs (Local Area Networks) and WANs (Wide Area Networks). If we focus on the nature of the communication links, networks can be classified into point-to-point networks and bus networks [59].

Interconnection networks consisting of some processors and connections between pairs of them are called point-to-point networks. They are usually modeled by graphs. A bus networks consists in a set of processors and a set of buses providing communication channels between subsets of processors. Bus networks are represented by hypergraphs. In both cases, the communication

*e-mail: matdfc@mat.upc.es
†e-mail: dferrero@fing.edu.uy

links can be directed or not. To model networks based on unidirectional links, directed graphs (digraphs) and directed hypergraphs (hyperdigraphs) are used.

In spite of their reliability and performance, networks of multiple bus architectures have not been as much studied as those based on point-to-point connections.

This work deals with some problems about graphs and hypergraphs related to this modelization.

When designing a communication system, most of the requirements related to the topology of the interconnection network can be stated in terms of graphs or hypergraphs. The first of these requirements is the need of connecting a large number of nodes with the minimum communication delay, when there is a limitation on the number of connections supported by every node. To evaluate the communication delay it is important to consider the number of processors that should be traversed to send a message from any processor to another arbitrary one. Although in some cases the minimization of this number is not enough to assure the communication efficiency, it is always relevant in order to simplify the network administration.

Another requirement for communication systems is the fault-tolerance. That is, the network should still communicate with high performance, if possible, when some processors or links are faulty. Then, for interconnection network models with the aforementioned conditions, an interesting property to study is the vulnerability. That is, the incidence of some faults in the network communications. Traditionally, it was the variation in the maximum number of links needed to connect any two processors, the parameter that has been used for this purpose [5, 45, 53].

In practice, apart from theoretical measures of the fault-tolerance capability, it is important the routing facilities of a model. Specially, the possibility to route efficiently in the presence of faults. For this reason, the studies of the fault-tolerance by finding all the better alternatives to route, depending on the faulty elements have particular interest [21].

Finally, a communication network is desired to be versatile, in the sense that it may provide easy ways to add or remove some processors without loss of performance. The expandability is related to the capability of a network to increase or decrease its number of processors. This concepts measures this facility in terms of the links that must be affected [10].

For point-to-point networks the basic requirements mean that the system may have a high level of connectivity and also may consider the maximum number of connections that a processor can admit to work efficiently. Different conditions related to graphs have been presented to model this need. They are basically expressed in terms of the order, the diameter and the maximum degree. The goal is to find graphs with arbitrary large order and minimum diameter for a given maximum degree. This generates a conflict between the concepts involved. So, this situation gives rise to many optimization problems concerning graphs. Basically, they consist in prioring a requirement and trying to optimize the second one. For such problems, some particular families of graphs and directed graphs have been presented. The De Bruijn and Kautz digraphs [17, 52], the bipartite digraphs $BD(d, n)$ [35] and the De Bruijn and Kautz generalized cycles [40] are some of them. More generally, for directed graphs there are of two techniques that were shown to be good for such problems. The line digraph technique provide facilities to construct large digraphs with arbitrary diameter and given minimum degree [36]. To obtain digraphs with small diameter for fixed values of the order and maximum degree, the partial line digraph technique has a good behaviour [33].

In relation to the fault-tolerance, there are particular studies for some of the good particular families of digraphs mentioned in the above paragraph [48, 54]. They are based on similar techniques, consisting in finding containers between any pair of processors. All these families are iterated line digraphs [35, 36], and the containers are based on this fact. Other methods were required for the same problem in general digraphs obtained from the line digraph technique. In this case, the problem was studied with the same point of view used before to study the connectivity. So, theoretical bounds were presented, but not describing containers, for example in [53].

Bus networks are a generalization of point-to-point-networks. However, an additional parameter must be considered, and it is the number of processors that a bus can connect. This impose restrictions on the number of vertices connected to a hyperarc, and on the number of hyperarcs connected to a vertex. These two values are bounded by the physical nature of the components. So, in this case, we focus on hypergraphs with arbitrary order and minimum diameter for fixed values of the minimum processor degree and the minimum bus size. Again, it is impossible to impose all the desired conditions at the same time. As for graphs, the approach to such problem is to treat it by optimization problems. In one case all the parameters are fixed except the order, and the problem consist in finding large hypergraphs with such parameters. The other situation is when for given values of all the parameters, except the diameter, it must be minimized [22]. As for graphs, there are also some particular proposed solutions. For the directed case the generalized De Bruijn and the Kautz hyperdigraphs were introduced [8]. For the case of size equal to one, these families coincide with other solutions proposed before for the same problems restricted to digraphs. Also the line digraph technique was generalized to hyperdigraphs [6]. The same good results in order to construct large hyperdigraphs are obtained.

In the present work we study some problems related to the fault-tolerance of digraphs. First for the particular good models for interconnection networks, and then for general iterated line digraphs. Besides, we deal with other aspects on hyperdigraphs. For example, the connectivity and the fault-tolerance. We also propose a method for constructing hyperdigraphs of minimum diameter for fixed order, degree and size. Some properties of such technique are given.

Next, we give a scheme of the organization of this work.

We present in Section 1 the definitions of the main concepts and the notation that will be used in the rest of the work. The definitions and the main known results about the families of digraphs and hyperdigraphs that are studied in this thesis are also given in this section.

Sections 2 and 3 are devoted to the study of point-to-point networks.

In Section 2, we study the fault-tolerance of the De Bruijn and Kautz generalized cycles. As it was mentioned, they were proposed as good models for designing interconnection networks. In fact, they can connect a large number of vertices in relation to the degree and the diameter. We deal with the problem by finding containers between every pair of vertices. The values obtained show that the fault-tolerance capability of these families is optimal. The results of this section correspond to the publications [26, 27].

The fault-tolerance of interconnection network models defined by the iteration of the line digraph technique is studied in Section 3. There, some new parameters are introduced in order to improve the known theoretical bounds on fault-diameters. The general bounds obtained, when calculated for some particular digraphs, coincide with the exact values. So, the bounds are optimal at least for some cases. The results in this section have been presented also in our paper [28].

We treat some problems related to bus networks in Sections 4 and 5.

In Section 4, we define some parameters in order to study the connectivity and fault-tolerance of directed hypergraphs defined by the iteration of the line hyperdigraph technique. For the connectivity, similar results that for iterated line digraphs are obtained. In order to analise the fault-tolerance, the fault-diameters are introduced. We obtain theoretical bounds, as a generalization of the known results for digraphs. The results stated in this section were presented in [24, 29, 30].

The partial line digraph technique is defined for directed hypergraphs in Section 5. Similar results that for digraphs are proved. The partial line hyperdigraph is shown to generalize also the line hyperdigraph technique.Partial line hyperdigraphs are shown to have large order for their diameter, minimum degree and minimum size. Besides, they present good connectivity and expandibility. Also a characterization of line hyperdigraphs in terms of line digraphs is given. The above characterization, in the case of line hyperdigraphs, is a proof of a conjecture introduced in [6]. Our paper [31] collects also the results in this section.

Finally, we include a section about De Bruijn sequences. Particularly, the De Bruijn sequences of maximum period length are interesting because of their randomness properties [14, 15, 19, 51]. They have a great number of applications in many areas of computer science and abstract algebra. Specially for stream ciphers, needed to provided security services [18, 56]. Actually, the Asynchronous Transfer Mode (ATM) and the Broadband Integrated Services Digital Network (B-ISDN) are the most promising techniques for high speed networks, and both scenarios require ciphering services [57].

In Section 6 we expose the problem of finding all De Bruijn sequences of maximum period length in terms of digraphs. By a matricial analysis of the problem we obtain some interesting properties. Also a test to decide whether or not a sequence is a De Bruijn one of maximum period length is presented. These results in this section correspond to [32].

2 Graphs, hypergraphs and interconnection networks

2.1 Basic definitions about graphs

We present here some definitions used in the following sections. For details and more information, see for example [16, 41]. A *directed graph* $G = (V, A)$ consists of a set of *vertices* V and a set A of ordered pairs of vertices called *arcs*. Usually, they are also called *digraphs* for short. The arcs in the form (x, x) are called *loops*. The cardinality of V is the *order* of the digraph. If (x, y) is an arc, it is said that x is *adjacent to* y and that y is *adjacent from* x. The set of vertices which are adjacent from(to) a given vertex v is denoted by $\Gamma^+(v)(\Gamma^-(v))$ and its cardinality is the *out-degree* of v, $d^+(v) = |\Gamma^+(v)|$(*in-degree* of v,$d^-(v) = |\Gamma^-(v)|$). Its minimum value over all vertices is the *minimum out-degree*,d^+,(*minimum in-degree*,d^-) of the digraph G. The *minimum degree* of G is $d = \min\{d^+, d^-\}$. The *maximum degree* d is defined analogously.

A *path* of length h from a vertex x to a vertex y is a sequence of vertices $x = x_0, x_1, \ldots, x_{h-1}, x_h = y$ where (x_i, x_{i+1}) is an arc. A digraph G is *strongly connected*, or simply *connected*, if for any pair of vertices x, y there exists a path from x to y. The length of a shortest path from x to y is the *distance from* x *to* y, and it is denoted by $d(x, y)$. Its maximum value over all pairs of vertices is the *diameter* of the digraph, $D(G)$. If G is not strongly connected, $D(G) = \infty$.

The *vertex-connectivity* $\kappa = \kappa(G)$ of a digraph $G = (V, A)$ is the minimum cardinality of the subsets of vertices $F \subset V$ such that $G - F$ is not strongly connected or is trivial. The *arc-connectivity* $\lambda = \lambda(G)$ is the minimum number of arcs whose deletion disconnects the digraph. For more information see [38].

A *cycle* in a digraph G is a path starting and ending at the same vertex. A *hamiltonian cycle* is one that contains all the vertices of G exactly once. An *eulerian cycle* contains all the arcs of G exactly once (but it could repeat vertices).

Given two digraphs, G and G', on N and N' vertices, respectively, $N \leq N'$, the *index of expandability* of G to G', $e(G, G')$, is defined as the minimum number of arcs that have to be deleted from G to obtain G' by adding $N' - N$ vertices and some appropriate arcs. [10].

2.2 The line digraph and the (d, D)-digraph problem

The (d, D)-digraph problem consists in finding digraphs with order as large as possible for fixed values of the maximum out-degree d and the diameter D [12, 34, 40]. The order, let say N of a digraph with maximum out-degree d and diameter D is upper bounded by the *Moore bound*, $M(d, D)$,

$$M(d, D) = 1 + d + d^2 + \ldots + d^D = \begin{cases} D + 1, & \text{if } d = 1; \\ (d^{D+1} - 1)/(d - 1), & \text{if } d > 1. \end{cases}$$

Due to the non-attainability of the Moore bound [13], the study of the

(d, D)-digraph problem is based in finding digraphs with order d, diameter D and order as close to $M(d, D)$ as it would be possible.

In the *line digraph* [36] LG of a digraph G each vertex represents an arc of G, that is, $V(LG) = \{uv \mid (u, v) \in A(G)\}$. A vertex uv is adjacent to a vertex vw if $v = w$, that is, whenever the arc (u, v) of G is adjacent to the arc (w, z). The maximum and minimum out and in-degrees of LG are equal to those of G. Therefore, if G is d-regular with order n, then LG is d-regular and has order dn. Besides, if G is a strongly connected digraph different from a directed cycle, the diameter of LG is the diameter of G plus one unit. So, in the same conditions for G, $L^k G$ is d-regular, has diameter $D + k$ and order $d^k n$, that is, the order increases in an asymptotically optimal way in relation to the diameter. As a consequence, the iteration of the line digraph operation is a good method for the (d, D)-digraph problem.

The set of vertices of the iterated line digraph $L^k G$ can be considered as the set of all paths of length k in G, that is, the set of the sequences of vertices of G with length $k + 1$, $x_0 x_1 \ldots x_k$, where (x_i, x_{i+1}) is an arc of G. A vertex $\mathbf{x} = x_0 x_1 \ldots x_k$ in $L^k G$ is adjacent to the vertices $\mathbf{y} = x_1 \ldots x_k x_{k+1}$ for all x_{k+1} adjacent from x_k. A path of length h in $L^k G$ can be written as a sequence of $k + h + 1$ vertices of G. The vertices of this path are the subsequences of $k + 1$ consecutive vertices of G. Because of this notation, iterated line digraphs admit very simple algorithms to find short paths between vertices. Observe that between any pair of vertices of $L^k G$ there exists at most one path of length $h \leq k + 1$.

2.3 The partial line digraph and the (d, N)-digraph problem

The (d, N)-digraph problem consists in finding digraphs G with minimum diameter for fixed values of the maximum out-degree d and the order N [12, 34, 40]. Since this problem has sense only for $d > 1$, by the Moore bound, if the diameter is D,

$$N \leq M(d, D) = \frac{d^{D+1} - 1}{d - 1}$$

From this inequality it is easy to find a lower bound for the diameter D,

$$D \geq \lceil \log_d (N(d-1) + 1) \rceil - 1$$

So, the (d, N)-digraph problem is based in finding digraphs with maximum degree d, order N and diameter equal, or at least close, to the minimum possible value.

A partial line digraph [33] of a digraph $G = (V, E)$ with minimum degree at least 1, is defined from a set E' of arcs of G with $V = \{v : (u, v) \in E'\}$. It is denoted by $\mathcal{L}G = (V(\mathcal{L}G), E(\mathcal{L}G))$ with $V(\mathcal{L}G) = \{uv : (u, v) \in E'\}$ and a vertex uv adjacent to the vertices $v'w$, for each w adjacent from v, where $v' = v$ if $vw \in V(\mathcal{L}G)$, or any other arbitrary vertex adjacent to w if not.

Since G has minimum degree at least 1, always exists a set E' with the conditions asked to construct the partial line digraph. Also, if $E' = E$, then $\mathcal{L}G$ coincides with LG. That is, the order of $\mathcal{L}G$ is between the order of G and the order of LG.

Partial line digraphs are shown to preserve the minimum degree and increase the diameter in at most one unit. Then, this technique is a good strategy for the (d, N)-digraph problem.

Besides, partial line digraphs tend to increase the connectivity (with a natural bound given by the minimum in-degree which is not preserved). Besides, easy routing algorithms can be defined in them. Finally, the technique is versatile in the sense that simple methods can be used to increase or decrease the order, maintaining the maximum out-degree constant, and with variations in the diameter of at most one unit.

2.4 Some interesting families of digraphs

2.4.1 De Bruijn digraphs and sequences

The De Bruijn digraph [17] denoted by $B(d, D)$ has set of vertices $\mathbf{Z}_d^D$. Any vertex $x_0 x_1 \ldots x_{D-1}$ is adjacent to the vertices $x_1 \ldots x_{D-1} x_D$, for every x_D in $\mathbf{Z}_d$.

Besides of this alphabetical definition of the De Bruijn digraphs, it is possible to define them in terms of iterated line digraphs. In fact, if we denote by K_d the complete digraph (i.e. with order d and an arc joining every two vertices), then $B(d, D) = L^{D-1} K_d$.

From any of the above definitions it is easy to conclude that $B(d, D)$ is d-regular, has diameter D and order d^D. Then, since:

$$d^D > \frac{d-1}{d} M(d, D)$$

these family of digraphs has good properties for the (d, D)-problem.

Since $B(d, D) = L^{D-1} K_d$, there paths can be represented by sequences of vertices of K_d. Particularly, cycles can be represented by sequences. A De Bruijn sequence of order D over $\mathbf{Z}_d$ is the corresponding sequence for a hamiltonian cycle in $B(d, D)$.

2.4.2 Reddy-Pradhan-Kuhl digraphs

The De Bruijn digraph $B(d, D)$ can also be arithmetically defined as the digraph with vertex set $\mathbf{Z}_n$, with $n = d^D$, and a where the vertex x is adjacent to the vertices $dx + t$ for any value of t in $\mathbf{Z}_d$. If we remove the condition $n = d^D$, and let n to be any positive integer, we obtain the *generalized De Bruijn* or *Reddy-Pradhan-Kuhl* digraphs [55], $RPK(d, n)$. Then, $RPK(d, n) = B(d, d^D)$.

For any values of d and n, the digraph $RPK(d, n)$ has naturally, order n and degree d. It is also known that its diameter is $\lceil \log_d n \rceil$. That is, the diameter is minimum whenever $(d^D - d)/(d-1) < n < d^D$, and exceed the minimum value in one unit if $d^{D-1} + 1 \leq n \leq (d^D - d)/(d-1)$.

They are also iterated line digraphs. More precisely, $LRPK(d,n) = RPK(d,dn)$.

2.4.3 Kautz digraphs

The Kautz digraph [52] denoted by $K(d,D)$ has set of vertices $\mathbf{Z}_d^D$. Any vertex $x_0x_1\ldots x_{D-1}$ is adjacent to the vertices $x_1\ldots x_{D-1}x_D$, for every x_D in $\mathbf{Z}_d$, with the condition $x_{D-1}\neq x_D$.

Let K_d^* denote the complete digraph without loops (i.e. with order d and an arc joining every two different vertices). Another possible definition of the Kautz digraphs is $K(d,D) = L^{D-1}K_{d+1}^*$.

From any of the above definitions it is easy to conclude that $K(d,D)$ is d-regular, has diameter D and order $d^D + d^{D-1}$. Then, they are closer to the Moore bound that the De Bruijn digraphs. In fact,

$$d^D + d^{D-1} > \frac{d^2-1}{d^2}M(d,D)$$

so, these family of digraphs has better properties for the (d,D)-problem.

Another interesting property is that between any two vertices of $K(d,D)$ there exists a unique path of length $D-1$ or D.

2.4.4 Imase-Itoh digraphs

In the *Imase-Itoh* digraph $II(d,n)$ [46, 47], $n\leq d\leq 2$, the set of vertices is $\mathbf{Z}_n$, and a vertex x is adjacent to the vertices $-dx-t$, for $t=1,2,\ldots,d$. These digraphs are also called *generalized Kautz*, since $II(d,d^D+d^{D-1}) = K(d,D)$.

For every values of d and n, $II(d,n)$ is d-regular and has order n. It only has loops when n is not a multiple of $d+1$. If D is the diameter of $II(d,n)$, it can be shown:

$$\lfloor \log_d n\rfloor \leq D \leq \lceil \log_d n\rceil.$$

As a consequence, the diameter of $II(d,n)$ never exceeds the diameter of the digraph $RPK(d,n)$. Then the diameter of $II(d,n)$ is minimum or exceeds the minimum possible value in at most one unit, but is in general, smaller than the diameter of the digraph $RPK(d,n)$.

2.4.5 $BD(d,n)$ digraphs

A *bipartite digraph* [35] is a digraph whose set of vertices can be partitioned into two nonempty sets, such that all the arcs are adjacent from a vertex in one part to a vertex in the other one. The bipartite digraphs $BD(d,n)$ were introduced when studying the (d,D)-problem restricted to bipartite digraphs. In this case, in the same way that it was calculated the Moore bound for general digraphs, we can find a better one, let us say $M^B(d,D)$.

For any integers d,n, $n\geq d\geq 2$, the *bipartite digraph* $BD(d,n)$ has set of vertices $\mathbf{Z}_2\times\mathbf{Z}_n = \{(\alpha,i) : \alpha\in\mathbf{Z}_2, i\in\mathbf{Z}_n\}$, and every vertex (α,i) is adjacent to the vertices $(1-\alpha,(-1)^\alpha d(i+\alpha)+t)$, for every $t=0,1,\ldots,d-1$.

It is clear that $B(d,n)$ is d-regular, has order $2n$, and if D is the diameter, it holds:

$$\lfloor \log_d n \rfloor + 1 \leq D \leq \lceil \log_d n \rceil + 1.$$

For some values of n the diameter can be exactly calculated. Particularly, for $BD(d, d^{D-1} + d^{D-3})$ the diameter is D. Moreover

$$2(d^{D-1} + d^{D-3}) > \frac{d^4 - 1}{d^4} M^B(d, D),$$

so we conclude that the digraphs $BD(d, d^{D-1} + d^{D-3})$ are a good solution to the (d, D)-problem for bipartite digraphs.

Also in relation to the (d, N)-digraph problem in the bipartite case, the family $BD(d, N)$ has good properties. In fact, it was shown that its diameter exceeds the minimum possible value arising from $M^B(d, D)$ in at most one unit. Particularly, if $N = d^{D-1} + d^{D-4k-3}$, for some integer k, $0 \leq k \leq \lfloor (D-3)/4 \rfloor$, the minimum value for the diameter is attained.

Finally, $BD(d, d^{D-1} + d^{D-3}) = L^{D-3} BD(d, 1 + d^2)$, and in general, $LBD(d, n) = BD(d, dn)$.

2.4.6 Large generalized cycles

A generalized p-cycle is a digraph whose set of vertices is partitioned in p parts that can be cyclically ordered in such a way that any vertex is adjacent only to vertices in the next part. That is, $V(G) = \bigcup_{\alpha \in \mathbf{Z}_p} V_\alpha$ and the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbf{Z}_p$. Observe that, for instance, a digraph is a 1-cycle or a bipartite digraph is a generalized 2-cycle.

