## Power domination in honeycomb networks

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**Abstract**

Electric power networks must be continuously monitored. Such monitoring can be efficiently accomplished by placing phase measurement units (PMUs) at selected network locations. Due to the high cost of the PMUs, their number must be minimized. The power domination problem consists of finding the minimum number of PMUs needed to monitor a given electric power system. The power dominating problem is NP-hard, but closed formulas for the power domination number of certain networks, such as rectangular meshes [4] have been found. In this work we extend the results for rectangular meshes to honeycomb meshes.

***Keywords:*** *honeycomb mesh, power domination.*
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## 1. Introduction

For electric power companies the continuous monitoring of their systems represents a crucial task. One way to accomplish it, consists of placing phase measurement units (PMU) at selected locations in the system. Because of the high cost of a PMU, it is desirable to minimize the

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number of PMUs used, while maintaining the ability to monitor the entire system. The power system monitoring problem, as introduced in [1], asks for the minimum number of PMUs, and their locations, needed to monitor an electric power system. This problem has been formulated as a graph domination problem by Haynes *et al.,* in [6]. However, this type of domination has a different flavor than the standard domination type problem, since the application of the domination rules can be iterated. Next we give a formal description of the power domination problem in graph theory.

Let $G = (V,E)$ be a graph representing an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. A PMU monitors, or dominates, the vertex at which it is placed and its incident edges and their end vertices. The other domination rules are as follows:

(1) Any vertex that is incident to a dominated edge is dominated.

(2) Any edge joining two dominated vertices is dominated.

(3) If a vertex is incident to $k > 1$ edges and if $k - 1$ of these edges are dominated, then all $k$ of these edges are dominated.

Note that we gave the rules as presented in [6]. In [3] the authors present the propagation rules in a different way, that ultimately, as observed in [4], is equivalent to the given formulation.

A set $S \subseteq V$ is defined to be a power dominating set of $G$ if every vertex and every edge in $G$ is dominated by $S$ according to the previous domination rules. The power domination number of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of a power dominating set of G. Notice that in the standard theory of domination, a set $S \subseteq V$ is a dominating set in $G$ if every vertex in V \S has at least one neighbor in S. The minimum cardinality of a dominating set of $G$ is its domination number, denoted by $\gamma(G)$. Since any dominating set is also a power dominating set, we have $1 \leq \gamma_p(G) \leq \gamma(G)$ for every graph $G$.

Given an arbitrary graph $G$ and an integer $k$, the problem of deciding if $G$ has a power dominating set of cardinality $k$ has been shown to be NP-complete even when restricted to bipartite graphs or chordal graphs [6], or even split graphs [8], a subclass of chordal graphs. However, Liao and Lee gave a linear algorithm for this problem in the case of interval graphs, provided that the interval ordering of the graph is known [8] . If the interval order is not given, they gave an algorithm of O(n log n) that they proved to be asymptotically optimal. Other efficient algorithms have been presented for trees [7] and more generally, for graphs with bounded treewidth [7].

The power domination number and minimal power dominating sets for grid graphs were obtained by Dorfling and Henning [4]. In [3], Dorbec *et al.*, determined $\gamma_p(G)$ when $G$ is the direct product of paths or $G$ is the lexicographic product of any two graphs. Later, Barrera and Ferrero obtained upper bounds for $\gamma_p(G)$ when is a cylinder, a torus, or a generalized Petersen graph, and identifies many cases where their bounds coincide with the power domination number [2]. More generally, upper bounds for $\gamma_p(G)$ for an arbitrary graph $G$ were given by Zhao, Kang and G.J. Chang [11]. Other upper bounds have been given for block graphs [12] and claw-free graphs [11].

In this paper we use a similar technique to that employed by Dorfling and Henning on the grid graph, and as a result we determine the power domination number for the honeycomb mesh.

## 2. Definitions and notation

In this paper we deal with honeycomb meshes, first studied by Stojmenovic [9]. Honeycomb meshes are closely related to grid graphs in the sense that they originate on different plane tessellations: hexagonal and square, respectively. Indeed, honeycomb meshes offer a model for multiprocessor interconnection networks with similar properties to those of mesh-connected computer networks, also referred to as grid graphs [10].

To define the honeycomb mesh we will use the following notation: for a given $n \in Z$, we denote by $[n]$ the set $\{-n+1, -n+2, \ldots, -1, 0, 1, 2, \ldots n\}$.

