On the edge sums of deBruijn graphs*

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ABSTRACT

An interconnection network is a highly symmetrical connected graph of order n nodes, size m edges, connectivity κ and diameter d, where n and κ are large but m and d are small. Many interconnection networks are defined algebraically in such a way that each node has an integer value. Then, every edge can be assigned an edge sum, defined as the sum of the two nodes it joins. The edge sum problem of a graph consists of the characterization of the set of edge sums over all edges. This problem was introduced by Graham and Harary who presented the solution for hypercubes. Our goal is to solve the edge sum problem for deBruijn interconnection networks.

* This work is the result of a research project carried out by students while attending a Summer Math Camp at Southwest Texas State University. This paper provides the starting point for other research projects also suitable for undergraduates.

Journal of Discrete Mathematical Sciences & Cryptography
Vol. 5 (2002), No. 2, pp. 105-117

1. INTRODUCTION

An interconnection network is a highly symmetrical connected graph G of order n nodes, size m edges, connectivity κ and diameter d, where n and κ are large but m and d are small. Many interconnection networks are defined algebraically in such a way that each node has an integer value. With that numerical labeling of the nodes of G, the network $N(G)$ is constructed by assigning an integer weight to the edges of G as follows. The weight w_{ij} or *edge sum* of edge $ij \in E(G)$ is defined by $w_{ij} = i + j$. The edge sum problem of a graph is to characterize its set of edge sums. Our object is to solve the edge sum problem for deBruijn graphs.

We begin by describing the solution [2] to the edge sum problem for hypercubes. The hypercube Q_n has for its nodes the set V_n of all the binary sequences with n terms, two of which are adjacent whenever the sequences disagree in exactly one place. The following characterization was established:

THEOREM 1A [2]. *A positive integer x is an edge sum of some hypercube if and only if $x \not\equiv 3 \bmod 4$.*

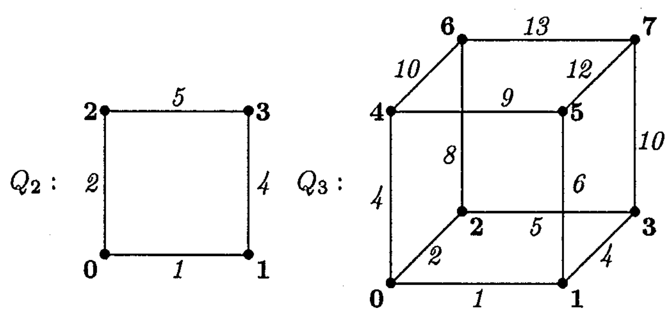

Figure 1. Graphs Q_2 and Q_3 with their edge sums

2. EDGE SUMS OF DEBRUIJN GRAPHS

To define the deBruijn graphs [1] **B(d, n)** we must first define the deBruijn digraph $B(d, n)$, $n \geq 2$. Given positive integers $d, n, d \geq 2$, $n \geq d$, the deBruijn digraph $B(d, n)$ has node set $\mathbf{Z}_d^n$, the set of sequences of length n on $\mathbf{Z}_d$. Two nodes α and β are adjacent whenever the last $n-1$ terms of α are identical with the first $n-1$ terms of β.

The well-known family of interconnection networks, the deBruijn graphs **B(d, n)**, is obtained from the digraph $B(d, n)$ by keeping the