The conjunction operator was introduced in [40]. It gives rise to generalized cycles. The *conjunction* of a directed cycle of length p, let say C_p, with a digraph $G = (V, A)$, $C_p \otimes G$, has set of vertices $\mathbf{Z}_p \times V$ and a vertex (α, x) is adjacent to the vertices $(\alpha + 1, y)$ for any y adjacent from x in the digraph G. Observe that $C_p \otimes G$ is a generalized p-cycle for any digraph G. The line digraph of $C_p \otimes G$ is isomorphic to $C_p \otimes LG$. In fact, it is not difficult to see that the mapping $(\alpha, x)(\alpha + 1, y) \mapsto (\alpha + 1, xy)$ defines a digraph isomorphism between $L(C_p \otimes G)$ and $C_p \otimes LG$.

Also in [40], making use of the conjunction operator, were introduced two families of generalized cycles. These are the De Bruijn and Kautz generalized cycles, $BGC(p, d, d^{k+1})$ and $KGC(p, d, d^{p+k} + d^k)$ respectively. They have large order for their degree and diameter.

The De Bruijn generalized cycle $BGC(p, d, d^{k+1})$ is defined to be $C_p \otimes B(d, k+1)$, where $B(d, k+1)$ is the De Bruijn digraph with degree d and diameter $k+1$. The De Bruijn digraph is an iterated line digraph, $B(d, k+1) = L^k K_d^*$, where K_d^* is the complete digraph with a loop on each vertex. Therefore, $BGC(p, d, d^{k+1})$ is also an iterated line digraph, $BGC(p, d, d^{k+1}) = C_p \otimes L^k K_d^* = L^k (C_p \otimes K_d^*) = L^k BGC(p, d, d)$. The set of vertices of the digraph

$BGC(p,d,d)$ is $\mathbf{Z}_p \times \mathbf{Z}_d$ and a vertex (α, x) is adjacent to $(\alpha+1, y)$ for any $y \in \mathbf{Z}_d$. This digraph is d-regular and has diameter p. The vertices of $BGC(p,d,d^{k+1})$, which is d-regular and has diameter $p+k$, can be seen as sequences of vertices of $BGC(p,d,d)$ $(\alpha, y_0)(\alpha+1, y_1)\dots(\alpha+k, y_k)$, where $\alpha \in \mathbf{Z}_p$ and $y_i \in \mathbf{Z}_d$, $i = 0, 1, \dots, k$.

The set of vertices of the Kautz generalized cycle $KGC(p,d,n)$ is $\mathbf{Z}_p \times \mathbf{Z}_n$. If $0 \leq \alpha \leq p-2$, the vertex (α, x) is adjacent to $(\alpha+1, dx+t)$ for any $t = 0, 1, \dots, d-1$. The vertex $(p-1, x)$ is adjacent to $(0, -dx-(d-t))$ for any $t = 0, 1, \dots, d-1$. The digraph $KGC(p,d,d^p+1)$ is d-regular and has diameter $2p-1$. The generalized cycle $KGC(p,d,d^{p+k}+d^k)$ is isomorphic to the iterated line digraph $L^k KGC(p,d,d^p+1)$. Then, it is d-regular and has diameter $D = 2p+k-1$. The vertices of $KGC(p,d,d^{p+k}+d^k)$ can be written as paths of length k in $KGC(p,d,d^p+1)$. That is, sequences $(\alpha, y_0)(\alpha+1, y_1)\dots(\alpha+k, y_k)$, where $\alpha \in \mathbf{Z}_p$ and y_i are vertices of the generalized cycle $KGC(p,d,d^p+1)$, $i = 0, 1, \dots, k$

Observe that $BGC(p,d,d^p)$ which has diameter $2p-1$ and order pd^p, is isomorphic to the *directed butterfly* $B_d(p)$ [1]. Besides, $K(d,D)$, the Kautz digraph of degree d and diameter D is the same that $KGC(1,d,d^D+d^{D-1})$, the Imase-Itoh digraph [46, 47], $GK(d,n)$, can be defined as $KGC(1,d,n)$, and the bipartite digraphs $BD(d,n)$ [35] coincides with $KGC(2,d,d^{D-p+1}+d^{D-2p+1})$.

For a p-generalized cycle of minimum degree d and diameter D, the Moore like bound is $M^{GC}(p,d,D) = \frac{p(d^{D+1}-d^r)}{d^p-1}$, being r an integer $0 \leq r \leq p-1$ such that $D-(p-1) = pm+r$. From this inequality can be obtained that the minimum diameter of a p-generalized cycle of minimum degree d and order n is $D^{GC}(p,d,n) = \lceil \log_d(n(d^p-1)+1) \rceil - 1$.

If D_1, D_2 are respectively the diameters of $KGC(p,d,n)$ and $BGC(p,d,n)$, then $D^{GC}(p,d,n) \leq D_2 \leq D_1 \leq D^{GC}(p,d,n)+1$.

The digraphs $BGC(p,d,d^{k+1})$, $0 \leq k \leq p-2$ attain the bound $M^{GC}(p,d,D)$ when $p \leq D \leq 2p-2$. Also the bound $M^{GC}(p,d,2p-1)$ is attained by the digraph $KGC(p,d,d^p+1)$. More generally, if $k > p$, for the digraphs $KGC(p,d,d^{p+k}+d^k) = L^k KGC(p,d,d^p+1)$ hold:

$$p(d^{p+k}+d^k) > \frac{d2p-1}{d^{2p}} M^{GC}(p,d,D)$$

so the bound is attained when $2p-1 \leq D \leq 3p-1$.

2.5 Containers, wide and fault-diameters

Let x and y be two vertices of a digraph G. Two paths from x to y are said to be *vertex-disjoint* or *disjoint* if they do not have any internal vertex in common. A *container* from a vertex x to another vertex y is a set $C(x,y)$ of disjoint paths from x to y. The *width* $w(C(x,y))$ of a container $C(x,y)$ is the number of disjoint paths that it contains, and its *length*, $l(C(c,y))$, is the maximum length of its paths. For an integer s, $0 \leq s \leq \kappa(G)$, the *s-width-distance* from x to y, $d_s(x,y)$, is the minimum length of all containers of width s from x to y. Finally, the *s-wide-diameter* of the digraph G, $d_s(G)$, is the maximum s-wide-distance among all pairs of different vertices in G. The $(s-1)$-*vertex-fault-diameter*, $D_s(G)$, of a digraph G is the maximum of the diameters of the digraphs obtained by removing at most $s-1$ vertices from G. The $(s-1)$-*arc-fault-diameter*, $D'_s(G)$, is defined analogously [43, 44, 45].

In general, the following relations hold between the wide-diameter and the fault-diameters: $d_s(G) \geq D_s(G)$ and $d_s(G) \geq D'_s(G)$. From the definition, $d_1 = D_1(G)$, $d_1 = D'_1(G)$ and coincide with the diameter of G. Clearly, $D_s(G) \leq D_{s+1}(G)$ and $D'_s(G) \leq D'_{s+1}(G)$. Also there exist some relations between these two parameters, the connectivities and the diameter. If $\kappa = \kappa(G)$ there is a container of width κ between every pair of distinct nodes. In particular, since $D(G) = \infty$ if G is not strongly connected, the vertex-connectivity $\kappa = \kappa(G)$ and the arc-connectivity $\lambda = \lambda(G)$ are, respectively, the minimum values of s satisfying $D_{s+1}(G) = \infty$ and $D'_{s+1}(G) = \infty$. Also from Menger's Theorem $d_{s+1}(G) = \infty$ if $s = \kappa(G)$, and $d'_{s+1}(G) = \infty$ if $s = \lambda(G)$.

Some fault-diameters have been calculated by finding disjoint paths between any pair of vertices. For example, for the de Bruijn and Kautz digraphs [20, 48], the bipartite digraphs $BD(d,n)$ [54], and Flip-trees in [50].

The fault-diameters of general iterated line digraphs were considered in [53]. It was proved there that, if an iterated line digraph L^kG has maximum connectivity, its fault-diameter is bounded by $D(L^kG) + C$, where C depends on some properties of the digraph G, but does not depend on the number of iterations k.

2.6 Basic definitions about hypergraphs

We present some of the most relevant concepts we are going to use. For additional information, see for instance [6, 7, 8, 37]. A *hyperdigraph* H is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of *vertices* or *nodes*, and $\mathcal{E}(H)$ is a set of ordered pairs of nonempty subsets of $\mathcal{V}(H)$, called *hyperarcs*. If $E = (E^-, E^+)$ is a hyperarc, we say that E^- is the *in-set*, E^+ is the *out-set* of E, and that E joins vertices in E^- to vertices in E^+. Its *in-size*(*out-size*) is the cardinal of E^-, $|E^-|(|E^+|)$. If v is a vertex, the *in-degree*(*out-degree*) of v is the number of hyperarcs containing v in the out-set(in-set), and it is denoted by $d^-(v)(d^+(v))$.

If H is a hyperdigraph, its *order* is the number of vertices, $|\mathcal{V}(H)|$, denoted by $n(H)$. The number of hyperarcs is usually denoted by $m(H)$. The *maximum in-size* and *maximum out-size* of H are respectively defined by

$$s^-(H) = \max\{|E^-| : E \in \mathcal{E}(H)\} \ , \ s^+(H) = \max\{|E^+| : E \in \mathcal{E}(H)\}$$

Similarly, the *maximum in-degree* , *maximum out-degree* of H are

$$d^-(H) = \max\{d^-(v) : v \in \mathcal{V}(H)\} \ , \ d^+(H) = \max\{d^+(v) : v \in \mathcal{V}(H)\}$$

We denote $s(H) = \max\{s^+(H), s^-(H)\}$, $d(H) = \max\{d^+(H), d^-(H)\}$. We say that a hyperdigraph H is *d-regular* if $d^-(H) = d^+(H) = d$. Also H is *s-uniform* if $s^-(H) = s^+(H) = s$. Note that when $s = 1$, H is a digraph.

A *path* of *length* k from a vertex u to a vertex v in H is an alternating sequence of vertices and hyperarcs $u = v_0, E_1, v_1, E_2, v_2, \ldots, E_k, v_k = v$ such that $v_i \in E^-_{i+1}$, $(i = 0, \ldots, k-1)$ and $v_i \in E^+_i$, $(i = 1, \ldots, k)$. The *distance* from u to v, $d(u,v)$, is the length of the shortest path from u to v. The *diameter*, $D(H)$, is the maximum distance between every pair of vertices of H.

A hyperdigraph is *connected* if there exists at least one path from each vertex to any other vertex. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph H, is the minimum number of vertices to be removed to obtain a non-connected or trivial hyperdigraph (a hyperdigraph with only one vertex). Similarly is defined the *hyperarc-connectivity*, $\lambda(H)$.

Any two paths in H are *vertex-disjoint* if they have no internal vertices in common, and are *hyperarc-disjoint* if they do not share hyperarcs. The Menger's theorem stablish that the vertex(hyperarc)-connectivity is the number of vertex(hyperarc)-disjoint paths between any pair of vertices. In fact, such theorem was enunciated for graphs [49], but different proves can be easily adapted to hyperdigraphs [6].

The *dual hyperdigraph*, H^*, of a hyperdigraph H has its set of vertices in one-to-one correspondence with the set of hyperarcs of H, and for every vertex v of H, it has a hyperarc, (V^-, V^+), such that a vertex $e \in V^-$, if and only if, $v \in E^+$ and $e \in V^+$, if and only if, $v \in E^-$.

The *underlying digraph* of a hyperdigraph H is the digraph $\widehat{H} = (\mathcal{V}(\widehat{H}), \mathcal{A}(\widehat{H}))$ with $\mathcal{V}(\widehat{H}) = \mathcal{V}(H)$ and $\mathcal{A}(\widehat{H}) = \{(u,v) : \exists E \in \mathcal{E}(H), u \in E^-, v \in E^+\}$. That is, there is an arc from a vertex u to a vertex v in $\widehat{H}$ if and only if there is a hyperarc joining u to v in H. So, paths in $\widehat{H}$ and H are in correspondence, and this implies $D(\widehat{H}) = D(H)$ and $\kappa(\widehat{H}) = \kappa(H)$. We are going to denote $\widehat{\kappa} = \kappa(\widehat{H})$ and $\widehat{\lambda} = \lambda(\widehat{H})$.

The *bipartite representation* of a hyperdigraph H is a bipartite digraph $R = R(H) = (V(R), A(R))$ with set of vertices $V(R) = V_0(R) \cup V_1(R)$, where $V_0(R) = \mathcal{V}(H)$ and $V_1(R) = \mathcal{E}(H)$, and set of arcs

$$A(R) = \{(u,E) \mid u \in V_0, E \in V_1, u \in E^-\} \cup \{(F,v) \mid v \in V_0, F \in V_1, v \in F^+\}.$$

Observe that, if u, v are two vertices of H, a path of length h from u to v in H correspond to a path of length $2h$ in $R(H)$ and, then, $d_R(u,v) = 2d_H(u,v)$.

2.7 The line hyperdigraph and the (d, D, s)-hyperdigraph problem

The (d, D, s)-hyperdigraph [37] problem consists of finding hyperdigraphs with minimum degree d, diameter D, maximum size s, and order as large as possible. The maximum order for such hyperdigraphs is given by the *Moore bound* for hyperdigraphs. It is denoted by $M(d, D, s)$, and its value is:

$$M(d,D,s) = 1 + ds + (ds)^2 + \ldots + (ds)^D = ((ds)^{D+1} - 1)/(ds - 1).$$

This bound cannot be attained if $D > 1$ unless H is a directed cycle [22]. Then, in relation to the (d, D, s)-hyperdigraph problem, it is interesting to find families of hyperdigraphs with order close to the corresponding Moore bound.

The *line hyperdigraph* of $H = (\mathcal{V}(H), \mathcal{E}(H))$ is defined in [6] as the hyperdigraph $LH = (\mathcal{V}(LH), \mathcal{E}(LH))$,

$$\mathcal{V}(LH) = \cup_{E \in \mathcal{E}(H)} \{(uEv) : u \in E^-, v \in E^+\}$$
$$\mathcal{E}(LH) = \cup_{v \in \mathcal{V}(H)} \{(EvF) : v \in E^+, v \in F^-\}$$

with $(EvF)^- = \{(wEv) : w \in E^-\}$ and $(EvF)^+ = \{(vFw) : w \in F^+\}$.

The iterated line hyperdigraph $L^k(H)$ is defined by $L^k(H) = LL^{k-1}(H)$, with $L^0(H) = L(H)$. In $L^k(H)$ the vertices are represented by paths of length k in H, $v_0E_1v_1E_2 \ldots E_kv_k$, the hyperarcs have the form $E_0v_1E_1v_2 \ldots v_kE_k$, and the paths of length l can be viewed as sequences $v_0E_1v_1E_2 \ldots E_{l+k}v_{l+k}$. Line hyperdigraphs iterations tend to increase the connectivities.

From its definition, and in a similar way than for digraphs, the iteration of the line hyperdigraph result a good method for the (d, D, s)-hyperdigraph problem.

2.8 The (d, N, s)-hyperdigraph problem

The (d, N, s)-hyperdigraph problem was introduced in [37] and consists in finding directed hypergraphs with order N, maximum out-degree d, maximum out-size s and minimum diameter.

Analogously than for digraphs, from the Moore bound, for a hyperdigraph with order N, maximum out-degree d, maximum out-size s and diameter D, it is easy to find a lower bound for the diameter D,

$$D \geq \lceil \log_{ds}(N(ds-1)+1) \rceil - 1$$

So, the (d, N, s)-hyperdigraph problem study is directed to finding hyperdigraphs with maximum degree d, maximum size s, order N and diameter close to the lower bound founded.

2.9 Generalized Kautz and De Bruijn hyperdigraphs

In [8, 6] were introduced the generalized De Bruijn and Kautz hyperdigraphs. There it was shown that they have good order in relation to their degree and size. In fact, for the case of hyperarc size 1 (digraphs), they are a generalization of the best known families according to the aforementioned criteria.

The generalized Kautz hyperdigraphs, are defined as it follows. Let n be the number of vertices and d the vertex out-degree. Choose the number of hyperarcs m and the out-size s, with the conditions $dn \equiv_m 0$ and $sm \equiv_n 0$. The generalized Kautz hyperdigraph $GK(d, n, s, m)$ has as vertices the integers modulo n and as hyperarcs the integers modulo m. The incidence rules are:

1. A vertex v is incident to every hyperarc E: $E \equiv_m dv + \alpha$, $0 \leq \alpha \leq d-1$
2. The out-set of the hyperarc E is: $u \equiv_n -sE - \beta$, $1 \leq \beta \leq s$

If $GK(d,n,s,m)$ is the generalized Kautz hyperdigraph with degree d, order n, size s and m hyperarcs,

$$\widehat{GK}(d,n,s,m) = II(ds,n)$$

where $II(ds,n)$ is the generalized Kautz digraph.

The generalized De Bruijn hyperdigraphs were defined with the purpose of having a similar property for them. That is, let $GB(d,n,s,m)$ be a generalized De Bruijn hyperdigraph with degree d, order n, size s and m hyperarcs,

$$\widehat{GB}(d,n,s,m) = RPK(ds,n)$$

where $RPK(ds,n)$ is the generalized De Bruijn digraph.

Such property can be achieved by two ways. So, we have two non-isomorphic schemes to define a *generalized De Bruijn hyperdigraph*, let us say, $GB_1(d,n,s,m)$ and $GB_2(d,n,s,m)$.

In the first scheme, we give an alphabetical definition. Let A, B be two sets of sizes d and s respectively. If k is a positive integer, $[AB]^k$ denotes any sequence of $2k$ elements in the form $(a,b,\ldots,a,b)$ where $a \in A, b \in B$.
The vertices of $GB_1(d,n,s,m)$ are $[BA]^D$ and the hyperarcs $[A][BA]^{D-1}[A]$. If $E = (E^-, E^+)$ is a hyperarc labeled $(a_0, b_1, a_1, \ldots, b_{D-1}, a_{D-1}, a_D)$, then

$$E^- = \{(\beta, a_0, b_1, a_1, \ldots, b_{D-1}, a_{D-1}) : \beta \in B\}$$
$$E^+ = \{(b_1, a_1, \ldots, b_{D-1}, a_{D-1}, \beta, a_D) : \beta \in B\}$$

From the vertices point of view, the hyperarcs such that a vertex s is adjacent to them, are founded by shifting the vertex label to the left by one, disposing of b_1, and introducing a new element of A at the right end. The set or hyperarcs that a vertex is adjacent from is founded by shifting all letters (except A_D) to right by one (disposing of b_D) and adding an element of A from the left end.

In the second scheme, $GB_2(d,n,s,m)$ is defined arithmetically. First we impose $dn \equiv_m 0$ and $sm \equiv_n 0$. Assume that the vertices are numbered from 0 to $n-1$ and the hyperarcs from 0 to $m-1$. Then, the vertex v is adjacent to the hyperarcs $E_\alpha \equiv_m dv + \alpha$, for any $\alpha = 0, \ldots, d-1$. The out-set of the hyperarc E consists of the vertices $V_i \equiv_n sE + \beta$, for any $\beta = 0, \ldots, s-1$.

For the generalized Kautz hyperdigraph of degree d and size s on n vertices and m hyperarcs, the diameter D is such that $n = (ds)^D + (ds)^{D-1}$. In both schemes of the generalized De Bruijn digraph of degree d and size s on n vertices and m hyperarcs, $G_1(d,n,s,m)$ or $G_2(d,n,s,m)$, the diameter,D, has the property that $n = (ds)^D$.

3 Containers in large generalized cycles

3.1 Introduction

By Menger's theorem it is possible to obtain upper bounds for the fault-diameters from disjoint paths of bounded length between every pair of vertices.

To evaluate how good are they, is usual to find also lower bounds and then compare them with the upper ones. This technique was introduced in [48] for the De Bruijn and Kautz digraphs, and was also used in [54] for the bipartite digraphs $BD(d, d^{D-1}+d^{D-3})$. In [48] it was proved the existence of a container of width d and length at most the diameter plus two unit between any two vertices in the d-regular De Bruijn or Kautz digraphs. Besides, it was shown that at most two paths have length equal to the diameter plus two units. This result was improved in [20]. There it was proved that at most one path has length equal to the diameter plus two units. For the bipartite digraphs $BD(d, d^{D-1}+d^{D-3})$ the it was proved in [54] the same result that in [48] for the De Bruijn and Kautz digraphs.

In all the above cases, paths are given and proved to be disjoint making use of the fact that the families treated are iterated line digraphs. We find lower bounds for fault-diameters of any generalized cycle and calculate them for the above mentioned families. Then, we compare these values with the upper bounds obtained by giving disjoint paths in them. From this comparison we obtain the exact value for the wide and fault-diameters. Moreover, we show that such values are optimum for these families.