**Definition 2.1.** *The hexagonal honeycomb mesh of dimension* $n \geq 1, n \in Z$, $HM(n)$, *has vertex set* $V(HM(n)) = \{(x, y, z) \,/\, x, y, z \;\; [n]$ *and* $1 \leq x+y+z \leq 2\}$ *and two vertices (x1,y1,z1) and (x2,y2,z2) are adjacent if and only if* $|x_1 - x_2| + |y_1 - y_2| = 1$. *Let* $V_1 = \{(x, y, z \,/\, x, y, z \in [n]$ *and* $x+y+z = 1\}$ *and* $V_1 = \{(x, y, z \,/\, x, y, z \in [n]$ *and* $x+y+z = 2\}$.

Intuitively, $HM(1)$ is one simple hexagon. Then, the honeycomb mesh of dimension 2, $HM(2)$, is obtained by adding six hexagons to the boundary edges of $HM(1)$. In general, the honeycomb mesh of dimension $t, HM(t)$, is obtained by adding a layer of hexagons around the boundary of $HM(t\text{–}1)$. The dimension of *HM*(n) represents the number of layers of hexagons between HM(1) and the border of *HM*(n). The following figure shows the labeled version of the graph *HM*(3).

Note that $HM(n)$ is a bipartite graph. We denote its partite sets by $V_1$ and $V_2$. Also, we define the following diagonals in $HM(n)$.

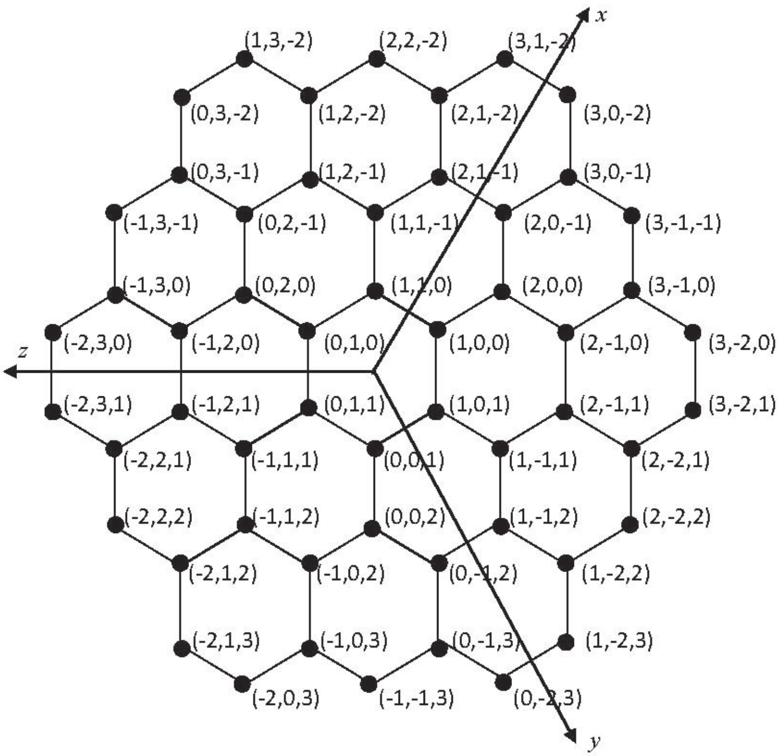


**Figure 1**

**The labeled honeycomb mesh *H*M(3)**

**Definition 2.2.** *For every* $k[n]$ *the x-diagonal at* $x = k$, *denoted by* $D_{x=k}$ *is defined as* $D_{x=k} = \{(k, y, z) \in HM(n) / 1 - k \leq y + z \leq 2 - k\}$.

Note that there are 2n $x$-diagonals in $HM(n)$. The $y$-diagonals and $z$-diagonals can be defined similarly. We say that a vertex v covers a diagonal $D$, if $v \in D$.

**Definition 2.3.** *For a graph G and a set* $T \subseteq V(G)$, *the closure of T in G is denoted by* $C_G(T)$ *is recursively defined as follows: Start with* $C_G(T) = T$. *As long as exactly one of the neighbors of some element of* $C_G(T)$ *is not in* $C_G(T)$, *add that neighbor to* $C_G(T)$.

**Definition 2.4.** *For a graph $G$ and a set $T \subseteq V(G)$, the star closure of $T$ in $G$ is denoted by $C_G^*(T)$ is recursively defined as follows: Start with $C_G^*(T) = T$. As long as exactly one of the neighbors of some vertex of $G$ is not in $C_G^*(T)$, add that neighbor to $C_G^*(T)$.*

If the graph $G$ is clear from the context, we simply write $C(T)$ and $C^*(T)$ rather than $C_G(T)$ and $C_G^*(T)$. Note that the set of vertices power dominated by a set $S$ is $C(N[S])$. In particular, if $S \in V$ is power dominating set of $G$, then $C_G(N[S]) = V$. Further, if $S$ power dominates $G$ and if $T$ is obtained from $S$ by adding all but one neighbor of every vertex in $S$ then $C_G(T) = V$.