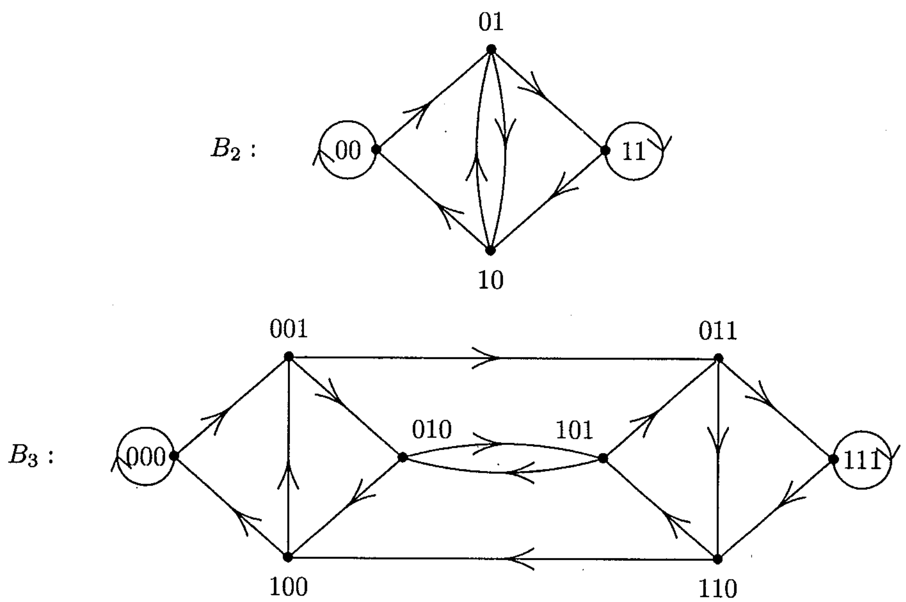

Figure 2. The two smallest deBruijn digraphs

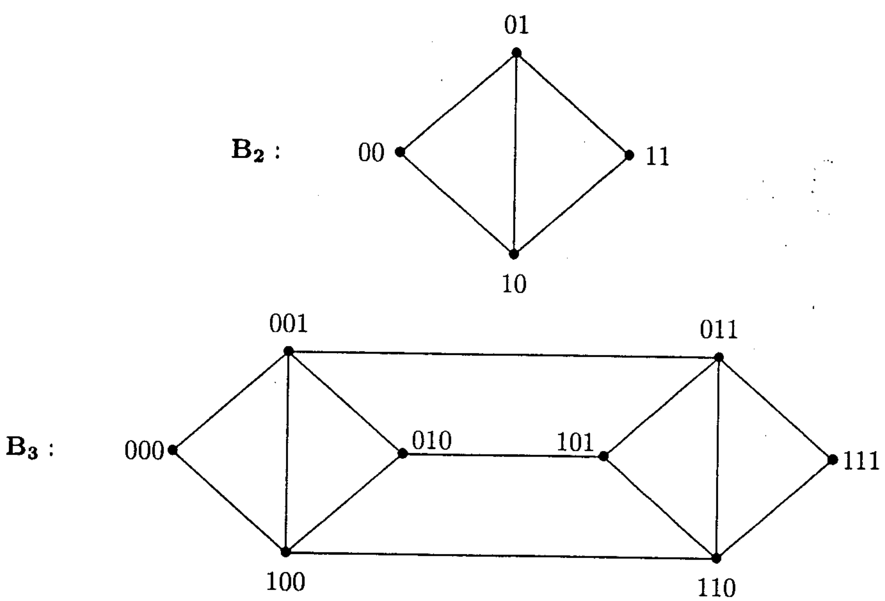

Figure 3. The two smallest deBruijn graphs

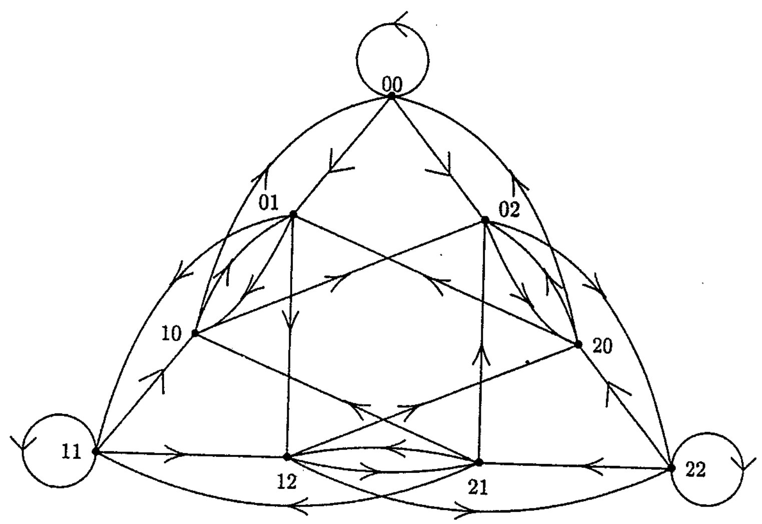

Figure 4. The deBruijn digraph B(3, 2)