3.2 Lower bounds

If G is a digraph with $D_s(G) = D'$, there exist at least s paths (not necessarily disjoint) of length at most D' between any pair of non-adjacent vertices of G.

Let G be a generalized p-cycle with maximum out-degree d and order n. Let r such that $D_s(G) = D'$, with $D' - (p-1) = pm + r$ and $0 \leq r \leq p-1$. Then, if $x \in V_{\alpha-r}$ and $y \in V_\alpha$ are two non-adjacent vertices of G, there must exist s paths of length at most $pm + r$ from x to y. There are at most $d^r(1 + d^p + d^{2p} + \ldots + d^{pm})$ paths of length less or equal than $pm + r$ from a vertex in $V_{\alpha-r}$ to the vertices in V_α. Therefore,

$$n = \sum_{\alpha \in \mathbf{Z}_p} |V_\alpha| \leq \begin{cases} p(1 + \lfloor (d^p + d^{2p} + \ldots + d^{pm})/s \rfloor), & \text{if } r = 0; \\ p(d + \lfloor (d^{p+1} + d^{2p+1} + \ldots + d^{pm+1})/s \rfloor), & \text{if } r = 1; \\ p(\lfloor d^r(1 + d^p + \ldots + d^{pm})/s \rfloor), & \text{if } r \neq 0, 1. \end{cases}$$

Then, $D_s(G) = D' = p(m+1) + r + 1 \geq \ell^r$, where

$$\ell^r = \begin{cases} \lceil \log_d(s(\frac{n}{p} - 1)(d^p - 1) + d^p) \rceil - 1, & \text{if } r = 0; \\ \lceil \log_d(s(\frac{n}{p} - d)(d^p - 1) + d^{p+1}) \rceil - 1, & \text{if } r = 1; \\ \lceil \log_d(\frac{n}{p} s(d^p - 1) + d^r) \rceil - 1, & \text{if } r \neq 0, 1. \end{cases}$$

Therefore, if G is a generalized p-cycle with maximum out-degree d and order n,then, $D_s(G) \geq \min_{0 \leq r \leq p-1} \ell^r = \ell^1$. We have found a lower bound for the $(s-1)$-vertex-fault-diameter.

Proposition 3.1 *Let G be a generalized p-cycle with maximum out-degree d and order n. Then, for any $s = 2, \ldots, d$,*

$$d_s(G) \geq D_s(G) \geq D_{\min}(s,p,d,n) = \left\lceil \log_d \left(s\left(\frac{n}{p} - d\right)(d^p - 1) + d^{p+1}\right)\right\rceil - 1. \ \square$$

If G is a digraph with $D'_s(G) = D'$, there must exist at least $s+1$ paths of length at most D' between any pair of different vertices. Reasoning in the same way as in the vertex case, we can obtain a lower bound for the $(s-1)$-arc-fault-diameter of a generalized cycle.

Proposition 3.2 *Let G be a generalized p-cycle with maximum out-degree d and order n. Then, for any $s = 2, \ldots, d$,*

$$d'_s(G) \geq D'_s(G) \geq D'_{\min}(s,p,d,n) = \left\lceil \log_d \left(s\left(\frac{n}{p} - 1\right)(d^p - 1) + d^{p}\right)\right\rceil - 1. \ \square$$

So, we have obtained lower bounds for the s-wide-diameter and the s-fault-diameter.

The values of these bounds for the families we are studying are:

- For $BGC(p,d,d^{k+1})$, if $s = 2, \ldots, d$:

$$d_s^{\min} \geq D_s^{\min}(p,d,pd^{k+1}) = p+k+1;$$
$$d'^{\min}_s \geq D'^{\min}_s(p,d,pd^{k+1}) = p+k+1.$$

- For $KGC(p,d,d^{p+k}+d^k)$, if $s = 2, \ldots, d$:

$$d_s^{\min} \geq D_s^{\min}(p,d,p(d^{p+k}+d^k)) = 2p+k;$$
$$d'^{\min}_s \geq D'^{\min}_s(p,d,p(d^{p+k}+d^k) = 2p+k.$$

That is, the bounds on vertex and arc-fault-diameters, are the diameter plus one unit in both cases.

3.3 Containers and fault-diameters

3.3.1 The De Bruijn generalized cycles $BGC(p,d,d^{k+1})$

Disjoint paths of bounded length between any two vertices of the generalized De Bruijn digraph $BGC(p,d,d^{k+1})$, $p \geq 2$, are given in this section. Particularly, we prove, in a constructive way, that there exists a container of width d and length at most $p+k+2$ between any pair of different vertices of $BGC(p,d,d^{k+1})$. Besides, only one path of this container has length $p+k+2$ and the other paths have length at most $p+k+1$.

First of all, we present these containers in the smallest digraph in this family: the generalized cycle $BGC(p,d,d) = C_p \otimes K_d^*$. Next, we show how to construct containers of bounded length between any pair of adjacent vertices of $BGC(p,d,d^{k+1})$. Finally, taking into account that these digraphs are iterated line digraphs, we prove that containers between any pair of different vertices can be founded from containers between vertices in a smaller digraph in the same family.

Proposition 3.3 *Let x and y be two (not necessarily different) vertices of the generalized cycle $BGC(p,d,d) = C_p \otimes K_d^+$, $p \geq 2$. There exists in this digraph a container of width d from x to y with length at most $p+1$.*

Proof: We can suppose that x is in the partite set $V_0 = \{0\} \times \mathbf{Z}_d$ and that $y \in V_h = \{h\} \times \mathbf{Z}_d$, $1 \leq h \leq p$ (of course, $V_p = V_0$). If $p \geq 3$ and $h \neq 1, 2$, there exist many different ways to find a container with width d and length h from x to y. Any path of these containers is in the form $x\alpha_1^i \ldots \alpha_{h-1}^i y$, $1 \leq i \leq d$, where $\alpha_r^i \in V_r$ and $\alpha_r^i \neq \alpha_r^j$ if $i \neq j$. If $p \geq 2$ and $h = 1$, we have a path of length 1, the arc (x,y), and we can take $d-1$ disjoint paths with length $p+1$ in the form $x\alpha_1^i \ldots \alpha_{p-1}^i \alpha_p^i y$, $1 \leq i \leq d-1$, where $\alpha_r^i \in V_r$, $\alpha_r^i \neq \alpha_r^j$ if $i \neq j$, $\alpha_1^i \neq y$ and $\alpha_p^i \neq x$. Finally, if $h = 2$ there are exactly d paths of length 2 from x to y, which are disjoint.□

Let $\mathbf{x}$, $\mathbf{y}$ be a pair of adjacent vertices of $BGC(p,d,d^{k+1})$. Since this digraph is the iterated line digraph $L^k(C_p \otimes K_d^*)$, we can put $\mathbf{x} = x_0 x_1 \ldots x_k$ and $\mathbf{y} = x_1 \ldots x_k x_{k+1}$, where x_r, $0 \leq r \leq k+1$, is a vertex of $C_p \otimes K_d^*$. Besides, we can suppose that $x_r = (r,j) \in \mathbf{Z}_p \times \mathbf{Z}_d$, that is, that x_r is a vertex in the partite set V_r of the generalized cycle $C_p \otimes K_d^*$. We want to construct d disjoint paths with length at most $p+k+2$ from $\mathbf{x}$ to $\mathbf{y}$. The first of these paths is the arc $(\mathbf{x},\mathbf{y}) = x_0 x_1 \ldots x_k x_{k+1}$. The other paths are going to be constructed from disjoint paths from x_k to x_1 in $C_p \otimes K_d^*$ and will have the form $x_0 x_1 \ldots x_k a_{k+1} a_{k+2} \ldots a_{k+r} x_1 \ldots x_k x_{k+1}$, with $r \leq p+1$. Since these paths can not contain the arc $(\mathbf{x},\mathbf{y})$, we must take $a_{k+1} \neq x_{k+1}$ and $a_{k+r} \neq x_0$. That is, we have to find a container of width $d-1$ and length at most $p+2$ from x_k to x_1 in $C_p \otimes K_d^*$ such that all the paths in it have their first and last arcs, respectively, different from (x_k, x_{k+1}) and (x_0, x_1).

If $p \geq 3$ and $k \equiv h \pmod{p}$, $1 \leq h \leq p-2$, then $x_k = x_1$ or $3 \leq d(x_k, x_1) = p-h+1 \leq p$. In this case, we consider $d-1$ paths from x_k to x_1 in the following form: $x_k \alpha_{h+1}^s \alpha_{h+2}^s \ldots \alpha_p^s x_1$, where $1 \leq s \leq d-1$, $\alpha_{h+1}^s \neq x_{k+1}$, $\alpha_p^s \neq x_0$ and $\alpha_r^s \neq \alpha_r^{s'}$ if $s \neq s'$. These paths are disjoint and have length $p-h+1$.

If $k \equiv p-1 \pmod{p}$, then $d(x_k, x_1) = 2$ and there are exactly d paths of length 2 from x_k to x_1. If $x_{k+1} = x_0$, we consider the $d-1$ paths of length 2 that avoid the vertex x_0: the paths $x_k \alpha_p^s x_1$, where $1 \leq s \leq d-1$, $\alpha_p^s \neq x_0$ and $\alpha_p^s \neq \alpha_p^{s'}$ if $s \neq s'$. If $x_{k+1} \neq x_0$, we can take only $d-2$ of these paths: the paths $x_k \alpha_p^s x_1$, where $1 \leq s \leq d-2$, $\alpha_p^s \neq x_0, x_{k+1}$ and $\alpha_p^s \neq \alpha_p^{s'}$ if $s \neq s'$. In this case, we have to consider also a path with length $p+2$: $x_k x_0 \alpha_1^{d-1} \ldots \alpha_{p-1}^{d-1} x_{k+1} x_1$.

If $k \equiv 0 \pmod{p}$, then $d(x_k, x_1) = 1$. In this case, we take $d-1$ paths of length $p+1$: $x_k \alpha_1^s \ldots \alpha_p^s x_1$, where $1 \leq s \leq d-1$, $\alpha_1^s \neq x_{k+1}$, $\alpha_p^s \neq x_0$ and $\alpha_r^s \neq \alpha_r^{s'}$ if $s \neq s'$.

Then, we can construct d paths from $\mathbf{x}$ to $\mathbf{y}$ in $BGC(p,d,d^{k+1})$:

- A path of length 1, the arc $A = x_0 x_1 \ldots x_k x_{k+1}$.

- $d-1$ or $d-2$ paths with length $D-h+1 \leq D+1$,

$$Q_s = x_0 x_1 \dots x_k \alpha^s_{h+1} \alpha^s_{h+2} \dots \alpha^s_p x_1 \dots x_k x_{k+1},$$

where $k \equiv h \pmod p$ and $0 \leq h \leq p-1$.
If $k \equiv p-1 \pmod p$, we may need one path with length $D+2$,

$$R = x_0 x_1 \dots x_k x_0 \alpha^{d-1}_1 \dots \alpha^{d-1}_{p-1} x_{k+1} x_1 \dots x_k x_{k+1}$$

Proposition 3.4 *If $s \neq t$, the paths Q_s and Q_t are disjoint.*

Proof: If $q_{s,j}$ is the j-th vertex of the path Q_s,

$$q_{s,j} = \begin{cases} x_j \dots, x_k \alpha^s_{h+1} \dots \alpha^s_{h+j} & j = 1, \dots, p-h \\ x_j \dots x_k \alpha^s_{h+1} \dots \alpha^s_p x_1 \dots x_{j-p+h} & j = p-h+1, \dots, k \\ \alpha^s_{h+j-k} \dots \alpha^s_p x_1 \dots x_{j-p+h} & j = k+1, \dots, k+p-h \end{cases}$$

and, of course, we have the analogous expressions for the vertices of the path Q_t. We have to prove that $q_{s,j} \neq q_{t,i}$ for any $i, j = 1, \dots, k+p-h$. By the symmetry of the paths, it suffices to compare $q_{s,j}$ with $q_{t,i}$ when $j \leq i$. The case $i = j$ is trivial because $\alpha^s_r \neq \alpha^t_r$ for any $r = h+1, \dots, p$. Besides, since the paths are in a p-cycle, it is only necessary to prove that $q_{s,j} \neq q_{t,i}$ when $i \equiv j \pmod p$ and $i \geq j+p$.

Let us suppose that there exist i, j, where $i \equiv j \pmod p$ and $i \geq j+p$, such that $q_{s,j} = q_{t,i}$.

If $1 \leq j \leq p-h$ and $p-h+1 \leq i \leq k$, we have that

$$q_{s,j} = x_j, \dots, x_k, \alpha^s_{h+1}, \dots, \alpha^s_{h+j} = x_i, \dots, x_k, \alpha^t_{h+1}, \dots, \alpha^t_p, x_1 \dots, x_{i-p+h} = q_{t,i}$$

Let us consider the subsequences formed by the vertices of $C_p \otimes K^*_d$ in the partite set V_{h+j}:

$$x_{h+j} x_{h+p+j} \dots x_{k-p+j} \alpha^s_{h+j} = x_{h+i} \dots x_{k-p+i} \alpha^t_{h+j} x_{h+j} \dots x_{i-p+h}$$

Observe that $h+j \leq p$. We consider the *equivalence relation digraph* [48] given by this equality. The arcs of the equivalence relation digraph join a symbol appearing in the first sequence with the symbol that appears in the same place in the second sequence. In this digraph, the vertices x_r have in-degree and out degree equal to one, α^s_{h+j} have in-degree 0 and out-degree 1 and α^t_{h+j} have in-degree 1 and out-degree 0. Then, there exists in the equivalence relation digraph a path from α^s_{h+j} to α^t_{h+j}. That means that $\alpha^s_{h+j} = \alpha^t_{h+j}$, a contradiction.

If $p-h+1 \leq j < i \leq k$, we consider j_0 such that $j_0 \equiv j \equiv i \pmod p$ and $0 \leq j_0 \leq p-1$. From the equality $q_{s,j} = q_{t,i}$, we take the subsequences formed by the vertices in the partite set V_p of $C_p \otimes K^*_d$:

$$x_{j+p-j_0} \dots x_{k-h} \alpha^s_p x_p \dots x_{j-j_0-\ell} = x_{i+p-j_0} \dots x_{k-h} \alpha^t_p x_p \dots x_{i-j_0-\ell}$$

where $\ell = 0$ if $j_0 \geq p-h$ and $\ell = p$ otherwise. As before, using the equivalence relation digraph, we obtain that $\alpha^s_p = \alpha^t_p$, a contradiction.

The remaining case, $p-h+1 \leq j \leq k$ and $k+1 \leq i \leq k+p-h$, is solved analogously.□

The following propositions are proved in a similar way.

Proposition 3.5 *The paths Q_s and R do not contain the arc A.*

Proposition 3.6 *The path R is disjoint with any path Q_s.*

Therefore, we have constructed d disjoint paths between any pair of adjacent vertices of $BGC(p,d,d^{k+1})$:

one of length 1, $d-2$ of length at most $p+k+1$ and one of length at most $p+k+2$.

Theorem 3.7 *Let $\mathbf{x}$, $\mathbf{y}$ be any pair of different vertices of the generalized cycle $BGC(p,d,d^{k+1})$, $p \geq 2$. There exists a container from $\mathbf{x}$ to $\mathbf{y}$ of width d and length less than or equal to $p+k+2 = D+2$ composed by one path with minimum length $d(\mathbf{x},\mathbf{y})$, $d-2$ with length at most $D+1$ and one of length at most $D+2$.*

Proof: We are going to prove this theorem by induction on k. If $k=0$, the result is true by Proposition 3.3. Let $\mathbf{x}$, $\mathbf{y}$ be two different vertices of $BGC(p,d,d^{k+1})$, $k \geq 1$. We have proved the existence of these paths if $d(\mathbf{x},\mathbf{y}) = 1$. Let us suppose that $d(\mathbf{x},\mathbf{y}) \geq 2$. Since $BGC(p,d,d^{k+1})$ is isomorphic to the line digraph $LBGC(p,d,d^k)$, we can put $\mathbf{x} = x_0x_1$ and $\mathbf{y} = y_0y_1$, where x_0, x_1, y_0 and y_1 are vertices of $BGC(p,d,d^k)$. Besides, $x_1 \neq y_0$, because $\mathbf{x}$ is not adjacent to $\mathbf{y}$. Then, in $BGC(p,d,d^k)$ there exists a container from x_1 to y_0 with width d and length at most $p+k+1$. In this container there are one path of length $d(x_1,y_0)$, $d-2$ of length at most $p+k$, one of length at most $p+k+1$. These paths induce in the line digraph $LBGC(p,d,d^k) = BGC(p,d,d^{k+1})$ a container of width d from $\mathbf{x} = x_0x_1$ to $\mathbf{y} = y_0y_1$, with one path of minimum length $d(\mathbf{x},\mathbf{y}) = d(x_1,y_0)+1$, $d-2$ of length at most $p+k+1 = D+1$ and one path of length at most $p+k+2 = D+2$.□

As a corollary of Theorem 3.7 we obtain the values of the s-wide-diameter and the s-fault-diameters of $BGC(p,d,d^{k+1})$. We can see that these values are almost optimal by comparing them with the lower bounds given in Propositions 3.1 and 3.2. In effect, for any $s = 2,\ldots,d$,

$$D_{\min}(s,p,d,pd^{k+1}) = D'_{\min}(s,p,d,pd^{k+1}) = p+k+1$$

Theorem 3.8 *Let G be the generalized cycle $BGC(p,d,d^{k+1})$, $p \geq 2$. Then*

- $d_s(G) = D'_s(G) = p+k+1 = D+1$ *if* $2 \leq s \leq d-1$ *or* $s = d$ *and* $0 \leq k \leq p-2$.
- $d_d(G) = D'_d(G) = p+k+2 = D+2$ *if* $k \geq p-1$.
- $d_s(G) = D_s(G) = p+k+1 = D+1$ *if* $2 \leq s \leq d-1$ *or* $s = d$ *and* $0 \leq k \leq p-1$.

- $d_d(G) = D_d(G) = p + k + 2 = D + 2$ *if* $k \geq p$.

Proof: It is obvious from Propositions 3.1 and 3.2 and Theorem 3.7 that $d_s(G) = D_s(G) = D'_s(G) = p + k + 1 = D + 1$ if $2 \leq s \leq d - 1$. The minimum value of k for which we need a path with length $D + 2$ (the path R) in order to construct the d disjoint paths is $k = p - 1$. Then, $D'_d(G) = D_{d-1}(G) = D + 1$ if $0 \leq k \leq p - 2$. Since in the digraph $BGC(p, d, d^{p-1})$ there are containers of width d and length at most $p+k-1$ between any pair of different vertices, in the digraph $BGC(p, d, d^p) = LBGC(p, d, d^{p-1})$ we can find a container of width d and length at most $p + k$ between any pair of non-adjacent vertices. Therefore, $D_d(G) = D + 1$ if $k = p - 1$. Finally, let us consider in $BGC(p, d, d^p)$ the vertices $\mathbf{x} = x_0 x_1 \dots x_{p-1}$ and $\mathbf{y} = x_1 \dots x_{p-1} x_p$, where $x_0 = (0, 0) \in \mathbf{Z}_p \times \mathbf{Z}_d$ and $x_p = (0, d - 1)$. If we remove from $BGC(p, d, d^p)$ the $d - 1$ arcs $\mathbf{e}_i = x_0 x_1 \dots x_{p-1} \alpha_p^i$,where $\alpha_p^i = (0, i)$, $1 \leq i \leq d - 1$, the distance from $\mathbf{x}$ to $\mathbf{y}$ in the resulting digraph will be equal to $2p + 1 = D + 2$. Using the line digraph technique, it is not difficult to find, for any $k > p - 1$, $d - 1$ vertices or arcs to be removed from $BGC(p, d, d^{k+1})$ in order to obtain a digraph with diameter $p+k+2$. Therefore, $D'_d(G) = D+2$ if $k \geq p-1$ and $D_d(G) = p+k+2 = D+2$ if $k \geq p$.□

3.3.2 The Kautz generalized cycles $KGC(p, d, d^{p+k} + d^k)$

Proceeding in the same way as in Section 3.3.1, first we find disjoint paths between vertices in the digraph $KGC(p, d, d^p + 1)$, which is the smallest digraph in this family. Next, we construct disjoint paths of bounded length between any pair of adjacent vertices of $KGC(p, d, d^{p+k} + d^k)$.

We recall now some properties in [40] of the generalized cycle $KGC(p, d, d^p + 1)$. If x and y are two different vertices in the same partite set of $KGC(p, d, d^p + 1)$, then $d(x, y) = p$ and there is only one shortest path form x to y. Besides, there is no cycle of length p in this digraph. For any vertex x there are exactly $1 + d + d^2 + \ldots + d^p$ vertices y such that $d(x, y) \leq p$. That is, if $d(x, y) \leq p$ there is only one path from x to y with length at most p.