## 3. Honeycomb mesh

In this section we are going to prove that $\gamma_P(HM(n)) = \left\lceil \frac{2n}{3} \right\rceil$, for every positive integer $n$. We begin by showing that the previous expression gives an upper bound.

**Lemma 3.1.** *If $G = HM(n)$, then $\gamma_P(G) \leq \left\lceil \frac{2n}{3} \right\rceil$.*

***Proof:*** We consider three possibilities and give a power dominating set for each.

(i) If $n = 3k$:

$D = (0, 3i, 2 - 3i){:}\, 1 \leq i \leq k(0, 3i - 2, 3 - 3i) : 1 \leq i \leq k\}$.

In this case, $|D| = 2k$. Also, $\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{2(3k)}{3} \right\rceil = 2k$.

(ii) If $n = 3k + 1$:

$D = \{(0, 3i - 2, 4 - 3i) : 1 \leq i \leq k + 1\} \cup \{(0, 3i - 1, 2 - 3i) :$
$1 \leq i \leq k\}$.

In this case, $|D| = 2k + 1$. Also, $\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{2(3k+1)}{3} \right\rceil = 2k + 1$.

(iii) If $n = 3k + 2$

$D = \{(0, 3i - 1, 3 - 3i) : 1 \leq i \leq k + 1\} \cup \{(0, 3i - 3, 4 - 3i) :$
$1 \leq i \leq k + 1\}$.

In this case, $|D| = 2k + 1$. Also, $\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{2(3k+1)}{3} \right\rceil = 2k + 1$.

In each case $D$ is a power dominating set of cardinality $\frac{2n}{3}$.

An illustration of a power dominating set used in Lemma 3.1 for the honeycomb mesh $HM(3)$ is given in Figure 2. The vertices in the power dominating set have been circled.

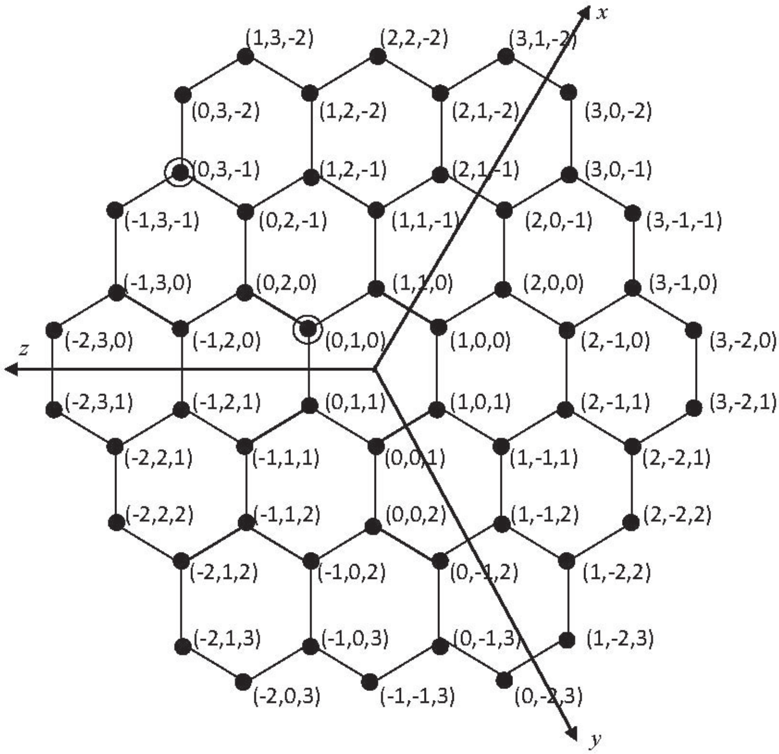


**Figure 2**

**The power dominating set of Lemma 3.1 for *HM*(3)**

To prove that the upper bound is also a lower bound we need the following result.

**Lemma 3.2.** *Let* $G = HM(n)$. *If* $T \subseteq V_1$ *and* $|T| < 2n$, *then* $C^*(T)$ *covers at most* $|T|$ *diagonals.*

**Proof:** Let $G'$ be the graph with vertex set $V(G') = V(G)$ where $uv \in E(G')$ if and only if $d_G(u,v) = 2$. For disjoint subsets $U_1, U_2 \subseteq V_1$, if no vertex of $C^*_G(V_1)$ is adjacent in $G'$ to any vertex of $C^*_G(V_2)$, then $C^*_G(V_1 \cup V_2) = C^*_G(V_1)C^*_G(V_2)$. We may therefore assume that $C^*_G(T)$ is connected in $G'$. Now, let us prove the statement by induction on $|T|$.