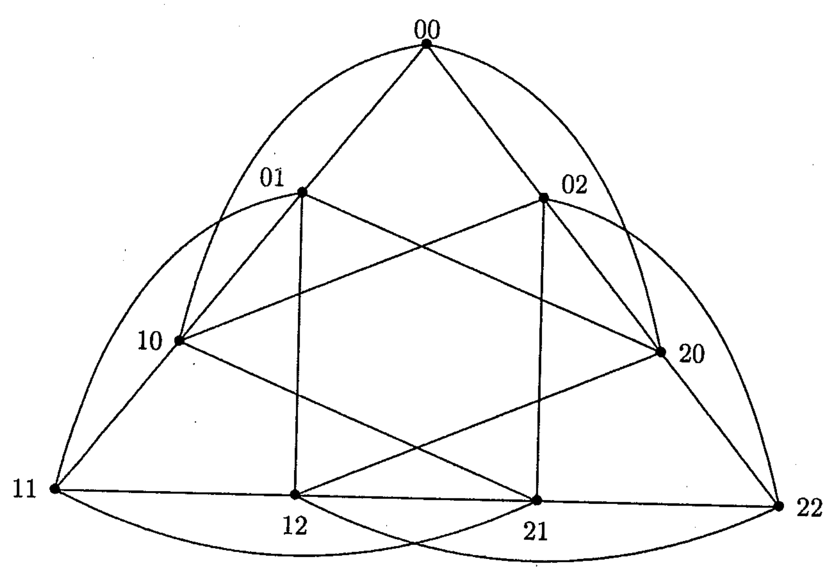

Figure 5. The deBruijn graph B(3, 2)

same node set, but replacing each single arc joining two nodes of $B(d, n)$ by an undirected edge, and also replacing each symmetric pair of arcs in $B(d, n)$ by a single edge. If there are loops, they are removed.

With this labeling on $\mathbf{B(d, n)}$ we consider the network $N(\mathbf{B(d, n)})$ in which the weight of the edge ij_k is $wij_k = i + (id) \bmod d^n + k$.

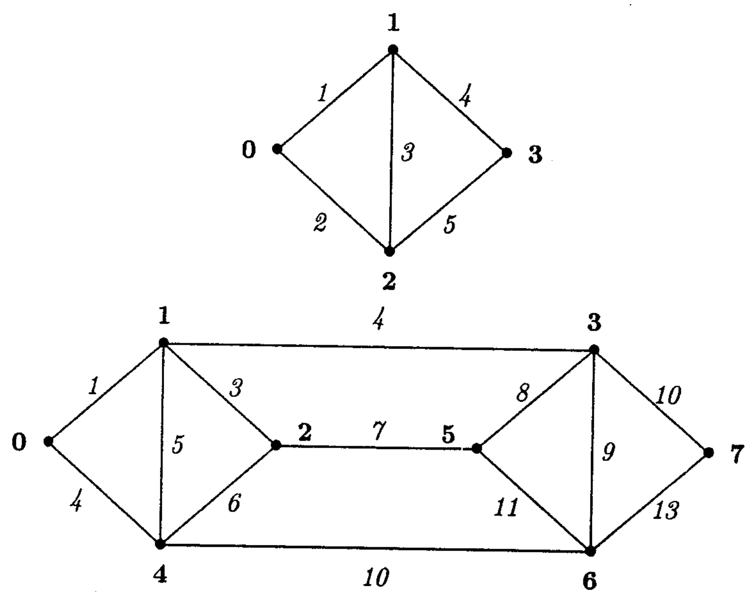

Figure 6. Graphs B(2, 2) and B(2, 3) with their edge sums

2.1. *Edge sums of deBruijn graphs* **B(2, n)**

To study the edge sums for the network $N(\mathbf{B(2, n)})$ we distinguish the following four classes of edges in $\mathbf{B(2, n)}$:

I) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$, and at least one term x_i differs from 0.

II) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$.

III) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$ where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$.

IV) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$, and at least one term x_i differs from 1.