Proposition 3.9 *Let x, y be any pair of vertices of $KGC(p, d, d^p + 1))$. There exists a container of width d from x to y with length at most $2p = D + 1$.*

Proof: We can suppose that $x \in V_0$ and $y \in V_h$, where $1 \leq h \leq p$. let $\Gamma^+(x) = \{z_1, z_2, \ldots, z_d\}$ be the set of the vertices that are adjacent from x and $\Gamma^-(y) = \{v_1, v_2, \ldots, v_d\}$ be the set of vertices that are adjacent to y.

If $h = 1$ and (x, y) is not an arc, Since there is a unique path of length p from any z_i to y, we have exactly d paths of length $p + 1$ from x to y: the paths $x z_i \ldots y$, $1 \leq i \leq d$. Using the properties of $KGC(p, d, d^p + 1))$, it is not difficult to see that these paths are disjoint. If (x, y) is an arc, we can suppose that $z_1 = y$. In this case, we have a path of length 1, the arc (x, y), and $d - 1$ paths of length $p + 1$: the paths $x z_i \ldots y$, $2 \leq i \leq d$. As before, these paths are disjoint.

If $h \geq 2$ and $d(x, y) = h$, we can suppose that the unique path of minimum length from x to y has the form $xz_1 \ldots v_1 y$ (where $z_1 = v_1$ if $h = 2$). Let σ be any permutation in $\{2, \ldots, d\}$. For any $i = 2, \ldots, d$, let $w_i \in V_{h-1}$ be any vertex such that $d(z_i, w_i) = h-2$ and consider the path $xz_i \ldots w_i \ldots v_{\sigma i} y$, which is a path from x to y with length $p + h \leq 2p$. Observe that, since $w_i \neq v_{\sigma i}$ if $2 \leq i \leq d$, there is a unique path of length p from w_i to $v_{\sigma i}$. By the properties of the generalized cycle $KGC(p, d, d^p + 1))$, these d paths from x to y are disjoint. If $d(x, y) = p + h$ or $h = p$ and $x = y$, we can consider any permutation σ in $\{1, 2, \ldots, d\}$ and construct d paths from x to y with length $2p$: the paths $xz_i \ldots w_i \ldots v_{\sigma i} y$, where $i = 1, \ldots, d$ and w_i is a vertex in V_{h-1} such that $d(z_i, w_i) = h - 2$. As before, these paths are disjoint.□

Let $\mathbf{x}$ and $\mathbf{y}$ be any pair of adjacent vertices of the generalized cycle $KGC(p, d, d^{p+k} + d^k)$. As it was done in Section 3.3.1 for $BGC(p, d, d^{k+1})$, we construct a container from $\mathbf{x}$ to $\mathbf{y}$ of width d and bounded length . Since that digraph is isomorphic to the iterated line digraph $L^k KGC(p, d, d^p + 1)$, its vertices can be written as paths of length k in $KGC(p, d, d^p + 1)$. Then, we can write $\mathbf{x} = x_0 x_1 \ldots x_k$ and $\mathbf{y} = x_1 \ldots x_k x_{k+1}$, where x_i, $i = 0, 1, \ldots, k+1$ are vertices of the generalized cycle $KGC(p, d, d^p + 1)$. We are going to find $d - 1$ disjoint paths from $\mathbf{x}$ to $\mathbf{y}$ that do not contain the arc $A = (\mathbf{x}, \mathbf{y})$. These paths are going to be to be constructed from disjoint paths from x_k to x_1 in $KGC(p, d, d^p + 1)$ and will have the form $x_0 x_1 \ldots x_k a_{k+1} a_{k+2} \ldots a_{k+r} x_1 \ldots x_k x_{k+1}$, where $r \leq 2p$, $a_{k+1} \neq x_{k+1}$ and $a_{k+r} \neq x_0$. Therefore, we must find in $KGC(p, d, d^p + 1)$ a container $C(x_k, x_1)$ with width $d - 1$ and length at most $2p + 1$ such that their first and last arcs must be, respectively, different from (x_k, x_{k+1}) and (x_0, x_1).

Lemma 3.10 *Let x and y be two vertices of $KGC(p, d, d^p + 1)$ such that $d(x, y) \not\equiv 1 \pmod{p}$. Let us consider $z \in \Gamma^+(x)$ and $v \in \Gamma^-(y)$. Then, there exists a container of width $d - 1$ and length at most $2p$, avoiding the arcs (x, z) and (v, y).*

Proof: We can suppose that $x \in V_0$ and $y \in V_h$, where $2 \leq h \leq p$. If $d(x, y) = h$, let $xz_1 \ldots v_1 y$ be the unique shortest path from x to y (if $h = 2$, then $z_1 = v_1$). We have to distinguish three cases.

Case 1: $h = p$ and $x = y$; or $d(x, y) = p + h$; or $d(x, y) = h$, $z = z_1$ and $v = v_1$; or $d(x, y) = h$, $z \neq z_1$ and $v \neq v_1$. By the proof of Proposition 3.9, we can find in this case a set of d disjoint paths from x to y with length at most $2p$ containing a path in the form $xz \ldots vy$. The other $d - 1$ paths are the paths we are looking for.

Case 2: $d(x, y) = h$, $z = z_1$ and $v \neq v_1$. Let us consider a vertex $z_2 \in \Gamma^+(x)$, $z_2 \neq z$ and a vertex $u \in V_1$ such that $u \neq z_1$ and $d(u, v_1) = h - 1$. Then, $u \notin \Gamma^+(x)$ and there is a path with length p from z_2 to u. Let us consider the following $d - 1$ paths with length $p + h$ from x to y: the path $xz_2 \ldots u \ldots v_1 y$,and the paths $xz_i \ldots w_i \ldots v_{\sigma i} y$, $3 \leq i \leq d$, constructed as in the proof of Proposition 3.9, where $v_{\sigma i} \neq v$. It is not difficult to prove that these paths are disjoint and do not contain neither the vertex z nor the vertex v.

Case 3: $d(x,y)=h$, $z\neq z_1$ and $v=v_1$. This case is analogous to Case 2.□

Lemma 3.11 *Let x and y be two vertices of $KGC(p,d,d^p+1)$ such that $d(x,y)\equiv 1 \pmod p$. Let us consider $z\in\Gamma^+(x)$ and $v\in\Gamma^-(y)$. Then, there exists a container of width $d-1$ and length at most $2p+1$ from x to y such that all the paths in it have their first and last arcs are, respectively, different from (x,z) and (v,y). Besides, at most one of the paths in the container have length $2p+1$.*

Proof: We can suppose that $x\in V_0$ and $y\in V_1$. As we have seen in the proof of Proposition 3.9, there are exactly d paths, which are disjoint, of length at most $p+1$ from x to y.

Case 1: $d(z,v)=p-1$, or $x=v$ and $y=z$. If $d(z,v)=p-1$, the path $xz\ldots vy$ is one of the d disjoint paths from x to y with length at most $p+1$. The other $d-1$ paths are disjoint and avoid the arcs (x,z) and (v,y). If $x=v$ and $y=z$, the two forbidden arcs are equal to (x,y). The other $d-1$ paths with length $p+1$ from x to y are the paths we are looking for.

Case 2: $d(z,v)=2p-1$, $x\neq v$ and $y\neq z$. In this case, there are two different paths from x to y with length $p+1$ containing one of the forbidden arcs: the paths $xz\ldots v'y$ and $xz'\ldots vy$. Then, there are $d-2$ paths from x to y with length at most $p+1$ avoiding the arcs (x,z) and (v,y). Let $w\in V_0$, $w\neq v$, be a vertex such that $d(z',w)=p-1$. Then, $w\notin\Gamma^-(y)$ and there exists a path of length p from w to v'. The path $xz'\ldots w\ldots v'y$, which has length $2p+1$, is disjoint with the above $d-2$ paths and do not contain neither z nor v.

Case 3: (x,y) is an arc, $x=v$ and $y\neq z$. Then, the d paths from x to y with length at most $p+1$ are: the arc (x,y), the path $xz\ldots v'y$ and $d-2$ paths with length $p+1$ that do not contain any of the forbidden arcs. Let $w\in V_1$, $w\neq z$, be a vertex such that $d(w,v')=p-1$. Then, $w\notin\Gamma^+(x)$ and there is a path with length p from y to w. The path $xy\ldots w\ldots v'y$, which has length $2p+1$, is disjoint with the other $d-2$ paths, its first arc is different from (x,z) and its last arc is different (v,y).

Case 4: (x,y) is an arc, $x\neq v$ and $y=z$. Analogously to Case 3, we have $d-2$ paths with length $p+1$ that do not contain any of the forbidden arcs. Let z' be the vertex such that $xz'\ldots vy$ is a path of length $p+1$. Let $w\in V_0$, $w\neq v$, be a vertex such that $d(z',w)=p-1$. Then, $w\notin\Gamma^-(y)$ and there is a path with length p from w to x. The path $xz'\ldots w\ldots xy$, which has length $2p+1$, and the above $d-2$ paths are the paths we are looking for.□

Let $\mathbf{x}=x_0x_1\ldots x_k$ and $\mathbf{y}=x_1\ldots x_kx_{k+1}$ be any pair of adjacent vertices of the generalized cycle $KGC(p,d,d^{p+k}+d^k)$. Let h be the integer such that $h\equiv k \pmod p$ and $1\leq h\leq p$. By Lemmas 3.10 and 3.11, there exist a container $C(x_k,x_1)$ in $KGC(p,d,d^p+1)$ with width $d-1$ and length at most $2p+1$ such that every path in the container has the first and the last arcs are, respectively, different from (x_k,x_{k+1}) and (x_0,x_1). Using these paths, we construct $d-1$

paths from $\mathbf{x}$ to $\mathbf{y}$ with length at most $2p+k+1=D+2$. By doing that, we have obtained d paths from $\mathbf{x}$ to $\mathbf{y}$ that will be proved to be disjoint. The first of these paths is the arc

$$A=(\mathbf{x},\mathbf{y})=x_0x_1\dots x_kx_{k+1}.$$

There can be one path with length $k+p-h+1\le D-p+1$,

$$P=x_0x_1\dots x_k\alpha^1_{h+1}\alpha^1_{h+2}\dots\alpha^1_px_1\dots x_kx_{k+1},$$

$d-1$, $d-2$ or $d-3$ paths with length $k+2p-h+1\le D+1$,

$$Q_s=x_0x_1\dots x_k\alpha^s_{h+1}\alpha^s_{h+2}\dots\alpha^s_p\dots\alpha^s_{2p}x_1\dots x_kx_{k+1},$$

and, if $h=p$, we may need one path with length $D+2$,

$$R=x_0x_1\dots x_k\alpha^{d-1}_1\dots\alpha^{d-1}_p\dots\alpha^{d-1}_{2p}x_1\dots x_kx_{k+1}.$$

It can be proved that these paths are disjoint by using the same techniques as in Section 3.3.1. The proof of the following theorem is the same as in Theorem 3.7

Theorem 3.12 *Let* $\mathbf{x}$, $\mathbf{y}$ *be any pair of different vertices of the generalized cycle* $KGC(p,d,d^{p+k}+d^k)$, $p\ge 2$. *There exists a container from* $\mathbf{x}$ *to* $\mathbf{y}$ *with width* d *and length less than or equal to* $2p+k+1=D+2$. *Moreover, in such container there is one path with minimum length* $d(\mathbf{x},\mathbf{y})$, *and there are* $d-2$ *paths with length at most* $D+1$ *and one of length at most* $D+2$. □

As a corollary of Theorem 3.12 we obtain the value of the s-wide-diameter $d_s(G)$, and the s-fault-diameters $D_s(G)$, $D'_s(G)$ for $G=KGC(p,d,d^{p+k}+d^k)$. The following theorem can be proved in the same way as Theorem 3.8.

Theorem 3.13 *Let* G *be the generalized cycle* $KGC(p,d,d^{p+k}+d^k)$, $p\ge 2$. *Then*

- $d_s(G)=D'_s(G)=2p+k=D+1$ *if* $2\le s\le d-1$ *or* $s=d$ *and* $0\le k\le p-1$.
- $d_d(G)=D'_d(G)=2p+k+1=D+2$ *if* $k\ge p$.
- $d_s(G)=D_s(G)=2p+k=D+1$ *if* $2\le s\le d-1$ *or* $s=d$ *and* $0\le k\le p$.
- $d_d(G)=D_d(G)=2p+k+1=D+2$ *if* $k\ge p+1$.

Again, the bounds obtained are almost optimal by comparing them with the lower ones given in Propositions 3.1 and 3.2. In effect, for any $s=2,\dots,d$,

$$D_{\min}(s,p,d,p(d^{p+k}+d^k))=D'_{\min}(s,p,d,p(d^{p+k}+d^k))=2p+k.$$

3.4 Fault-tolerant routings

Using the containers constructed in Sections 3.3.1 and 3.3.2, routing algorithms for $BGC(p, d, d^k)$ and $KGC(p, d, d^{p+k}+d^k)$ are respectively given. We avoid details of such algorithms that depend on the implementation of the network [21], to focus on possibilities to make use of the containers. [27]

We assume that before the routing algorithms, other algorithms were running on the network. These algorithms recognize the faulty elements (nodes and links), giving a list of them as output. Note that this is not a restriction since is the most common way in which routers work when no acknowledge messages are sent [59].

3.5 The De Bruijn generalized cycles

Let u, v be any two vertices of the $BGC(p, d, d^{k+1})$, let say,

$$u = (c_1, \ldots, c_r, a_0, a_1, \ldots, a_{k-r}),\; v = (a_0, a_1, \ldots, a_{k-r}, b_1, \ldots, b_r)$$

with all their coefficients in $C_p \otimes K_d^+$, and r the distance from u to v.

If $r = 1$ we just have a description of the minimum length paths from u to v. Otherwise, we consider vertices u' and v' in $BGC(p, d, d^{k-r})$,

$$u' = (c_1, a_0, a_1, \ldots, a_{k-r})$$
$$v' = (a_0, a_1, \ldots, a_{k-r}, b_r)$$

with all coefficients in $C_p \otimes K_d^+$.
Now $d(u', v') = 1$, so we know paths from u' to v' and can go from u to v by the paths from u' to v'.

Then, a brief description of the routing algorithm could be:

Input: u, v vertices in $BGC(p, d, d^{k+1})$

- Calculate r, the distance from u to v.
- If $r = 1$ choose the path of minimum length not intersecting the list of faulty nodes.
- If $r \neq 1$ take u', v' as above. Construct paths between u' and v' and extend them to go from u to v. Choose the one of minimum length from the paths which does not have any faulty element.

To calculate the distance between the input vertices, the most natural way is to compare the corresponding sequences to have:

$$r = d(u, v)$$
$$u = c_1, \ldots, c_r, a_1, \ldots, a_{k-r+1}$$
$$v = a_1, \ldots, a_{k-r+1}, b_1, \ldots, b_r$$

Knowing the distance, we know also d disjoint paths from u to v. In fact, if they are adjacent, we have a direct description of them. If not, we take two adjacent vertices:

$$u' = c_r, a_1, \dots, a_{k-r+1}$$
$$v' = a_1, \dots, a_{k-r+1}, b_1$$

and from paths between them, arise the ones we want.

At this point, we have to choose a path of minimum length in the above set which does not contain neither a faulty node nor a faulty arc. We can do it in several ways. A first idea could be to construct all paths by increasing order of length, and start with the shortest until we find one with the conditions desired. Another option could be to compare the nodes and arcs of each path with the faulty ones during its construction. That is, could be not necessary to construct the whole container. So, we construct one path, check the conditions, and only if it is necessary, we proceed constructing another one. Also, we can improve this idea, checking the conditions during the construction. That is, in the precise moment we add a node -and obviously an arc- we check that it is not a faulty one. If there are no other alternative, we discard this construction and start with another one. Naturally, we should start trying with paths in increasing order of length.

Example: Let $d = 7$, $p = 4$ and $k = 5$. We want a path from u to v, vertices of $BGC(4, 7, 117.649) = L^5(C_4 \otimes K_7^+)$, with:

$$u = (3,1)(4,6)(1,3)(2,0)(3,2)(4,6), v = (1,3)(2,0)(3,2)(4,6)(1,1)(2,3)$$

Since $d(u, v) = 2$, the path must be constructed recursively from other between vertices at distance 1. Let:

$$u' = (4,6)(1,3)(2,0)(3,2)(4,6),\ v' = (1,3)(2,0)(3,2)(4,6)(1,1)$$

Now, $d(u', v') = 1$, so we already know 7 paths from u' to v':

- The arc (u', v'):

$$[(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)]$$

- Paths based on others from $(4,6)$ to $(1,3)$:

$$[(4,6)(1,3)];$$
$$[(4,6)(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)(1,3)], \text{ with } \alpha_1^s \neq 1,3 \text{ and } \alpha_4^s \neq 6.$$

These paths give rise to:

$$[(4,6)(1,3)(2,0)(3,2)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)]$$
$$[u'(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)v'], \text{ with } \alpha_1^s \neq 1,3 \text{ and } \alpha_4^s \neq 6.$$

From these 7 paths from u' to v', by recursion we obtain the following 7 from u to v:

- $[(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(2,3)]$

- $[(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(2,3)]$
- $[u(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)v]$, with $\alpha_i^s = 1,\ldots,5$, $\alpha_1^s \neq 1,3$, $\alpha_4^s \neq 6$ and $\alpha_i^s \neq \alpha_i^t$ if $s \neq t$

Now, we have constructed d paths from u to v, and it only remains to select one of minimum (or minimal) length not containing faulty elements.

Let F be the set of faulty nodes and L the set of faulty links. For example, consider F and E respectively:

$$\{(2,0)(3,2)(4,6)(1,3)(2,0)(3,2),(4,5)(1,4)(2,0)(3,2)(4,5)(1,3)\}.$$
$$\{(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)\}.$$

The algorithm discards the first two paths and one of the third class.

Now, if F and E are respectively:

$$\{(2,1)(3,2)(4,3)(1,5)(2,0)(3,1),(4,5)(1,4)(2,0)(3,2)(4,5)(1,3)\}$$
$$\{(2,0)(3,2)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(2,3)\}.$$

The algorithm discards the second path and one or two of the third class.

3.5.1 The Kautz generalized cycles

Let u, v be any two vertices of $KGC(p,d,d^{p+k}+d^k)$, let say,

$$u = (c_1,\ldots,c_r,a_0,a_1,\ldots,a_{k-r})$$
$$v = (a_0,a_1,\ldots,a_{k-r},b_1,\ldots,b_r)$$

with all their coefficients in $KGC(p,d,d^{p+1})$ and r the distance from u to v.

If $r = 1$ we just have a description of the minimum length paths from u to v. Otherwise, we consider vertices u' and v' in $KGC(p,d,d^{p+k}+d^k)$:

$$u' = (c_1,a_0,a_1,\ldots,a_{k-r})$$
$$v' = (a_0,a_1,\ldots,a_{k-r},b_r)$$

with their all coefficients in $KGC(p,d,d^{p+1})$.

Now, $d(u',v') = 1$ and we know the paths from u' to v'. These allow us to go from u to v.

A short description of the routing algorithm could be:

Input u, v in $KGC(p,d,d^{p+k}+d^k)$:

- Calculate r, the distance from u to v.
- If $r = 1$ choose the path of minimum length which does not intersect the list of faulty nodes.
- If $r \neq 1$ find vertices u', v' as above. Construct paths between u' and v' and extend them to paths from u to v. Choose one of minimum length which does not intersect the list of faulty nodes.

That is, the routing strategy is the same that for $BGC(p,d,d^k)$.

Example: Let take $d=4$, $p=5$ and $k=6$. We want a path from u to v, vertices of $KGC(5,4,1025)=L^4(C_5 \otimes GK(4,1025))$, with:

$$u = (5,77)(1,715)(2,814)(3,182)(4,297)(5,170)(1,681)$$
$$v = (3,182)(4,297)(5,170)(1,681)(2,860)(3,660)(4,435)$$

Since $d(u,v)=3$, we have to construct the paths recursively, from paths between vertices at distance 1. So, we determine:

$$u' = (2,814)(3,182)(4,297)(5,170)(1,681)$$
$$v' = (3,182)(4,297)(5,170)(1,681)(2,860)$$

Now, $d(u',v')=1$ and applying the base construction, we obtain 5 paths from u' to v':

- The arc (u',v'):

$$[(2,814)(3,182)(4,297)(5,170)(1,681)(2,860)]$$

- Paths from others between $(1,681)$ and $(3,182)$:

$$[(1,681)(2,\alpha_2^{s_1})(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})(3,182)]$$

  giving rise to:

$$[u'(2,\alpha_2^{s_1})(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})v']$$

From these paths from u' to v', by recursion we obtain the following ones from u to v:

- If $((1,681),(2,860))$,$((2,814),(3,182))$ are in the same path, we discard it.
- If $((1,681),(2,860))$,$((2,814),(3,182))$ are in different paths, we discard both, and add the path obtaining by replacing the arc $((1,681),(2,860))$ by a cycle of length p, in the path that contain it.

$$[u(2,\alpha_2^{t})(3,\alpha_3^{t})(4,\alpha_4^{t})(5,\alpha_5^{t})(1,\alpha_1^{t})(2,860)(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})v]$$

Now, there are d paths from u to v, and it only remains to select the one of minimum (or minimal) length without faulty elements.