If $|T| = 1$, the result clearly holds. Now, let us consider $T \subseteq V_1$ with $|T| = 1$. We can assume $C^*_G(T)$ is connected in $G'$. Also, since the number of $x, y$ or $z$-diagonals in $HM(n)$ is exactly $2n$, we can assume $|T| < 2n$. By inductive hypothesis, the result holds for all $T' \subseteq T$. In particular, for a maximal proper subset $T' \subseteq T$ such that $C^*_G(T')$ is connected in $G'$. Since $C^*_G(T)$ is connected, some vertex of $T', C^*_G(T \backslash T')$ is adjacent in $G'$ to some vertex of $C^*_G(T \backslash T')$. By maximality of $T', C^*_G(T \backslash T')$ is connected. Since the the inductive hypothesis also applies to $(T \backslash T')$, we have the following:

(1) The number of diagonals covered by $C^*_G(T') \leq |T'|$.
(2) The number of diagonals covered by $C^*_G(T') \leq |T'|$.

Therefore, from (1) and (2) we conclude that the number of diagonals covered by $C^*_G(T) = C^*_G(T') \cup C^*_G(T \backslash T')$ is at most $|T'| + |T \backslash T'| = |T|$.

Figure 3 shows a set $T$ (red vertices) in $HM(3)$ and the corresponding set $C^*_G(T)$ (blue vertices).

**Lemma 3.3.** If $G = HM(n)$, then $\gamma_p(G) \geq \left\lceil \frac{2n}{3} \right\rceil$.

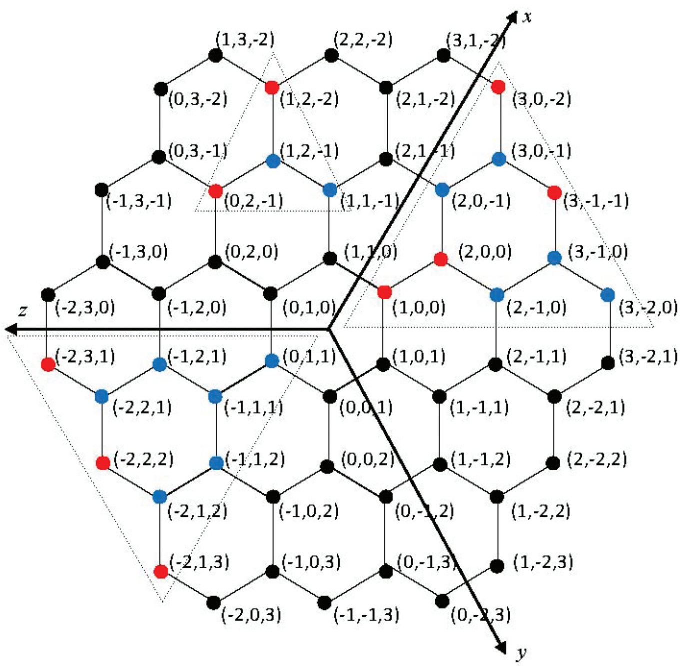


**Figure 3**
**$C^*_G(T)$ (blue vertices) for a set $T$ (red vertices) in $HM(3)$**

***Proof:*** Let $G = HM(n)$ and let $S \subseteq V(G)$ be a power dominating set of $G$. Let $T$ be obtained from $S$ by adding the neighbors of every vertex in $S$ that is $T = N[S]$. Since $S$ is a power dominating set of $G$, then $CG(T) = V(G)$. Notice that in a bipartite graph $H$ with partite sets $H_1$ and $H_2$, $C_H(W) \cap H_1 \subseteq C_H((W \cap H_1) \cup H_2 \cap H_1 = C^*_H(W \cap H_1)$, for any $W \subseteq V(H)$. Thus we have, $C_G(T) \cap V_1 \subseteq C_G((T \cap V_1) \cup V_2) \cap V_1 = C^*_G(T \cap V_1)$ and therefore $C^*_G(T \cap V_1)$ covers all diagonals. Hence it follows from Lemma 3.2 that $|T \cup V_1| \geq 2n$ which implies $|T| \geq 2n$. For any $v \in G$, we have $\deg(v) \leq 3$ and so $|T| \leq 3|S|$. Thus we have, $3|S| \geq |T| \geq 2n$. $\therefore |S| \geq \frac{2n}{3}$. ■

Now we can state our main result.

**Theorem 3.4.** *If $G = HM(n)$, then $\gamma_P(G) = \left\lceil \frac{2n}{3} \right\rceil$.*

**Proof:** It follows from Lemma 3.1 and Lemma 3.3. ■

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