Let x be the integer whose binary representation is given by the sequence $x_1 \dots x_{n-1}$, then $0 \le x \le 2^{n-1} - 1$ and we can state:

I) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 0\}$ has edge sum $3x$.
II) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 1\}$ has edge sum $3x + 1$.
III) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 0\}$ has edge sum $3(x + 2^{n-1}) - 2^n$.
IV) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 1\}$ has edge sum $3(x + 2^{n-1}) + 1 - 2^n$.

Observe that in cases I) and IV) we must ask $x \neq 0$ and $x \neq 2^{n-1} - 1$ respectively, because the graph has no loops.

PROPOSITION 2.1. *Let n be a positive integer and y an integer, $0 \le y \le 2^n - 2$. Then, $1 + y$ is an edge sum of* **B(2, n)** *if and only if $2^{n+1} - 3 - y$ is an edge sum of* **B(2, n)**.

PROOF. We start proving that $1 + y$ is the edge sum of a type I) edge, if and only if $2^{2n+1} - 3 - y$ is the edge sum of a type IV) edge. Indeed, $1 + y$ is the edge sum of an edge $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 0\}$ if and only if $1 + y = 3x$, where x is the integer represented by $x_1 \dots x_{n-1}$. Then, $y = 3x - 1$ and so $2^{2n+1} - 3 - y = 2^{n+1} - 3 - (3x - 1)$, which is exactly the edge sum of the type IV) edge $\{1(1 - x_1) \dots (1 - x_{n-1}), (1 - x_1) \dots (1 - x_{n-1})1\}$. In the same way we can show that $1 + y$ is the edge sum of a type II) edge $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1} 1\}$ if and only if $2^{n+1} - 3 - y$ is the edge sum of a type III) edge $\{1(1 - x_1) \dots (1 - x_{n-1}), (1 - x_1) \dots (1 - x_{n-1})0\}$, which is sufficient to conclude the proof. □

If we look at the edge sums module 3,

- The sums for type I) edges are all integers congruent to 0 (mod 3), between 3 and $3(2^{n-1} - 1)$.
- The sums for type II) edges are all integers congruent to 1 (mod 3), between 1 and $3(2^{n-1} - 1) + 1$.

For the next two classes we need to consider two cases depending on n being even or odd, since $2^m \equiv_3 2$ if m is even and $2^m \equiv_3 1$ if m is odd.

If n is even:

- The sums for type III) edges are all integers congruent to 2 (mod 3) between 2^{n-1} and $3(2^{n-1} - 1) + 2^{n-1}$.
- The sums for type IV) edges are all integers congruent to 0 (mod 3) between $1 + 2^{2n-1}$ and $3(2^{n-1} - 1) + 2^{n-1} + 1$.

If n is odd:

- The sums for type III) edges are all integers congruent to 1 (mod 3) between 2^{n-1} and $3(2^{n-1} - 1) + 2^{n-1}$.

- The sums for type IV) edges are all integers congruent to 2(mod 3) between $1 + 2^{n-1}$ and $3(2^{n-1} - 1) + 2^{n-1} + 1$.

Now we are able to state the following two Theorems that give a complete description of the edge sums of $\mathbf{B}_n$, depending on n even or odd.

THEOREM 2.2. *Let n be a positive even integer, then the edge sums set of* **B(2, n)** *is*

$$L_n = \{m : m \equiv_3 0 \text{ or } m \equiv_3 1,\ 1 \le m \le 3(2^{n-1} - 1) + 1\}$$
$$\cup \{m : m \equiv_3 2 \text{ or } m \equiv_3 0,\ 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}.$$

PROOF. The edge sums of type I) edges in **B(2, n)** is the set $\{m : m \equiv_3 0, 1 \le m \le 3(2^{n-1} - 1) + 1\}$ and the edge sums of type II) edges is the set $\{m : m \equiv_3 1, 1 \le m \le 3(2^{n-1} - 1) + 1\}$. Also if n is even the edge sum of type III) edges in **B(2, n)** is the set $\{m : m \equiv_3 2, 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}$ and the edge sums of type IV) edges is the set $\{m : m \equiv_3 0, 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}$. The union of those four sets gives the set L_n of edge sums of **B(2, n)**. □