4 Fault-diameter of iterated line digraphs

4.1 Introduction

The fault-diameters digraphs were considered in [5, 9]. For the case of general iterated line digraphs there are particular studies. The best known result was proved in [53]. It says that, if an iterated line digraph L^kG has maximum connectivity, its fault-diameter is bounded by $D(L^kG) + C$, where C depends on some properties of the digraph G, but does not depend on the number of iterations k.

Here we introduce two parameters in order to find new bounds on the fault-diameters of iterated line digraphs. The bounds presented here, are not only in general tighter than the ones given in [53]. They also improve some other aspects. First of all, they do not need L^kG to be maximally connected to be applied. Besides, instead of dealing only with the worst case, that is, when the number of faulty elements is just one unit less than the connectivity, our bounds depend on the number of faulty elements. Finally, the bounds given in [53] can take different values when they are calculated for $H_1 = L^kG$ or for $H_2 = L^{k'}(L^{k-k'}G)$, being these two digraphs isomorphic. The bounds here avoid this problem. Also for some digraphs, their values are shown to be optimal.

4.2 Preliminaries

Let x and y be two different vertices of a digraph G. If the shortest path from x to y is unique, it will be denoted by $x \to y$. Its first vertex after x will be $v(x \to f)$ and its last one before y will be $v(y \leftarrow x)$. Now, if F is a set of vertices of G and $x \notin F$, $v(x \to F)$ is the set formed by $v(x \to f)$ for every vertex of $f \in F$, such that the shortest path from x to f is unique, and $v(x \leftarrow F)$ is defined analogously. When x is a vertex and $e = (u, v)$ is an arc such that the shortest path from x to u is unique, we denote by $a(x \to e)$ its first arc. If $x = u$, $a(x \to e) = e$. Also $a(x \leftarrow e)$ is the last arc of the unique shortest path from v to x. If $x = v$, $a(x \leftarrow e) = e$. If F is a set of arcs of G, we define as before the sets $a(x \to F)$ and $a(x \leftarrow F)$.

In [23] Fiol and Fàbrega introduced the following parameter: for a digraph G with minimum degree d and diameter D, $\ell = \ell(G)$ is the greatest integer, $1 \leq \ell \leq D$, such that, for any $x, y \in V(G)$,

a) if $d(x, y) < \ell$, the shortest $x \to y$ path is unique and there are no paths of length $d(x, y) + 1$;

b) if $d(x, y) = \ell$, there is only one shortest $x \to y$ path.

In [23] it was also proved that if κ and λ denote respectively the vertex and arc-connectivity of G, then they are maximum where the diameter D is $D \leq 2l - 1$ and $D \leq 2l$, respectively.

The parameter ℓ was used in [53] to study the fault-diameters of maximally connected loopless digraphs. For the same purpose but for digraphs with loops, they introduced a variation of this parameter, ℓ_1^* as following: for a digraph G with diameter D, $\ell_1^* = \ell_1^*(G)$ is the greatest integer, $1 \le \ell_1^* \le D$, such that, for any $x, y \in V(G)$,there exist two unique vertices $x^+ \in \Gamma^+(x)$,$y^- \in \Gamma^-(y)$, not necessarily different, such that

a) if $d(x,y) < \ell_1^*$, the shortest $x \to y$ path is unique and if there exists a path of length $d(x,y)+1$, it is unique and its first and last arcs are, respectively, (x, x^+) and (y^-, y);

b) if $d(x,y) = \ell_1^*$, there is only one shortest $x \to y$ path.

Making use of the above parameters, in [53] the following two results for fault-diameters were presented.

Let G be a digraph with minimum degree $d \ge 2$, diameter $D = D(G)$ and $\ell = \ell(G)$. Then,

- $D_s(L^k G) \le D(L^k G) + C$, if $k \ge D - 2\ell + 1$
- $D'_s(L^k G) \le D(L^k G) + C$, if $k \ge D - 2\ell$

for $s = 1, \ldots, d-1$, where $C = \max\{D+1, 2(D-\ell)\}$.

If G is a loopless digraph with minimum degree $d \ge 2$ diameter $D = D(G)$ and $\ell_1^* = \ell_1^*(G)$. Then,

- $D_s(L^k G) \le D(L^k G) + C$, if $k \ge D - 2L_{1,1} + 1$
- $D'_s(L^k G) \le D(L^k G) + C$, if $k \ge D - 2L_{1,1}$

for $s = 1, \ldots, d-1$, where $C = \max\{D+1, 2(D-\ell_1^*)\}$.

4.3 Parameters $L_{\pi,r}$ y $M_{\pi,r}$

This section is devoted to the introduction of some new parameters and their main properties, together with some notation for the following. We begin introducing the parameter $L_{\pi,r}$, which will allow us to stablish bounds according to the number of faulty elements. While doing it, we present the notation $\Phi^+_{\pi,r}, \Phi^-_{\pi,r}$ that will be helpful in the next.

Let G be a digraph with minimum degree $d \ge 2$ and diameter $D = D(G)$. Let π be an integer, $0 \le \pi \le d-2$. For any positive integer r we define $L_{\pi,r} = L_{\pi,r}(G)$ as the greatest integer, $0 \le L_{\pi,r} \le D$, such that for each vertex x there exist sets $\Phi^+_{\pi,r}(x) \subset \Gamma^+(x)$,$\Phi^-_{\pi,r}(x) \subset \Gamma^-(x)$, with $|\Phi^+_{\pi,r}(x)|, |\Phi^-_{\pi,r}(x)| \le \pi$, satisfying:

1. if $d(x,y) < L_{\pi,r}$, there is only one shortest path from x to y and any other path with length lesser than or equal to $d(x,y)+r$ has its first vertex in $\Phi^+_{\pi,r}(x)$ and its last one in $\Phi^-_{\pi,r}(y)$.
2. if $d(x,y) = L_{\pi,r}$, the shortest path from x to y is unique.

This parameter is a generalization of the parameters ℓ_0 [23] and ℓ_1^* [53]. In fact, $L_{0,1}(G) = \ell_0(G)$ and $L_{1,1}(G) = \ell_1^*(G)$.

Proposition 4.1 *Let G be a digraph with minimum degree $d \geq 2$. Let π be an integer with $0 \leq \pi \leq d-2$ and r a positive integer such that $L_{\pi,r} = L_{\pi,r}(G) \geq 1$. Then, for any positive integer k, $L_{\pi,r}(L^k G) = L_{\pi,r}(G) + k$.*

Proof: As $L^k(G) = LL^{k-1}G$, it is enough to consider the case $k = 1$. Let $\mathbf{x} = x_0x_1$ and $\mathbf{y} = y_0y_1$ be two different vertices of LG. If $d_{LG}(\mathbf{x}, \mathbf{y}) \leq L_{\pi,r}(G) + 1$, then $d_G(x_1, y_0) \leq L_{\pi,r}(G)$ and there is in G only one shortest path from x_1 to y_0. Therefore, in LG, the shortest path from $\mathbf{x}$ to $\mathbf{y}$ is unique. Let us consider $\Phi^+_{\pi,r}(x_1) = \{w_1, \ldots, w_s\}$ and $\Phi^-_{\pi,r}(y_0) = \{u_1, \ldots, u_t\}$, where $1 \leq s, t \leq \pi$. If $d_{LG}(\mathbf{x}, \mathbf{y}) < L_{\pi,r}(G) + 1$, then $d_G(x_1, y_0) < L_{\pi,r}(G)$ and any non-shortest path from x_1 to y_0 with length at most $d(x_1, y_0) + r$ has its first vertex in $\Phi^+_{\pi,r}(x_1)$ and its last one in $\Phi^-_{\pi,r}(y_0)$. Therefore, all non-shortest paths from $\mathbf{x}$ to $\mathbf{y}$ with length at most $d(\mathbf{x}, \mathbf{y}) + r$ have their first vertices in $\Phi^+_{\pi,r}(\mathbf{x}) = \{x_1w_1, \ldots, x_1w_s\}$ and their last ones in $\Phi^-_{\pi,r}(\mathbf{y}) = \{u_1y_0, \ldots, u_ty_0\}$. If $\mathbf{x} = \mathbf{y} = x_0x_1$, and there is a cycle $\mathbf{x}\mathbf{x}_1 \ldots \mathbf{x}_{h-1}\mathbf{x}$ with length $h \leq r$, then, there is a cycle $C = x_0x_1x_2 \ldots x_{h-1}x_0$ in the digraph G. Since $L_{\pi,r}(G) \geq 1$, we have that $x_2 \in \Phi^+_{\pi,r}(x_1)$ and $x_{h-1} \in \Phi^-_{\pi,r}(x_0)$. Then, $\mathbf{x}_1 = x_1x_2$, which is the first vertex after $\mathbf{x}$ in the cycle C, is in $\Phi^+_{\pi,r}(\mathbf{x})$, and the last vertex of the cycle is $\mathbf{x}_{h-1} = x_{h-1}x_0 \in \Phi^-_{\pi,r}(\mathbf{x})$. Therefore, we have proved that $L_{\pi,r}(LG) \geq L_{\pi,r}(G) + 1$.

On the other hand, since $d \geq 2$, for any two vertices x_1, y_0 of G there exist vertices x_0, y_1 of G such that $\mathbf{x} = x_0x_1$ and $\mathbf{y} = y_0y_1$ are vertices of LG with $d(\mathbf{x}, \mathbf{y}) = d(x_1, y_0) + 1$. Then, it is not difficult to prove that $L_{\pi,r}(G) \geq L_{\pi,r}(LG) - 1$.□

Lemma 4.2 *Let G be a digraph with minimum degree $d \geq 2$, and $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π with $0 \leq \pi \leq d-2$ and a positive integer r. If x, y are two vertices of G, then*

- *(a) if $d(x, y) < L_{\pi,r}$: for all $x_1 \in \Gamma^+(x)\Phi^+_{\pi,r}(x)$ such that $x_1 \neq v(x \to y)$, $d(x_1, y) \geq d(x, y) + r$; for all $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y)$ such that $y_1 \neq v(y \leftarrow x)$, $d(x, y_1) \geq d(x, y) + r$;*
- *(b) if $d(x, y) = L_{\pi,r}$: for all $x_1 \in \Gamma^+(x)\{v(x \to y)\}$, $d(x_1, y) \geq L_{\pi,r}$; for all $y_1 \in \Gamma^-(y) - \{v(y \leftarrow x)\}$, $d(x, y_1) \geq L_{\pi,r}$.*

Proof: If $d(x, y) < L_{\pi,r}$, $x_1 \notin \Phi^+_{\pi,r}(x)$ and $x_1 \neq v(x \to y)$, then, the length of any path $xx_1 \to y$ is greater than $d(x, y) + r$. Therefore, $d(x_1, y) \geq d(x, y) + r$. In the same way, $d(x, y_1) \geq d(x, y) + r$.

If $d(x, y) = L_{\pi,r}$ the shortest path from x to y is unique. A shortest path from $x_1 \neq v(x \to y)$ to y determines a path from x to y. Then, $d(x_1, y) + 1 \geq d(x, y) + 1 = L_{\pi,r} + 1$. Analogously, $d(x, y_1) \geq L_{\pi,r}$. □

Iterating the application of the Lemma 4.2 we obtain the following:

Lemma 4.3 *Let G be a digraph with minimum degree $d \geq 2$, and $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π with $0 \leq \pi \leq d-2$ and a positive integer r. Let F be a set of vertices of G, $1 \leq |F| \leq d - \pi - 1$, and x, y two vertices of G, $x, y \notin F$. Then, for every $m \geq 1$:*

(a) there exists a path $xx_1 \dots x_m$ such that for all $f \in F$:
$d(x_i, f) \geq \min\{d(x,f) + rm, L_{\pi,r}\}$;

(b) there exists a path $y_m \dots y_1 y$ such that for all $f \in F$:
$d(f, y_i) \geq \min\{d(f,y) + rm, L_{\pi,r}\}$. □

Let G be a digraph with minimum degree $d \geq 2$. Let π be an integer with $0 \leq \pi \leq d-2$ and r a positive integer such that $L_{\pi,r}(G) \geq 1$. A (π, r)-*double detour* is a set of four paths $\{C_1, C_1', C_2, C_2'\}$ such that

- C_1 and C_1' are paths from x to f, with lengths s and s', respectively, where $s' \geq s$ and $s' \geq 1$. C_2 and C_2' are paths from f to y, with lengths t and t', respectively, where $t' \geq s$ and $t' \geq 1$. Besides, $\max\{s,t\} \geq 1$.
- If $(x, x'1)$ is the first arc of C_1', then $x_1' \notin \Phi^+_{\pi,r}(x)$. If $s \neq 0$ and (x, x_1) is the first arc of C_1, then $x_1' \neq x_1$.
- If (y_1', y) is the last arc of C_2', then $y_1' \notin \Phi^-_{\pi,r}(y)$. If $t \neq 0$ and (y_1, y) is the last arc of C_2, then $x_1' \neq x_1$.

The *length* of a (π, r)-double detour is defined to be $s' + t'$. We define $M_{\pi,r} = M_{\pi,r}(G)$ as the minimum length of a (π, r)-double detour of G.

It is not difficult to check that, for any digraph G, $M_{1,1}(G) \geq 4$ and $M_{0,1} \geq 4$ if G is loopless.

Proposition 4.4 *Let G be a digraph with minimum degree $d \geq 2$. Let π be an integer with $0 \leq \pi \leq d-2$ and r a positive integer such that $L_{\pi,r}(G) \geq 1$. Then, for any positive integer k, $M_{\pi,r}(L^k G) = M_{\pi,r}(G) + k$.*

Proof: As before, it is enough to prove the proposition for $k = 1$. Let $\{C_1, C_1', C_2, C_2'\}$ be a (π, r)-double detour in LG with length $s' + t'$, where C_1 and C_1' are paths from $\mathbf{x} = x_0x_1$ to $\mathbf{f} = f_0f_1$ and C_2 and C_2' are paths from $\mathbf{f} = f_0f_1$ to $\mathbf{y} = y_0y_1$. If $s, t \geq 1$, it is not difficult to prove that there exist a (π, r)-double detour in G with length $s' + t' - 1$. This double detour consists in two paths from x_1 to f_0 and two paths from f_0 to y_0.

Now, we assume that $s = 0$ and $t \geq 1$, that is, $\mathbf{x} = \mathbf{f} = f_0f_1$. In this case, we can find a (π, r)-double detour $\{\mathcal{C}_1, \mathcal{C}_1', \mathcal{C}_2, \mathcal{C}_2'\}$ in the digraph G, where $\mathcal{C}_1$ and $\mathcal{C}_1'$ are paths from f_1 to f_1 with lengths $s = 0$ and s', respectively, and $\mathcal{C}_2$ and $\mathcal{C}_2'$ are paths from f_1 to y_0 with lengths $t - 1$ and $t' - 1$, respectively.

Therefore, if there is a (π, r)-double detour in LG with length $s' + t'$, then there exists a (π, r)-double detour in G with length $s' + t' - 1$. On the other hand, for any (π, r)-double detour in G with length h, it is easy to find a (π, r)-double detour in LG with length $h + 1$.□

Lemma 4.5 *Let G be a digraph with minimum degree $d \geq 2$. Let π and r be two integers with $0 \leq \pi \leq d-2$, and $M_{\pi,r} = M_{\pi,r}(G)$. Let x, y, f be any three vertices. If $x_1 \in \Gamma^+(x) - \Phi^+_{\pi,r}(x), x_1 \neq v(x \to f)$ and $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y), y_1 \neq v(y \leftarrow f)$, then $d(x_1, f) + d(f, y_1) \geq M_{\pi,r} - 2$.*

Proof: Since $x_1 \notin \Phi^+_{\pi,r}(x), x_1 \neq v(x \to f)$ and $y_1 \notin \Phi^-_{\pi,r}(y), y_1 \neq v(y \leftarrow f)$, we can consider a (π, r)-double detour in G with C_1 the shortest path from x to f, C_2 the shortest path from f to y, $C'_1 = xx_1 \to f$ and $C'_2 = f \to y_1 y$. Then, $M_{\pi,r}(G) \leq d(x_1, f) + d(f, y_1) + 2$, and $d(x_1, f) + d(f, y_1) \geq M_{\pi,r} - 2$. □

The following result can be proved analogously.

Lemma 4.6 *Let G be a digraph with minimum degree $d \geq 2$. Let π and r be two integers with $0 \leq \pi \leq d-2$, and $M_{\pi,r} = M_{\pi,r}(G)$. Let x, y be two vertices and (f, g) an arc. If $x_1 \in \Gamma^+(x) - \Phi^+_{\pi,r}(x), x_1 \neq v(x \to f)$ and $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y), y_1 \neq v(y \leftarrow g)$, then $d(x_1, f) + d(g, y_1) \geq M_{\pi,r} - 3$.* □

4.4 New bounds

In this section we present upper bounds for both, vertex and arc-fault-diameters of iterated line digraphs, making use of the results of Section 4.3.

Theorem 4.7 *Let G be a digraph with minimum degree $d \geq 2$, diameter $D = D(G)$, and parameters $M_{\pi,r} = M_{\pi,r}(G)$, $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π, $0 \leq \pi \leq d-2$ and a positive integer r. If $D \leq 2L_{\pi,r} - 1$, the $(s-1)$-vertex-fault-diameter of G is*

$$D_s(G) \leq D(G) + C$$

for $s = 1, \ldots, d - \pi$ with $C = \max\{\left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, \left\lceil \frac{2(D - L_{\pi,r})}{r} \right\rceil\}$.

Proof: Let F be a non-empty set of faulty vertices of G, $|F| = s \leq d - \pi - 1$. Let x, y be two different vertices of G which are not in F. As $|F| \leq d - \pi - 1$, there exist x_1 in $\Gamma^+(x) - \Phi^+_{\pi,r}(x) - v(x \to F)$ and y_1 in $\Gamma^-(y) - \Phi^-_{\pi,r}(y) - v(y \leftarrow F)$. From Lemma 4.5, $d(x_1, f) + d(f, y_1) \geq M_{\pi,r} - 2$, for all $f \in F$. By Lemma 4.2: $d(x_1, f) \geq d(x, f) + r$ or $d(x_1, f) \geq L_{\pi,r}(L^k G)$ and $d(f, y_1) \geq d(f, y) + r$ or $d(f, y_1) \geq L_{\pi,r}(G)$, for all $f \in F$. Also, as $D \leq 2L_{\pi,r} - 1$: $2L_{\pi,r}(G) \geq D + 1$. By Lemma 4.3, there exist paths $x_1 x_2 \ldots x_m$ and $y_n \ldots y_2 y_1$ with $x_i, y_i \notin F$, such that $d(x_m, f) \geq \min\{d(x_1, f) + r(m-1), L_{\pi,r}\}$ and $d(f, y_n) \geq \min\{d(f, y_1) + r(n-1), L_{\pi,r}\}$, for all $f \in F$. Now, if $(m+n)r \geq D - M_{\pi,r} + 3 + 2r$ and $mr, nr \geq D - L_{\pi,r}$ in any case we have $d(x_m, f) + d(f, y_n) \geq D + 1 = D + 1$. Then, a shortest path from x_m to y_n (with length at most $D + k$) does not contain any vertex of F. Therefore, we have found a path from x to y with length at most $D + m + n$ avoiding F. Considering m and n such that $m + n = \max\{\left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, \left\lceil \frac{2(D - L_{\pi,r})}{r} \right\rceil\}$ we obtain the desired bound.□

For iterated line digraphs, using Proposition 4.4 and 4.1, we can state the following:

Corollary 4.8 *Let G be a digraph with minimum degree $d \geq 2$, diameter $D = D(G)$, and parameters $M_{\pi,r} = M_{\pi,r}(G)$, $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π, $0 \leq \pi \leq d-2$ and a positive integer r. For any integer k, such that $k \geq D-2L_{\pi,r}+1$, the $(s-1)$-vertex-fault-diameter of L^kG is*

$$D_s(L^kG) \leq D(L^kG) + C$$

for $s = 1, \ldots, d-\pi$ with $C = \max\{\left\lceil \frac{D-M_{\pi,r}+3+2r}{r} \right\rceil, \left\lceil \frac{2(D-L_{\pi,r})}{r} \right\rceil\}$. □

Making use of Lemma 4.6 instead of 4.5, we obtain:

Theorem 4.9 *Let G be a digraph with minimum degree $d \geq 2$, diameter $D = D(G)$, and parameters $M_{\pi,r} = M_{\pi,r}(G)$, $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π, $0 \leq \pi \leq d-2$ and a positive integer r. If $D \leq 2L_{\pi,r}$, the $(s-1)$-arc-fault-diameter of G is*

$$D'_s(G) \leq D(G) + C$$

for $s = 1, \ldots, d-\pi$ with $C = \max\{\left\lceil \frac{D-M_{\pi,r}+3+2r}{r} \right\rceil, \left\lceil \frac{2(D-L_{\pi,r})}{r} \right\rceil\}$.□

Corollary 4.10 *Let G be a digraph with minimum degree $d \geq 2$, diameter $D = D(G)$, and parameters $M_{\pi,r} = M_{\pi,r}(G)$, $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π, $0 \leq \pi \leq d-2$ and a positive integer r. For any integer k, such that $k \geq D - 2L_{\pi,r}$, the $(s-1)$-arc-fault-diameter of L^kG is*

$$D'_s(L^kG) \leq D(L^kG) + C$$

for $s = 1, \ldots, d-\pi$ with $C = \max\{\left\lceil \frac{D-M_{\pi,r}+3+2r}{r} \right\rceil, \left\lceil \frac{2(D-L_{\pi,r})}{r} \right\rceil\}$.□

Theorems 4.1 and 4.2 in [53] are a consequence of the following corollary, which is proved by taking $\pi = 1$ and $r = 1$ in the previous theorems

Corollary 4.11 *Let G be a digraph with minimum degree $d > 2$, diameter $D = D(G)$ and $\ell_1^* = L_{1,1} = L_{1,1}(G)$. Then,*

- $D_s(L^kG) \leq D(L^kG) + C$, *if* $k \geq D - 2L_{1,1} + 1$
- $D'_s(L^kG) \leq D(L^kG) + C$, *if* $k \geq D - 2L_{1,1}$

for $s = 1, \ldots, d-1$, where $C = \max\{D - M_{1,1} + 5, 2(D - L_{1,1})\}$. □

If we take $\pi = 0$ and $r = 1$, we obtain the following result, from which Theorems 3.1 and 3.2 in [53] follow.

Corollary 4.12 *Let G be a digraph without loops and with minimum degree $d \geq 2$, diameter $D = D(G)$ and $\ell_0 = L_{0,1} = L_{0,1}(G)$. Then,*

- $D_s(L^kG) \leq D(L^kG) + C$, *if* $k \geq D - 2L_{0,1} + 1$
- $D'_s(L^kG) \leq D(L^kG) + C$, *if* $k \geq D - 2L_{0,1}$

for $s = 1, \ldots, d$ where $C = \max\{D - M_{0,1} + 5, 2(D - L_{0,1})\}$.□

4.5 Applications

Since $BGC(p, d, d^{p+k} + d^k) = L^k(C_p \otimes K_d^*)$, we can apply the bounds obtained to this family.

First, let us see the values obtained from the bounds in [53]. There it was proved that for a digraph G with minimum degree $d \geq 2$, diameter D and $\ell = \ell(G)$,

- $D_s(L^k G) \leq D(L^k G) + C$, if $k \geq D - 2\ell + 1$;
- $D'_s(L^k G) \leq D(L^k G) + C$, if $k \geq D - 2\ell$.

where $C = \max\{D + 1, 2(D - \ell)\}$ for any $s = 2, \ldots, d$.

The digraph $C_p \otimes K_d^*$, has diameter $D = p$ [40] and parameter $\ell = 1$ if $p > 1$ (it is easy to obtain from [40]). With these values, the bound arising from the results in [53] is:

$$C = \max\{2p, 2(p - 1)\} = 2p.$$

The results in Section 4.4 state that if G is a digraph with minimum degree $d \geq 2$, diameter D, and $M_{\pi,r} = M_{\pi,r}(G)$, $L_{\pi,r} = L_{\pi,r}(G)$ for an integer π, $0 \leq \pi \leq d - 2$ and a positive integer r.

- $D_s(L^k G) \leq D(L^k G) + C$, if $k \geq D - 2L_{\pi,r} + 1$;
- $D'_s(L^k G) \leq D(L^k G) + C$, if $k \geq D - 2L_{\pi,r}$.

where $C = \max\{\left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, \left\lceil \frac{2(D - L_{\pi,r})}{r} \right\rceil\}$ for $s = 1, \ldots, d - \pi$.

The values of the parameters $M_{\pi,r}$ and $L_{\pi,r}$ are simple to calculate for $C_p \otimes K_d^*$ when $\pi = 0$ and $r = p - 1$. In fact, from some properties presented in [40] and in Section 3.3.1, it can be easily stated the following proposition.

Proposition 4.13 *For any positive integer p and any integer $d \geq 2$,*

$$M_{0,p-1}(C_p \otimes K_d^*) = 4 \text{ and } L_{0,p-1}(C_p \otimes K_d^*) = 1. \ \square$$

With these values,

$$C = \max\{\left\lceil \frac{p - 4 + 3 + 2(p - 1)}{p - 1} \right\rceil, \left\lceil \frac{2(p - 1)}{p - 1} \right\rceil\} = 3$$

for $s = 2, \ldots, d$.

These bounds hold for example, for the directed butterfly $B_d(p) = BGC(2, d, d^p)$, where no results were known.

In Section 3.3.1 it was proved the existence of a container of width d and maximum length $D + 2$ between any two vertices of the generalized cycles $BGC(p, d, d^{k+1})$. So, the general bounds obtained exceed in one unit the exact value for the family of digraphs $BGC(p, d, d^{k+1})$.

Also $KGC(p,d,d^{p+k}+d^k) = L^k KGC(p,d,d^p+1)$ and we can apply the bounds obtained to this family.

Let us see previously, the values obtained from the bounds in [53].

The digraph $KGC(p,d,d^p+1)$, has diameter $D = 2p-1$ [40] and parameter $\ell = p$ if $p > 1$ (it is easy to obtain from [40]). With these values, the bound arising from the results in [53] is:

$$C = \max\{2p, 2(2p-1-p)\} = 2p$$

for any $s = 2, \ldots, d$.

Instead, by the results in Section 4.4,

$$C = \max\{\left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, \left\lceil \frac{2(D - L_{\pi,r})}{r} \right\rceil\}$$

for $s = 2, \ldots, d - \pi$.

The values of the parameters $M_{\pi,r}$ and $L_{\pi,r}$ are simple to calculate for $KGC(p,d,d^p+1)$ when $\pi = 0$ and $r = p-1$. In fact, from some properties presented in [40] and in Section 3.3.2, it can be easily stated the following proposition.

Proposition 4.14 *For any positive integer p and any integer $d \geq 2$,*

$$M_{0,p-1}(KGC(p,d,d^p+1)) = 2p+2 \text{ and } L_{0,p-1}(KGC(p,d,d^p+1)) = p. \;\square$$

With these values,

$$C = \max\{\left\lceil \frac{2p-1-2p-2+3+2(p-1)}{p-1} \right\rceil, \left\lceil \frac{2(2p-1-p)}{p-1} \right\rceil\} = 2$$

for $s = 2, \ldots, d$.

These bounds coincide with the exact values given in [54] for the fault-diameters of the bipartite digraphs $BD(d,n)$, where it was proved the existence of a container of width d and maximum length $D+2$ between any two vertices. We recall that $KGC(2,d,d^{p+k}+d^k) = BD(2,d^{p+k}+d^k)$.

Also in Section 3.3.2 it was proved the existence of a container of width d and maximum length $D+2$ between any two vertices of the generalized cycles $KGC(p,d,d^{p+k}+d^k)$.

As a conclusion, the general bounds obtained are optimal for the families of digraphs $BD(d,n)$ and more generally, the $KGC(p,d,d^{p+k}+d^k)$.

5 Connectivity and fault-tolerance of hyperdigraphs

5.1 Introduction

Some results about the fault-tolerance of bus interconnection networks modeled by directed hypergraphs are presented in this section. In particular, we study the connectivities and fault-diameters of hyperdigraphs.

The main results we present in this section are related to the fault-tolerance of iterated line hyperdigraphs. We prove that, for any hyperdigraph H, the iterated line digraph L^kH is maximally connected if the number of iterations k is large enough. That generalizes the results in [23] about the connectivity of iterated line digraphs.

The results in [53] and in Section 4 about the fault-diameters of iterated line digraphs are also generalized here for hyperdigraphs. We prove that, if the number of iterations is large enough, the diameter of an iterated line hyperdigraph L^kH increases in at most a constant value when some vertices or hyperarcs are deleted. This constant value depends only on the properties of the hyperdigraph H and does not depend on the number of iterations k.

Some results about the connectivity and the fault-diameter of Kautz and de Bruijn hyperdigraphs are derived.

5.2 Basic results on connectivity

We say that a hyperdigraph H is *simple* if its underlying digraph $\widehat{H}$ has no parallel arcs. That is, a hyperdigraph H is simple if and only if there does not exist any pair of hyperarcs E_1, E_2 of H with $E_1^- \cap E_2^- \neq \emptyset$ and $E_1^+ \cap E_2^+ \neq \emptyset$.

Proposition 5.1 *Let H be a hyperdigraph. Then, its line digraph LH is a simple hyperdigraph.*

Proof: Let E, F be hyperarcs of LH such that $E^- \cap F^- \neq \emptyset$. Suppose that $E = (E_1 v_1 F_1)$ and $F = (E_2 v_2 F_2)$. If $E^- \cap F^- \neq \emptyset$, then $E_1 = E_2$ and $v_1 = v_2$. Then, $E^+ = \{(v_1 F_1 w_i) : w_i \in F_1^+\}$ and $F^+ = \{(v_1 F_2 z_i) : z_i \in F_2^+\}$. If $E \neq F$, it must be $F_1 \neq F_2$ and then, $E^+ \cap F^+ = \emptyset$. □

Let H be a hyperdigraph with minimum degree d and minimum size s. We denote by $\widehat{d} = d(\widehat{H})$ the minimum degree of the underlying digraph $\widehat{H}$. Let κ and λ be, respectively, the vertex and hyperarc-connectivities of H and let $\widehat{\kappa}$ and $\widehat{\lambda}$ be the vertex and arc-connectivities of the underlying digraph $\widehat{H}$.

It is clear that $\kappa = \widehat{\kappa}$ and, from the properties of the connectivities of digraphs, $\widehat{\kappa} \leq \widehat{\lambda} \leq \widehat{d}$. On the other hand, it is obvious that $\lambda \leq d$.

If the hyperdigraph H is s-uniform, we have that $\widehat{d} \leq ds$. If, besides, H is simple, $\widehat{d} = ds$. Then, in the uniform case, $\kappa = \widehat{\kappa} \leq \widehat{\lambda} \leq \widehat{d} \leq ds$. Another relation between the connectivities of a hyperdigraph is given in next proposition for the uniform case.

Proposition 5.2 *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be an s-uniform hyperdigraph with vertex and hyperarc connectivities κ and λ, respectively. Then, $\kappa \leq \lambda s$.*

Proof: Let $\mathcal{F} = \{E_1, \ldots, E_\lambda\} \subset \mathcal{E}(H)$ be a cut-set of H. It is not difficult to see that at least one of the sets of vertices $\mathcal{F}^- = E_1^- \cup \cdots \cup E_\lambda^-$ or $\mathcal{F}^+ = E_1^+ \cup \cdots \cup E_\lambda^+$ is a cut-set of H. Observe that $|\mathcal{F}^-|, |\mathcal{F}^+| \leq \lambda s$.□

We say that a hyperdigraph H is *maximally connected* if $\kappa = \widehat{d}$ and $\lambda = d$. Observe that, if H is simple and s-uniform and has vertex-connectivity

$\kappa = ds$, then, from Proposition 5.2, $\lambda = d$. Therefore, a simple and s-uniform hyperdigraph H is maximally connected if and only if $\kappa = ds$.

5.3 Fault-tolerance under deletion of vertices

The vertex-connectivity and of a hyperdigraph H coincides with the vertex-connectivity of its underlying digraph $\widehat{H}$, that is, $\kappa = \kappa(H) = \widehat{\kappa} = \kappa(\widehat{H})$. The same occurs with the $(w-1)$-vertex-fault-diameter: $D_w(H) = D_w(\widehat{H})$ for any $w = 1, \ldots, \widehat{d}$, where $\widehat{d}$ is the minimum degree of $\widehat{H}$. Therefore, the numerous known results about this parameter for digraphs, some of them presented in the previous sections of this work, can be applied for hyperdigraphs just by considering the underlying digraph.

Next results are obtained by considering the results about vertex-connectivity of digraphs given in [23].

Proposition 5.3 *Let H be a simple hyperdigraph with diameter D and vertex-connectivity κ. Let $\widehat{d}$ be the minimum degree of the underlying digraph $\widehat{H}$ and consider $\ell_\pi = \ell_\pi(\widehat{H})$, where $0 \le \pi \le \widehat{d} - 2$. Then, $\kappa \ge \widehat{d} - \pi$ if $D \le 2\ell_\pi - 1$.*

Some interesting corollaries about the vertex-connectivity of iterated line hyperdigraphs are deduced from this proposition. The following one is proved by taking into account that $\widehat{L^k H} = L^k \widehat{H}$ and $\ell_\pi(L^k \widehat{H}) = \ell_\pi(\widehat{H}) + k$ whenever H is a simple digraph, $\widehat{H}$ is not a cycle and $\ell_\pi(\widehat{H}) \ge 1$.

Corollary 5.4 *Let H be a simple hyperdigraph with diameter D. Let $\widehat{d}$ be the minimum degree of the underlying digraph $\widehat{H}$ and consider $\ell_\pi = \ell_\pi(\widehat{H})$, where $0 \le \pi \le \widehat{d} - 2$ and $\ell_\pi(\widehat{H}) \ge 1$. Then, $\kappa(L^k H) \ge \widehat{d} - \pi$ if $k \ge D - 2\ell_\pi + 1$.*

The particular case $\pi = 0$ is specially interesting.

Corollary 5.5 *Let H be a simple hyperdigraph with diameter D such that its underlying digraph $\widehat{H}$ is loopless. Let us consider $\ell_0 = \ell_0(\widehat{H}) \ge 1$. Then, $\kappa(L^k H) = \widehat{d}$ if $k \ge D - 2\ell_0 + 1$.*

Since the line hyperdigraph LH is simple for any hyperdigraph H, we can see from the last corollary that, for any hyperdigraph H such that $\widehat{H}$ is loopless, the vertex-connectivity of $L^k H$ is maximum if the number of iterations k is large enough. If, besides, H is s-uniform, we have seen that H is maximally connected if and only if $\kappa = ds$. Therefore, in that case, the iterated line hyperdigraph $L^k H$ is maximally connected if k is large enough.

In a similar way, we can apply Theorem 4.7 and Corollary 4.8 in order to find bounds on the vertex-fault-diameter of hyperdigraphs. In particular, from Corollary 4.8, we can see that, if k is large enough, the $(w-1)$-vertex-fault-diameter of an iterated line hyperdigraph $L^k H$ is $D_w(L^k H) \le D(L^k H) + C$, where C is a constant that depends only on w and the properties of $\widehat{H}$, but does not depend on the number of iterations k.

5.4 Hyperarc-connectivity

Bounds on the hyperarc-connectivity λ of an s-uniform hyperdigraph H can be derived from bounds on its vertex connectivity κ because, in the uniform case, $\kappa \leq \lambda s$. In particular, we have seen that $\lambda = d$ if $\kappa = \widehat{d} = ds$.

The aim of this section is to present some bounds for the hyperarc-connectivity of a hyperdigraph H, even if H is not s-uniform. Sufficient conditions for a hyperdigraph to have maximum hyperarc-connectivity, that is $\lambda = d$, are derived.

Let us recall that the bipartite representation of a hyperdigraph H is a bipartite digraph $R = R(H) = (V(R), A(R))$ with set of vertices $V(R) = V_0(R) \cup V_1(R)$, where $V_0(R) = \mathcal{V}(H)$ and $V_1(R) = \mathcal{E}(H)$, and set of arcs

$$A(R) = \{(u, E) \mid u \in V_0, E \in V_1, u \in E^-\} \cup \{(F, v) \mid v \in V_0, F \in V_1, v \in F^+\}.$$

Observe that, if u, v are two vertices of H, a path of length h from u to v in H correspond to a path of length $2h$ in $R(H)$ and, then, $d_R(u, v) = 2d_H(u, v)$. Observe also that the bipartite repreentation of the line hyperdigraph LH is $R(LH) = L^2R(H)$.

The hyperarc-connectivity $\lambda = \lambda(H)$ of a hyperdigraph H can be expressed in terms of the bipartite representation of H. In effect, λ is the minimum cardinality of all the subsets $\mathcal{F} \subset V_1$ such that there exist two vertices $u, v \in V_0$ such that there is no path from u to v in $R - \mathcal{F}$.

We define next a parameter, similar to the parameter ℓ_π, that will be useful to find bounds on the hyperarc-connectivity. This parameter is defined for bipartite digraphs and will be applied to the bipartite representation of the hypergraph. Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph. Let us consider $d_0^+(R) = \min_{v \in V_0} d^+(v)$, the minimum out-degree of the vertices in V_0, and $d_0^-(R)$, the minimum in-degree of the vertices in V_0. Let us take $d_0 = d_0(R) = \min\{d_0^+, d_0^-\}$. Let π be an integer such that $0 \leq \pi \leq d_0 - 2$. We define $h_\pi = h_\pi(R)$ as the maximum integer, with $1 \leq h_\pi \leq D$, such that for any pair of vertices u, v, where $u \in V_i$ and $y \in V_j$ with $i \neq j$,

- if $d(x, y) < h_\pi$, there is only one shortest path from x to y and there are at most π paths from x to y with length $d(x, y) + 2$;
- if $d(x, y) = h_\pi$, there is only one shortest path from x to y.

Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph. Then, the iterated line digraph L^2R is a bipartite digraph and, in a natural way, we can put, for $i = 0, 1$, $V_i(L^2R) = \{x_0x_1x_2 \in V(L^2R) \mid x_0 \in V_i(R)\}$. In this situation, $d_0(L^2R) = d_0(R)$ and we can consider $h_\pi(R)$ and $h_\pi(L^2R)$ for the same values of π.

Proposition 5.6 *Let $R = (V_0(R) \cup V_1(R), A(R))$ be a bipartite digraph different from a cycle. Then, $h_\pi(L^2R) = h_\pi(R) + 2$ for any $\pi = 1, \ldots, d_0 - 2$. If there are no cycles of length 2 in R, then $h_0(L^2R) = h_0(R) + 2$*

Proof: Let us consider $\mathbf{x} = x_0x_1x_2 \in V_i(L^2R)$ and $\mathbf{y} = y_0y_1y_2 \in V_j(L^2R)$, where $i \neq j$. If $d(\mathbf{x},\mathbf{y}) \geq 3$ and $d(\mathbf{x},\mathbf{y}) \leq h_\pi(R)+2$, then $d(x_2,y_0) \leq h_\pi(R)$. Therefore, the shortest path from x_2 to y_0 is unique and so is the shortest path from $\mathbf{x}$ to $\mathbf{y}$. If $d(\mathbf{x},\mathbf{y}) \geq 3$ and $d(\mathbf{x},\mathbf{y}) < h_\pi(R)+2$, then $d(x_2,y_0) < h_\pi(R)$ and there are at most π paths from $\mathbf{x}$ to $\mathbf{y}$ with length $d(\mathbf{x},\mathbf{y})+2$. If $d(\mathbf{x},\mathbf{y}) = 1$, then the vertices x_1x_2 and y_0y_1 of LR are equal. Since in LR there is at most one cycle of length 2 on the vertex x_1x_2, in L^2R there is at most one path with length $3 = d(\mathbf{x},\mathbf{y})+2$ from $\mathbf{x}$ to $\mathbf{y}$. If R has no cycles of length 2, there is not any path of length 3 from $\mathbf{x}$ to $\mathbf{y}$. Therefore, $h_\pi(L^2R) \geq h_\pi(R)+2$ if $\pi \geq 1$ or $\pi = 0$ and R has no cycles of length 2. Since R is not a cycle, it is not difficult to see that $h_\pi(L^2R) \leq h_\pi(R)+2$.□

Proposition 5.7 *Let $R = (V_0 \cup V_1, A)$ be a bipartite digraph and let us consider $h_\pi = h_\pi(R)$, where $0 \leq \pi \leq d_0 - 2$. Let us consider a vertex $x \in V_0$, a subset $\mathcal{F} \subset V_1$, with $|\mathcal{F}| \leq d_0 - \pi - 1$, and a vertex $y \in \mathcal{F}$. Then,*

- *There exists a vertex $x_1 \in V_0$ and a path xy_1x_1 such that $y_1 \notin \mathcal{F}$ and $d(x_1,y) \geq \min\{d(x,y)+2, h_\pi\}$ and $d(x_1,y') \geq \min\{d(x,y'), h_\pi\}$ for any $y' \in \mathcal{F}$.*
- *There exists a vertex $x_{-1} \in V_0$ and a path $x_{-1}y_{-1}x$ such that $y_{-1} \notin \mathcal{F}$ and $d(y,x_{-1}) \geq \min\{d(y,x)+2, h_\pi\}$ and $d(y',x_{-1}) \geq \min\{d(y',x), h_\pi\}$ for any $y' \in \mathcal{F}$.*

Proof: We are going to prove the first statement. The second one is proved analogously. Since $|v(x \to \mathcal{F})| \leq d_0 - \pi - 1$, there exists a vertex $y_1 \in \Gamma^+(x) - v(x \to \mathcal{F})$ such that the first vertex of any path from x to y with length $d(x,y)+2$ is different from y_1. Let x_1 be any vertex in $\Gamma^+(y_1)$. It is not difficult to prove that this vertex satisfies the required conditions.□

Theorem 5.8 *Let H be a hyperdigraph with minimum degree d, diameter D and hyperarc-connectivity λ. Let $R = R(H)$ be its bipartite representation and consider $h_\pi = h_\pi(R)$. Then, $\lambda \geq d - \pi$ if $D \leq h_\pi - 1$.*

Proof: We are going to prove that, if $D \leq h_\pi - 1$, for any set of vertices of the bipartite representation $\mathcal{F} \subset V_1 = \mathcal{E}(H)$, with $|\mathcal{F}| \leq d - \pi - 1$, and for any pair of vertices $u, v \in V_0$, there exists a path from u to v in $R - \mathcal{F}$. Effectively, from Proposition 5.7, we can find in R a path $uE_1u_1E_2u_2 \ldots E_mu_m$ such that $E_i \notin \mathcal{F}$ and $d_R(u_m, \mathcal{F}) \geq h_\pi$. Equally, we can find a path $v_{-n}E_{-n} \ldots v_{-2}E_{-2}v_{-1}E_{-1}v$ such that $E_{-i} \notin \mathcal{F}$ and $d_R(v_{-n}, \mathcal{F}) \geq h_\pi$. Then, a shortest path from u_m to v_{-n}, which has length at most $2D < 2h_\pi$, will avoid $\mathcal{F}$.□

The following corollary is a direct consequence of Theorem 5.8 and Proposition 5.6.