THEOREM 2.3. *Let n be a positive odd integer, then the edge sums set of* **B(2, n)** *is*

$$L_n = \{m : m \equiv_3 0 \text{ or } m \equiv_3 1,\ 1 \le m \le 3(2^{n-1} - 1) + 1\}$$
$$\cup \{m : m \equiv_3 1 \text{ or } m \equiv_3 2,\ 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}.$$

PROOF. The edge sums of type I) edges in **B(2, n)** is the set $\{m : m \equiv_3 0, 1 \le m \le 3(2^{n-1} - 1) + 1\}$ and the edge sums of type II) edges is the set $\{m : m \equiv_3 1, 1 \le m \le 3(2^{n-1} - 1) + 1\}$. Also if n is odd the edge sum of type III) edges in **B(2, n)** is the set $\{m : m \equiv_3 1, 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}$ and the edge sums of type IV) edges is the set $\{m : m \equiv_3 2, 2^{n-1} \le m \le 3(2^{n-1} - 1) + 2^{n-1}\}$. The union of those four sets gives the set L_n of edge sums of **B(2, n)**. □

It is simple to observe that the edge sums of **B(2, n)** are integers in the interval $[1, 2^{n+1} - 3]$. The next two Corollaries provide a characterization of the integers in that interval that are not in the edge sums set of **B(2, n)**.

COROLLARY 2.4. *Let n be a positive even integer, the edge sums set of* **B(2, n)** *contains all the integer between* 1 *and* $2^{n+1} - 3$, *except by the* $2\left(\dfrac{2^{n-1} - 2}{3}\right)$ *integers in the set*

$$\{m : m \equiv_3 2,\ 0 \le m \le 2^{n-1} - 2\} \cup \{2^n - 2 - m : m \equiv_3 2,\ 0 \le m \le 2^{n-1} - 2\}.$$

PROOF. By Theorem 2.2 we know the set L_n of all the edge sums of $\mathbf{B(2, n)}$. It is easy to see that $\{m : 1 \le m \le 2^{2n+1} - 3\} - L_n$ is the union of the set $\{m : m \equiv_3 2, 0 \le m \le 2^{n-1} - 2\}$ and the set $\{2^n - 2 - m : m \equiv_3 2, 0 \le m \le 2^{n-1} - 2\}$. □

COROLLARY 2.5. *Let n be a positive odd integer, the edge sums set of* $\mathbf{B(2, n)}$ *contains all the integer between* 1 *and* $2^{n+1} - 3$, *except by the* $2\left(\frac{2^{n-1} - 1}{3}\right)$ *integers in the set*

$$\{m : m \equiv_3 2,\ 0 \le m \le 2^{n-1} - 1\} \cup \{2^n - 2 - m : m \equiv_3 2,\ 0 \le m \le 2^{n-1} - 1\}.$$

PROOF. By Theorem 2.3 we know the set L_n of all the edge sums of $\mathbf{B(2, n)}$. As in the previous Corollary, $\{m : 1 \le m \le 2^{n+1} - 3\} - L_n$ is union of the sets $\{m : m \equiv_3 2, 0 \le m \le 2^{n-1} - 1\}$ and $\{2^n - 2 - m : m \equiv_3 2, 0 \le m \le 2^{n-1} - 1\}$. □

The *multiplicity* of an integer s as edge sum of a certain $\mathbf{B(2, n)}$ is the number of edges whose edge sum is s.

THEOREM 2.6. *Let n be a positive even integer, the multiplicity of all the integers in the edge sums set of* $\mathbf{B(2, n)}$ *is* 1, *except by the* $\left(\frac{2^n - 4}{3}\right) + 1$ *integers in the set* $\{m : m \equiv_3 0, 1 + 2^{n-1} \le m \le 3(2^{n-1} - 1)\}$ *that have multiplicity* 2.