Corollary 5.9 *Let H be a hyperdigraph with minimum degree d, and diameter D. Let $R = R(H)$ be its bipartite representation and consider $h_\pi = h_\pi(R)$. Then,*

- $\lambda(L^k H) \geq d - \pi$ *if* $k \geq D - h_\pi + 1$.
- *If* R *has no cycles of length 2, then* $\lambda(L^k H) = d$ *if* $k \geq D - h_0 + 1$.

5.5 Hyperarc-fault-diameter

We present in this section some results about the *hyperarc-fault-diameter*, $D'_w(H)$, of a hyperdigraph H, which is defined as the maximum diameter of the hyperdigraphs obtained from H by removing at most $w-1$ hyperarcs.

In the same way as we did for the hyperarc-connectivity, we are going to use the bipartite representation $R(H)$ to study that parameter. In particular, we present a bound on $D'_w(H)$ in terms of $h_0(R)$ and the parameter $M_{0,1}(R)$, which has been defined in Section 4.3. We are going to use the following lemma, which is proved in a similar way as Proposition 5.7.

Lemma 5.10 *Let* $R = (V_0 \cup V_1, A)$ *be a bipartite digraph without cycles of length 2 and* $h_0 = h_0(R)$. *Let us consider a vertex* $x \in V_0$ *and a subset* $\mathcal{F} \subset V_1$, *with* $|\mathcal{F}| \leq d_0 - 1$. *Then,*

- *There exists a vertex* $x_1 \in V_0$ *and a path* xy_1x_1 *such that* $y_1 \notin \mathcal{F}$ *and* $d(x_1, y) \geq \min\{d(x,y)+2, h_\pi\}$ *for any* $y \in \mathcal{F}$.
- *There exists a vertex* $x_{-1} \in V_0$ *and a path* $x_{-1}y_{-1}x$ *such that* $y_{-1} \notin \mathcal{F}$ *and* $d(y, x_{-1}) \geq \min\{d(y,x)+2, h_\pi\}$ *for any* $y \in \mathcal{F}$.

Theorem 5.11 *Let* H *be a simple hyperdigraph with minimum degree* d *and diameter* D *and let* $R = R(H)$ *be its bipartite representation. Let us consider* $h = h_0(R)$ *and* $M = M_{0,1}(R)$. *Then, if* $D \leq h-1$, *for any* $w = 1, \ldots, d-1$, *the* w*-hyperarc-fault-diameter of* H *verifies* $D'_{w+1}(H) \leq D + C$, *where*

$$C = \max\left\{D - \left\lfloor \frac{M-1}{2} \right\rfloor + 4, 2\left(D - \left\lfloor \frac{h}{2} \right\rfloor\right)\right\}$$

Proof: Let $\mathcal{F} \subset \mathcal{E}(H) = V_1(R)$ be a set of faulty hyperarcs with $|\mathcal{F}| = w < d$. We are going to prove that, for any pair of vertices $x, y \in \mathcal{V}(H) = V_0(R)$, there exists in H a path from x to y with length at most $D + C$ avoiding the hyperarcs in $\mathcal{F}$. From Lemma 5.10, there exist paths xE_1x_1 and $y_{-1}E_{-1}y$ in R such that $E_1, E_{-1} \notin \mathcal{F}$ and $d_R(x_1, \mathcal{F}), d_R(\mathcal{F}, y_{-1}) \geq 3$ (observe that $h \geq D+1 \geq 2$). Besides, from the definition of the parameter $M_{0,1}(R)$, we have that $d_R(x_1, F) + d_R(F, y_{-1}) \geq M - 4$ for any $F \in \mathcal{F}$. Applying again Lemma 5.10, for any $m, n \geq 1$ we can find paths $xE_1x_1 \ldots E_mx_m$ and $y_{-n}E_{-n} \ldots y_{-1}E_{-1}y$ such that, for any $F \in \mathcal{F}$,

$$d_R(x_m, F) \geq \min\{d_R(x_1, F) + 2(m-1), h\}$$

and

$$d_R(F, y_{-n}) \geq \min\{d_R(F, y_{-1}) + 2(n-1), h\}$$

Then, if $m, n \geq D - \lfloor h/2 \rfloor$ and $m + n = C$, it is not difficult to see that $d_R(x_m, F) + d_R(F, y_{-n}) > 2D$ for any $F \in \mathcal{F}$. Therefore, any shortest path in R from x_m to y_{-n}, which has length at most $2D$, will avoid $\mathcal{F}$. Hence, we have found a path from x to y in H with length at most $D+m+n = D+C$ avoiding the faulty hyperarcs in $\mathcal{F}$.□

As a consequence of Theorem 5.11, we obtain the following result about the hyperarc-fault-diameter of iterated line hyperdigraphs.

Corollary 5.12 *Let H be a simple hyperdigraph with minimum degree d and diameter D and let $R = R(H)$ be its bipartite representation. Let us consider $h = h_0(R)$ and $M = M_{0,1}(R)$. Then, for any $k \geq D - h + 1$ and for any $w = 1, \ldots, d-1$, the w-hyperarc-fault-diameter of the iterated line hyperdigraph $L^k H$ verifies $D'_{w+1}(L^k H) \leq D(L^k H) + C$, where*

$$C = \max\left\{D - \left\lfloor \frac{M-1}{2} \right\rfloor + 4, 2\left(D - \left\lfloor \frac{h}{2} \right\rfloor\right)\right\}$$

Proof: Apply Theorem 5.11 by taking into account that $R(L^k H) = L^{2k} R(H)$ and that $h_0(L^{2k} R) = h_0(R) + 2k$ (Proposition 5.6) and $M_{0,1}(L^{2k} R) = M_{0,1}(R) + 2k$ (Proposition 4.4).□

5.6 Applications

We apply next the results in the above sections in order to study the fault-tolerance of the generalized de Bruijn and Kautz hyperdigraphs.

The vertex-connectivity of these hyperdigraphs can be derived from the results about the vertex-connectivity of the corresponding digraphs that are given in [42, 23]. It is proved in those papers that, if G is the generalized de Bruijn digraph $GB(d, n)$ or the generalized Kautz digraph $GK(d, n)$ and $D(G) \geq 3$, then $\kappa(G) \geq d - 1$. If $G = GB(d, n)$, or $G = GK(d, n)$ and $d + 1$ does not divide n, then G has loops. In this case, $\kappa(G) = d - 1$ if $D(G) \geq 3$. Besides, if $G = GK(d, n)$ has diameter $D \geq 5$, then

$$\kappa(G) = \begin{cases} d & \text{if } n \text{ is a multiple of } d+1 \text{ and } \gcd(d, n) \neq 1; \\ d - 1 & \text{otherwise.} \end{cases}$$

We can obtain from these results the vertex-connectivity of the generalized de Bruijn and Kautz hyperdigraphs.

Theorem 5.13 *Let us consider positive integers d, n, s, m, with $dn \equiv_m 0$ and $sm \equiv_n 0$. Let H be the generalized de Bruijn hyperdigraph $H = GB(d, n, s, m)$. Then, $\kappa(H) = ds - 1$ if $D(H) \geq 3$.*

Proof: We only have to take into account that $\widehat{H} = GB(ds, n)$ [8].□

Theorem 5.14 *Let us consider positive integers d, n, s, m, with $dn \equiv_m 0$ and $sm \equiv_n 0$. Let H be the generalized Kautz hyperdigraph $H = GK(d,n,s,m)$. Then, $\kappa(H) \geq ds-1$ if $D(H) \geq 3$. Besides, if $D(H) \geq 5$,*

$$\kappa(H) = \begin{cases} ds & \text{if } n \text{ is a multiple of } ds+1 \text{ and } \gcd(ds,n) \neq 1; \\ ds-1 & \text{otherwise.} \end{cases}$$

Proof: As before, $\widehat{H} = GK(ds,n)$ [8].□

If $dn = sm$, the hyperdigraphs $H_1 = GB(d,n,s,m)$ and $H_2 = GK(d,n,s,m)$ are s-uniform. In this case, we can find the hyperarc-connectivity of those hyperdigraphs because $\kappa(H_i) \leq \lambda(H_i)s$. Therefore, if $D(H_i) \geq 3$, we have that $\lambda(H_i)s \geq ds-1$ and, hence, $\lambda(H_i) = d$ if $s \geq 2$.

The vertex-fault-diameter of the de Bruijn hyperdigraphs,

$$HB(d,s,D) = GB(d,(ds)^D,s,d^2(ds)^{D-1})$$

and Kautz hyperdigraphs,

$$HK(d,s,D) = GK(d,(ds)^D+(ds)^{D-1},s,d^2((ds)^{D-1}+(ds)^{D-2}))$$

can be computed by taking into account that $\widehat{HB}(d,s,D) = B(ds,D)$ and $\widehat{HK}(d,s,D) = K(ds,D)$. Therefore, $D_w(HB(d,s,D)) = D+2$ for any $w = 2,\ldots,ds-1$ and $D_w(HK(d,s,D)) = D+2$ for any $w = 2,\ldots,ds$.

6 Partial line hyperdigraphs

6.1 Introduction

The partial line hyperdigraph technique is a generalization of the line hyperdigraph technique [6], the partial line digraph [33], and consequently, the line digraph [36].

First, we show the usefulness of such technique for the (d,N,s)-hyperdigraph problem.

We are going to show that the partial line hyperdigraph tends to increase the connectivity. Clearly, the minimum degree of the partial line hyperdigraph, is a natural lower bound.

Also we extend the definition of the index of expandability to hyperdigraphs. This allow to measure the capability of a bus network to increase its number of processors.

We present a characterization of line hyperdigraphs in terms of line digraphs, proving a conjecture [6] for the characterization of line hyperdigraphs.

Finally, we study the application of the partial line hyperdigraph technique to the generalized Kautz hyperdigraphs. Also some results concerning with digraphs are obtained.

6.2 The technique

Given a hyperdigraph $H = (\mathcal{V}(H), \mathcal{E}(H))$ with minimum in-degree at least 1 for any set $\mathcal{V}'$ of vertices of LH such that $\{v : \exists (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$, the partial line hyperdigraph of H will be the hyperdigraph $\mathcal{L}H = (\mathcal{V}(\mathcal{L}H), \mathcal{E}(\mathcal{L}H))$,

$$\mathcal{V}(\mathcal{L}H) = \mathcal{V}'$$
$$\mathcal{E}(\mathcal{L}H) = \{(EvF) : v \in F^-, \exists u \in \mathcal{V}(H) : (uEv) \in \mathcal{V}'\}$$

where

$$(EvF)^+ = \{(vFw) : (vFw) \in \mathcal{V}'\} \cup \{(v'F'w) : w \in F^+, (vFw) \notin \mathcal{V}'\}$$
$$(EvF)^- = \{(uEv) : u \in E^-, (uEv) \in \mathcal{V}'\}$$

That is, $(EvF)^+$ contain all the vertices in the form (vFw) of $\mathcal{V}'$, and one arbitrary vertex, $(v'F'w)$, if (vFw) is not in $\mathcal{V}'$.

That is, the partial line hyperdigraphs depends on the election of the set of vertices $\mathcal{V}'$ and also in the way that the out-sets of the hyperarcs are constructed.

Note that always exists a set $\mathcal{V}'$ with $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$, because the minimum degree of H is at least 1.

Particularly, observe that in the case $|\mathcal{V}'| \geq ds$, we can choose the vertices in $|\mathcal{V}'|$ in such a way that $\mathcal{E}(\mathcal{L}H) = \mathcal{E}(LH)$. In this situation, the out-set of any hiperarc $EvF \in \mathcal{E}(\mathcal{L}H) = \mathcal{E}(LH)$ can be taken as $(EvF)^+ = \{(vFw) : (vFw) \in \mathcal{V}'\} \cup \{(v'Fw) : w \in F^+, (vFw) \notin \mathcal{V}'\}$. That is, it is not necessary to consider hyperarcs $F' \neq F$ in order to determine the vertices in $(EvF)^+$.

Notice that if H is a digraph, $\mathcal{L}H$ coincides with a partial line digraph. Also, if $\mathcal{V}' = \mathcal{V}(LH)$ then $\mathcal{L}H$ is LH. So, the partial line hyperdigraph technique is a generalization of the line hyperdigraph technique [6], the partial line digraph [33], and consequently, the line digraph [36].

Next, we show some useful relations of this technique to digraphs.

Proposition 6.1 *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ with minimum $d > 1$, and $\mathcal{V}'$ a set of vertices of LH such that $\{v : \exists (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. For any vertex (uEv) and any hyperarc (EvF) of the partial line hyperdigraph of H, $\mathcal{L}H$,*

$$d^+_{\mathcal{L}H}(uEv) = d^+_H(v);$$

$$s^+_{\mathcal{L}H}(EvF) = s^+_H(F). \ \square$$

Proposition 6.2 *Let H be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of LH, and a set of arcs of $\widehat{H}$, such that with these sets $\widehat{\mathcal{L}H}$ and $\mathcal{L}\widehat{H}$ are isomorphic.*

Proof: Given a set $\mathcal{V}'$ of vertices of LH such that $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. Let E' be the set of arcs of $\widehat{H}$ defined by $E' = \{(u, v) : \exists E \in \mathcal{E}(H), (uEv) \in \mathcal{V}'\}$. With these sets there is a trivial isomorphism between $\widehat{\mathcal{L}H}$ and $\mathcal{L}\widehat{H}$. $\square$

6.3 The (d, s, N)-hyperdigraph problem

In [22] was presented a Moore like bound for the order of a hyperdigraph with diameter D, maximum out-degree d and maximum out-size s:

$$N \leq 1 + (d^+s^+) + (d^+s^+)^2 + \ldots + (d^+s^+)^D = \frac{(d^+s^+)^{D+1} - 1}{d^+s^+ - 1}$$

From this arises the following lower bound for the diameter:

$$(\log_{d^+s^+}(N(d^+s^+ - 1) + 1)) - 1 \leq D.$$

We will show the good behaviour of the proposed technique for such problem.

The order of the hyperdigraph $\mathcal{L}H$ is the cardinal of $\mathcal{V}'$, and it is chosen with the condition $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. Then,

$$|\mathcal{V}(H)| \leq |\mathcal{V}(\mathcal{L}H)| \leq |\mathcal{V}(LH)|.$$

The partial line hyperdigraph preserves the maximum out-degree of H. In fact, $d^+_{\mathcal{L}H}(uEv) = d^+_H(v)$ for any vertex (uEv) of $\mathcal{L}H$. Also, the out-size of H remains constant since for every hyperarc (EvF) of $\mathcal{L}H$, $s^+_{\mathcal{L}H}(EvF) = s^+_H(F)$ (the in-size of H is preserved too). Then, if H is d-regular, $\mathcal{L}H$ is also d-regular. So, if the out-size of all hyperarcs of H is s,

$$|\mathcal{V}(H)| \leq |\mathcal{V}(\mathcal{L}H)| \leq |\mathcal{V}(H)|ds$$

Since $D(H) = D(\widehat{H})$ for every hyperdigraph H, by Proposition 6.2 we have $D(\mathcal{L}H) = D(\widehat{\mathcal{L}H})$. Now, $\widehat{H}$ is a digraph and by [33]: $D(\widehat{H}) \leq D(\mathcal{L}\widehat{H}) \leq D(\widehat{H}) + 1$. So, $D(H) \leq D(\mathcal{L}H) \leq D(H) + 1 = D(LH)$.

From all the above considerations about the order, maximum out-degree and maximum out-size, we can state the following result:

Theorem 6.3 *Let H be a hyperdigraph with maximum out-degree $d^+ > 1$, maximum out-size s^+, order N and diameter D. Then the order $N_{\mathcal{L}}$, the maximum out-degree $d^+_{\mathcal{L}}$, the maximum out-size $s^+_{\mathcal{L}}$ and the diameter $D_{\mathcal{L}}$ of any partial line hyperdigraph $\mathcal{L}H$ satisfy:*

$$N \leq N_{\mathcal{L}} \leq Nds; \qquad d^+_{\mathcal{L}} = d^+;$$
$$D \leq D_{\mathcal{L}} \leq D + 1. \qquad s^+_{\mathcal{L}} = s^+.\square$$

6.4 Connectivity

To show that the partial line hyperdigraph tends to increase the connectivity (with the minimum degree of the partial line hyperdigraph as a lower bound), first, we extend a useful concept introduced in [33] for digraphs. A hyperdigraph H has *no redundant short paths* when there is at most one path of length one or two between every pair of vertices (different or not) of H. Notice that under this restriction we can still work with interesting hyperdigraphs. For instance, the generalized De Bruijn hyperdigraphs and the generalized Kautz hyperdigraphs [8] have no redundant short paths.

Lemma 6.4 *Let H be a hyperdigraph. Then, H has no redundant short paths if and only if $\widehat{H}$ has no redundant short paths.* □

Theorem 6.5 *Let H be a hyperdigraph with minimum in-degree $d > 1$ and minimum in-size s. If H has no redundant short paths:*

$$\min\{\kappa(H), d(\mathcal{L}H)s\} \leq \kappa(\mathcal{L}H)$$

Proof: By Lemma 6.4, $\widehat{H}$ has no redundant short paths, so by the bound on the connectivity of partial line digraphs [33], $\min\{\kappa(\widehat{H}), d(\mathcal{L}\widehat{H})\} \leq \kappa(\mathcal{L}\widehat{H})$. Since $\kappa(H) = \kappa(\widehat{H})$, then $\min\{\kappa(H), d(\mathcal{L}H)s\} \leq \kappa(\mathcal{L}H)$. □

For the hyperarc-connectivity the analogous bound holds, but to prove it, we need the following result of [6]:

Lemma 6.6 *Let H be a hyperdigraph with hyperarc-connectivity λ. Then, every vertex v in H is on λ hyperarc-disjoint cycles.*□

Theorem 6.7 *Let H be a hyperdigraph with minimum in-degree $d > 1$. Let $\mathcal{V}'$ be a set of vertices of LH, $|\mathcal{V}'| \geq ds$, and $\mathcal{L}H$ a partial line hyperdigraph with $\mathcal{E}(\mathcal{L}H) = \mathcal{E}(LH)$. Then, $\lambda(\mathcal{L}H) \geq \lambda(H)$.*

Proof: It is enough to prove that a set of $\lambda = \lambda(H)$ hyperarc-disjoint paths in H induces a set of λ hyperarc-disjoint paths in $\mathcal{L}H$. Let (uEv) and (xFy) be two different vertices of $\mathcal{L}H$. In order to construct λ hyperarc-disjoint paths from (uEv) to (xFy) in $\mathcal{L}H$ from λ hyperarc-disjoint paths from v to x in H, we consider two cases:

1. If $v \neq x$, we have λ hyperarc-disjoint paths from v to x in H:

   $$P_i = v, E_1^i, v_1^i, E_2^i, v_2^i, \ldots, E_{n_i-1}^i, v_{n_i-1}^i, E_{n_i}^i, x$$

   where $i = 1, \ldots, \leq \lambda$. Each path P_i gives rise to a path from (uEv) to (xFy), $\mathcal{L}P_i$ in $\mathcal{L}H$ defined by:

   $$\mathcal{L}P_i = (uEv), (EvE'^i_1), (v'E_1^i v_1^i), (Ei_1 v_1^i E_2^i), (v'^i_1 E_2^i v_2^i), \ldots \\ \ldots (v'^i_{n_i} E_{n_i}^i x), (E_{n_i}^i x' F), (xFy)$$

   It is not difficult to see that the paths $\mathcal{L}P_i$ are equally hyperarc disjoint.

2. If $v = x$, we proceed as in the above case but with hyperarc disjoint cycles in H. By Lemma 6.6, if the hyperarc-connectivity is λ, each vertex of H is in λ hyperarc-disjoint cycles. In the same way as we do with paths P_i, we can obtain λ paths in $\mathcal{L}H$ from these cycles in H. Again, since the original cycles are hyperarc-disjoint, these new paths are hyperarc-disjoint also. □

6.5 Expandability

Given two hyperdigraphs H and H', on N and N' vertices, respectively, $N \leq N'$, we define the *index of expandability* of H to H', $e(H, H')$, as the minimum number of hyperarcs that has to be modified or removed from H to obtain H' by adding $N' - N$ vertices and some appropriate hyperarcs, if it is necessary.

That is, the index of expandability measures the necessary modifications of hyperarcs of H, to obtain a sub-hyperdigraph H'.
Notice that this definition generalizes the corresponding one for digraphs.

If H is a hyperdigraph, $\mathcal{L}_n H$ will denote a partial line hyperdigraph of H with order n. Next we show that any hyperdigraph $\mathcal{L}_n H$ has good expandability to some $\mathcal{L}_{n+1} H$.

Theorem 6.8 *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with maximum in-degree $d > 1$. For any partial line hyperdigraph $\mathcal{L}_n H$ on n vertices, $|\mathcal{V}(H)| \leq n \leq |\mathcal{V}(LH)| - 1$, there exists a digraph $\mathcal{L}_{n+1} H$, such that the index of expandability of $\mathcal{L}_n H$ to $\mathcal{L}_{n+1} H$ satisfies:*

$$e(\mathcal{L}_n H, \mathcal{L}_{n+1} H) \leq d$$

Proof: Let $\mathcal{V}'$ be the set of vertices of $\mathcal{L}_n H$. The hyperdigraph $\mathcal{L}_{n+1} H$ can be obtained from $\mathcal{L}_n H$ by the following algorithm:

a) Choose a vertex (uEv) of $\mathcal{L}H$ which is not $\mathcal{V}'$. Since $|\mathcal{V}'| \leq |\mathcal{V}(LH)| - 1'$, always exist at least one.

b) Add the vertex (uEv) to $\mathcal{L}H$.

c) For every hyperarc of $\mathcal{L}H$ denoted by (FuE), replace in their out-sets, the vertex $(u'E'v)$ by the vertex (uEv).

d) For every hyperarc F of H, if (EvF) is not a hyperarc of $\mathcal{L}H$, add it, with $(EvF)^+ = \{(vFw) : (vFw) \in \mathcal{V}'\} \cup \{(v'F'w) : w \in F^+, (vFw) \notin \mathcal{V}'\}$.
$(EvF)^- = \{(uEv) : u \in E^-, (uEv) \in \mathcal{V}'\}$.

   For each F such that (EvF) is a hyperarc of $\mathcal{L}H$, put the vertex (uEv) in the corresponding in-set.

We only add new hyperarcs or replace the existing ones in steps c) and d), so the index of expandability is given by the number of changes there. Since the maximum degree is d, this number is at most d. □

The above proof gives an algorithm to expand partial line hyperdigraphs. With a few changes it can be used to decrease the number of vertices.

Also in some applications, it could also be useful to measure the number of vertex-to-vertex connections that have to be modified to add components. From the above algorithm:

Corollary 6.9 *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with maximum in-degree $d > 1$ and maximum out-size s. For any partial line hyperdigraph $\mathcal{L}_n H$ on n vertices, $|\mathcal{V}(H)| \leq n \leq |\mathcal{V}(LH)| - 1$, there exists a hyperdigraph $\mathcal{L}_{n+1}H$, such that the connections that have to be modified to transform $\mathcal{L}_n H$ to $\mathcal{L}_{n+1}H$ are at most ds.* □

6.6 Applications

As we have said before, the main goal of the partial line hyperdigraph technique is to construct hyperdigraphs with minimum diameter. An interesting family of such hyperdigraphs is obtained when this technique is applied to the de Bruijn hyperdigraphs,

$$HB(d, s, D) = GB(d, (ds)^D, s, d^2(ds)^{D-1})$$

and Kautz hyperdigraphs,

$$HK(d, s, D) = GK(d, (ds)^D + (ds)^{D-1}, s, d^2((ds)^{D-1} + (ds)^{D-2}))$$

By doing that, we obtain a new family of hyperdigraphs with minimum diameter that have other interesting properties in relation to the fault-tolerance and the routing algorithms. For instance, as a direct consequence of next theorem, we have that some of these hyperdigraphs are iterated line hyperdigraphs.

Theorem 6.10 *Let H be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of LH, and a set of vertices of L^2H, such that with these sets $L\mathcal{L}H$ and $\mathcal{L}LH$ are isomorphic.*

Proof: The vertices of $L\mathcal{L}H$ are in correspondence with the hyperarcs of $\mathcal{L}H$, so there are two kinds of vertices:

1. $(uEv)(EvF)(v'F'w)$, with $v \in F^-$, $w \in F^+$ and $(vFw) \notin V(\mathcal{L}H)$
2. $(uEv)(EvF)(vFw)$, with $v \in F^-$, $w \in F^+$ and $(vFw) \in V(\mathcal{L}H)$

Clearly, for any choice of vertices of LH, there are different digraphs $\mathcal{L}H$ and $L\mathcal{L}H$. For a given digraph $L\mathcal{L}H$, we construct a set of vertices of LH by the rules:

1. If $(uEv)(EvF)(v'F'w) \in V(L\mathcal{L}H)$, we take $(uEv)(EvF)(vFw)$
2. If $(uEv)(EvF)(vFw) \in V(L\mathcal{L}H)$, we take $(uEv)(EvF)(vFw)$

Now, applying the partial line technique to LH with this set of vertices, $L\mathcal{L}H$ and $\mathcal{L}LH$ are isomorphic. □

Corollary 6.11 *Let H be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of LH, and a set of vertices of $L^{k+1}H$, such that with these sets, for any integer $k \geq 1$, $L^k\mathcal{L}H$ and $\mathcal{L}L^kH$ are isomorphic.*

Proof: By the above theorem, there exists a set of vertices of LH, and a set of vertices of H, such that $L\mathcal{L}H$ and $\mathcal{L}LH$. (The result for $k = 1$.) We are going to prove by induction on k that the same holds for any $k \geq 1$. Let us assume that $L^i\mathcal{L}H$ and $\mathcal{L}L^iH$ are isomorphic for any integer i, $1 \leq i \leq k-1$, and we are going to prove $L^k\mathcal{L}H$ and $\mathcal{L}L^kH$ are isomorphic:

$$\mathcal{L}L^kH = \mathcal{L}L^{k-1}LH \sim L^{k-1}\mathcal{L}LH \sim L^l\mathcal{L}H. \ \square$$

6.7 On a conjecture of Bermond and Ergincan

In [6], Bermond and Ergincan conjecture that an equivalent condition for a directed hypergraph to be a directed line hypergraph, is that its underlying digraph and the underlying digraph of its dual, must be both line digraphs. We are going to prove it.

Proposition 6.12 *Let H be a hyperdigraph. If H is a line hyperdigraph then, its underlying digraph $\widehat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are both, line digraphs.*

Proof: Since $(LH)^*$ is isomorphic to LH^* and $\widehat{L}H$ is isomomorphic to $L\widehat{H}$ [6], if H is a partial line hyperdigraph, then $\widehat{H}$ and $\widehat{H}^*$ are partial line digraphs too, so we conclude that if H is a line hyperdigraph, both digraphs are also line digraphs. $\square$

Proposition 6.13 *Let H be a hyperdigraph. If its underlying digraph, $\widehat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are both, partial line digraphs, then H is a partial line hyperdigraph.*

Proof: If $\widehat{H}$ is a line digraph, its vertices, which are the set of vertices of H, can be labeled with ordered pairs of vertices of other digraph, let us say H_1, where $\widehat{H} = LH_1$. Moreover, we can assure that any two vertices u_iv_i, u_jv_j are adjacent in $\widehat{H}$ if and only if $v_i = u_j$.

Analogously, if $\widehat{H^*}$ is a line digraph, the set of vertices of $\widehat{H^*}$, corresponding with the set of hyperarcs of H, can be labeled with ordered pairs of vertices of other digraph, let us say H_2, such that $\widehat{H^*} = LH_2$. Besides, two vertices of $\widehat{H^*}$, E_iF_i and E_jF_j, are adjacent if and only if $F_i = E_j$. Then, by the definition of $\widehat{H^*}$, there exists a vertex v of H belonging to the out-set of the hyperarc labeled E_iF_i, $(E_iF_i)^+$, and to the in-set of the hyperarc with label E_jF_j, $(E_jF_j)^-$, if and only if $F_i = E_j$.

Now, we modify the labeling for the vertices of H introducing the labeling for the hyperarcs. That is, if a vertex labeled with u_iv_i is in $(E_jF_j)^-$, we re-label it with $u_iE_jv_i$, and if the vertex with label u_iv_i is in $(E_jF_j)^+$, we re-label it with $u_iF_jv_i$. This is consistent because if a vertex belongs to $(E_pF_p)^+$ and to $(E_qF_q)^-$ it must be $E_q = F_p$, because $\widehat{H^*}$ is a line digraph.

Then, we have defined a labeling in H with the line hyperdigraph conditions, so H it is a line hyperdigraph. $\square$

As a direct consequence of Propositions 6.12 and 6.13 we have proved:

Theorem 6.14 *Let H be a hyperdigraph. Then, H is a line hyperdigraph if and only if its underlying digraph, $\widehat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are line digraphs.* □

Corollary 6.15 *Let H be a hyperdigraph and k a positive integer. Then, H is a k-iterated line hyperdigraph if and only if its underlying digraph, $\widehat{H}$, and the underlying digraph of its dual, $\widehat{H^*}$, are both, k-iterated line digraphs.* □

7 De Bruijn sequences of maximum period length

7.1 Introduction

A feedback shift register of length k over $\mathbf{Z}_n$ is a k-tuple of elements in $\mathbf{Z}_n$ together with a feedback function $f : \mathbf{Z}_n^k \rightarrow \mathbf{Z}_n$. The tuple represent the state of the register, and the function f, the element introduced in the corresponding shift from a given state [39]. The case of linear feedback function was carefully studied by algebraic methods. On the contrary, for the non-linear case, only a few properties are known.

To a given feedback shift register over $\mathbf{Z}_n$, it is possible to associate the sequence of the feedback function for consecutive states of the register. Moreover, this is a one-to-one correspondence. These are called De Bruijn sequences.

Because of their randomness property, are specially interesting the De Bruijn sequences with maximum period length. It was shown that they cannot be obtained by linear feedback functions [58]. So, the feature is to find a way to obtain, or at least characterize, all the non-linear feedback functions generating De Bruijn sequences of maximum period length.

The De Bruijn sequences of maximum period were first introduced over $\mathbf{Z}_2$ [17]. There it was proved that if ℓ is the length of the register, the number of all the De Bruijn sequences of maximum period length that can be generated is $2^{2^{\ell}-1}$. In [32] we deal with this problem from a theoretical graph point of view.

Different matrices can be associated to a digraph. For a given digraph G, the *adjacency matrix*, M, has one row and one column by each vertex. If there is an arc from the vertex i to the vertex j in G, then (i,j) entry of the matrix is 1, and otherwise is 0. If k is a positive integer, a value 1 in the (i,j) entry of M^k means the existence of a path from i to j of length k. In the *incidence matrix*, I, rows represent the vertices and columns the arcs of G. Then, its (i,e) entry is 1, if e is incident to i, -1 if e is incident from i, and 0 otherwise. If I has n columns, m rows, and G has c connected components, then $rank(I) = n - c$ [11].

7.2 Analysis based on graphs

In the following two sections we are going to present some results based in the analysis of the adjacency and incidence matrix of a digraph. For a De Bruijn sequence over $\mathbf{Z}_n$, with register length ℓ and feedback function $f : \mathbf{Z}_n^\ell \to \mathbf{Z}_n$, first we consider another function $F : \mathbf{Z}_n^\ell \to \mathbf{Z}_n^\ell$ defined by:

$$F(x_0, x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, f(x_0, x_1, \ldots, x_{n-1})).$$

Now, we define a digraph with vertex set $\mathbf{Z}_n^\ell$, and a vertex x adjacent to another vertex y, if and only if, $y = f(x)$. We denote such digraph by G_F.

With this construction, note that G_F is a subdigraph of the De Bruijn digraph $B(n, \ell)$. A De Bruijn sequence has maximum period length, if and only if, the corresponding G_F for its feedback function defines a hamiltonian cycle in the corresponding De Bruijn digraph. Also note that, since the De Bruijn digraphs are iterated line digraphs, is the same to find hamiltonian or eulerian cycles. (Every eulerian cycle induces a unique hamiltonian cycle in its line digraph).

Reciprocally, every hamiltonian cycle is in one-to-one correspondence with some digraph G_f, which is a subdigraph of $B(n, \ell)$.

So, we are looking for all the hamiltonian cycles in a general De Bruijn digraph, let us say $B(n, \ell)$.

7.2.1 Application of the adjacency matrix of a digraph

Proposition 7.1 *Let G' be a subdigraph of a digraph G. If $M_{G'}$ and M_G are respectively their adjacency matrices. Then, if the (i, j) entry of M_G is 0, the (i, j) entry of $M_{G'}$ is also 0.* □

Proposition 7.2 *Let G be a digraph on n vertices and M its adjacency matrix. For any positive integer i, $1 \leq i \leq n$, there is a cycle of length ℓ containing the vertex i, if and only if, the (i, i) entry of the matrix M is 1.* □

As a consequence of the above two propositions we obtain the following one:

Proposition 7.3 *Let M be the adjacency matrix of a hamiltonian cycle in the De Bruijn digraph $B(n, \ell)$ with adjacency matrix M_B:*

a) is a permutation matrix;

b) the (i, j) entry of M can be 1 only if the (i, j) entry of M_B is 1;

c) for $i = 1, \ldots, \ell^n - 1$, the diagonal of M^i has only entries with value 0;

d) in M^{ℓ^n} the diagonal has only entries with value 1. □

To calculate all the De Bruijn sequences of maximum period length it is possible with the above conditions, but using large scale symbolic calculations. Perhaps are more interesting some properties that can be obtained from them.

For the binary case, for example, condition $a)$ implies that:

Given a function $F : \mathbf{Z}_n^2 \to \mathbf{Z}_n^2$, we denote by F' another function $F' : \mathbf{Z}_n^2 \to \mathbf{Z}_n^2$ defined by:

$$F(x_0, x_1, \ldots, x_n) = x_1, \ldots, x_n 0, \text{ then } F'(x_0, x_1, \ldots, x_n) = x_1, \ldots, x_n 1;$$

$$F(x_0, x_1, \ldots, x_n) = x_1, \ldots, x_n 1, \text{ then } F'(x_0, x_1, \ldots, x_n) = x_1, \ldots, x_n 0.$$

Proposition 7.4 *If $F : \mathbf{Z}_n^2 \to \mathbf{Z}_n^2$ defines a De Bruijn sequence of maximum period length, then $F' : \mathbf{Z}_n^2 \to \mathbf{Z}_n^2$ too.* □

This means that it is sufficient to calculate a half of the desired function.

7.2.2 Application of the incidence matrix of a digraph

We present some conditions on the function f to make the corresponding function F, to determine a digraph G_F to be a hamiltonian cycle in the corresponding De Bruijn digraph.

Proposition 7.5 *Let $F : \mathbf{Z}_n^\ell \to \mathbf{Z}_n^\ell$ be a function and G_F its associated digraph. Then, G_F is a hamiltonian cycle in $B(n, \ell)$ if and only if, the following three conditions hold:*

a) F is bijective;

b) If $x = x_0, x_1, \ldots, x_\ell$ and $y = F(x)$, then $y = x_1, \ldots, x_\ell, x_{\ell+1}$;

c) G_F is connected.□

Clearly, the conditions $a)$ and $b)$ of the above proposition are easy to check. To verify $c)$ we propose the following result:

Proposition 7.6 *Let $F : \mathbf{Z}_n^\ell \to \mathbf{Z}_n^\ell$ be a function and G_F its associated digraph, with incidence matrix I_F. Then, G_F is connected if and only if $rank(I_F) = n^\ell - 1$.* □

With this proposition, together with the above one, we have a test to decide whether or not, a De Bruijn sequence has maximum period length.

Example: Suppose $f : \mathbf{Z}_2^3 \to \mathbf{Z}_2$ defined by $f(x_0, x_1, x_2) = 1 + x_0 + x_2 + x_1 x_2$. Now, let us verify if the function $F : \mathbf{Z}_2^3 \to \mathbf{Z}_2^3$ defined by $F(x_0, x_1, x_2) = (x_1, x_2, f(x_0, x_1, x_2))$ verify the conditions of Proposition 7.5.

a): It is enough to show that F is injective, since it is clear that f is onto $\mathbf{Z}_2$. So, let us suppose that

$$F(x_0, x_1, x_2) = F(y_0, y_1, y_2).$$

Then, it must be:

$$(x_1, x_2, f(x_0, x_1, x_2)) = (y_1, y_2, f(y_0, y_1, y_2)).$$

Clearly it follows that $x_1 = y_1$, $x_2 = y_2$ and $f(x_0, x_1, x_2) = f(y_0, y_1, y_2)$, and from these three equations also $x_0 = y_0$, and this condition is verified.

b): It is obvious.

c): We have to calculate the rank of one incidence matrix for the corresponding digraph G_F, let us say, I_F.

Let us assume that the columns, whose correspond to the vertices are enumerated in the order:

$$000, 001, 010, 011, 100, 101, 110, 111$$

and the rows, in correspondence with the arcs are in the order:

$$0001, 0010, 0101, 1011, 0111, 1110, 1100, 1000.$$

Then,

$$I_F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The rank of this matrix is 7, so also this condition holds.

As a conclusion, f defines a hamiltonian cycle in $B(2,3)$.

Conclusions and open problems

In this work have been studied many problems related to the fault-tolerance of interconnection networks based on digraph models. Also the fault-tolerance, together with some basic problems in interconnection networks design have been treated for hyperdigraphs models.

The wide and fault-diameters of the best known generalized cycles for interconnection networks design have been given by finding containers in them. These families contain other proposed good models. The results we have obtained coincide with the known ones for them. They also shown the good fault-tolerance capability of these families.

More generally, we have introduced some terminology in terms of which we have given bounds for fault-diameters in iterated line digraphs. Our bounds improve the better known ones in many directions.

We have presented some results about hyperdigraphs. As a starting point we have studied the connectivity. In order to do it, we have introduced some terminology to determine bounds when a given number of elements are removed. We have presented a characterization for maximally connected hyperdigraphs. The fault-tolerance in hyperdigraphs has been studied. First, we have stated bounds for the fault-diameters of maximally connected hyperdigraphs, extending some known results on digraphs.

In relation to the (d, N, s)-hyperdigraph problem we have extended the partial line digraph, the line hyperdigraph, and of course, the line digraph techniques, by defining the partial line hyperdigraph. This technique is shown to be good for the mentioned problem. Moreover, the hyperdigraphs obtained have good connectivity, expandability and easy routings. The partial line digraph has been shown to give specially nice results when it is applied to the generalized Kautz hyperdigraphs. Also for line hyperdigraphs we have presented a characterization in terms of digraphs.

Finally, we include some properties we have obtained related to De Bruijn sequences of maximum period length. A test to decide if a function generates a sequence of maximum period length has been given. The digraph formulation of the problem, is different from others introduced before to study the problem.

The study of wide and fault-diameters in any De Bruijn and Kautz generalized cycle is an interesting open problem, to continue the working in the direction of Section 2. Another line of future work could be the application of the techniques presented in this section to some families of hyperdigraphs.

The bounds for the fault-diameters presented in Section 3 can also be applied to other interesting families of digraphs. The problem consists in the calculus of the involved parameters for other families of digraphs. The terminology introduced in this section can be useful also to study other problems. From the fault-tolerance point-of view, an open problem is to study the fault-diameters of generalized de Bruijn and Kautz digraphs and hyperdigraphs. Equally, it would be interesting to study this parameter for partial line hyperdigraphs, or at least for partial line digraphs.

It could be interesting to study the possibility of applying the partial line hyperdigraphs to the design of architectures for representing some optical networks. Particularly, the partial line of Kautz hyperdigraphs could be interesting.

The digraph models that have been introduced for dealing with the De Bruijn sequences, is a new point of view to continue studying the problem. Another interesting analysis could arise from comparing the complexity of this algorithm, with the complexity of the problem itself.

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