PROOF. The edge sums of type II) edges have all multiplicity 1 because each integer in the set $\{m : m \equiv_3 1, 1 \le m \le 3(2^{n-1} - 1) + 1\}$ is the edge sum of only one type II) edge and other type of edges have their sums not contained in that set because they are not congruent to 1 (mod 3). Similarly, the edge sums of type III) edges have also multiplicity 1. However, type II) and IV) edges have edge sums congruent to 0 (mod 3) in the integer intervals $[1, 3(2^{n-1} - 1) + 1]$ and $[2^{n-1}, 3(2^{n-1} - 1) + 2^{n-1}]$, respectively. Thus, the intersection of those intervals gives the only edge sums with multiplicity 2, and it is the set $\{m : m \equiv_3 0, 1 + 2^{n-1} \le m \le 3(2^{n-1} - 1)\}$. □

THEOREM 2.7. *Let n be a positive odd integer, the multiplicity of all the integers in the edge sum set of* $\mathbf{B(2, n)}$ *is* 1, *except by the*

$\left(\frac{2^n-2}{3}\right)+1$ *integers in the set* $\{m : m \equiv_3 0,\ 2^{n-1} \le m \le 3(2^{n-1}-1)+1\}$ *that have multiplicity* 2.

PROOF. The edge sums of type I) edges have all multiplicity 1 because each integer in the set $\{m : m \equiv_3 0,\ 1 \le m \le 3(2^{n-1}-1)+1\}$ is the edge sum of exactly one edge of type I) and other type of edges do not have their edge sums in that set since they are not congruent to 0 (mod 3). Analogously, the edge sums of type IV) edges have multiplicity 1. Type II) and III) edges have edge sums congruent to 1 (mod 3) in the integer intervals $[1, 3(2^{n-1}-1)+1]$ and $[2^{n-1}, 3(2^{n-1}-1)+2^{n-1}]$, respectively. The intersection of those intervals gives the set $\{m : m \equiv_3 0,\ 1+2^{n-1} \le m \le 3(2^{n-1}-1)\}$ containing all edge sums with multiplicity 2. □

2.2. *Edge sums of deBruijn graphs* **B(d, n)**

PROPOSITION 2.8. *Let* d, n *be two positive integers,* $d \ge 2$ *and* y *an integer,* $0 \le y \le d^n - 2$. *Then,* $1+y$ *is an edge sum of* **B(d, n)** *if and only if* $2^{n+1} - 3 - y$ *is an edge sum of* **B(d, n)**.

PROOF. If $1+y$ is the edge sum of an edge, let us say $\{ax_1 \dots x_{n-1}, x_1 \dots x_{n-1}b\}$, then $1+y = (d+1)x + ad^{n-1} + b$ where x is the integer represented by the sequence $x_1 \dots x_{n-1_{(2)}}$. Now we look at the edge sum of $\{(d-1-a)(d-1-x_1)\dots(d-1-x_{n-1}), (d-1-x_1)\dots(d-1-x_{n-1})(d-1-b)\}$. If x' is the integer represented by the sequence $(d-1-x_1)\dots(d-1-x_{n-1})$, the edge $\{(d-1-a)(d-1-x_1)\dots(d-1-x_{n-1}), (d-1-x_1)\dots(d-1-x_{n-1})(d-1-b)\}$ has edge sum $(d+1)x' + (d-1-a)d^{n-1} + (d-1-b)$. Now, because of the choice of x and x', $x' = d^{n-1} - 1 - x$, and so $(d+1)x' + (d-1-a)d^{n-1} + (d-1-b) = 2d^n - (d+1)x - ad^{n-1} - 2 - b$. This proves that $2^{n+1} - 3 - y$ is an edge sum of **B(d, n)**. The reciprocal can be proved in the same way. □

THEOREM 2.9. *Let* d, n *be two positive integers,* n *even and* $d \ge 2$. *The edge sums set of* **B(d, n)** *contains all the integer between* 1 *and* $2d^n - 3$, *except by the* $2\left(\frac{d^{n-1}-d}{d+1}\right)$ *integers in the set*

$$\{m : m \equiv_{(d+1)} 3d,\ 0 \le m \le d^{n-1} - d\} \cup \{2d^n - 2 - m : m \equiv_{(d+1)} 3d,\ 0 \le m \le d^{n-1} - d\}.$$

PROOF. By Proposition 2.8 it is enough to prove that the only integers that are not edge sums between 0 and $d^{n-1}-1$ are exactly the set $\{m : m \equiv_{(d+1)} d,\ 0 \le m \le d^{n-1}-d\}$. Now, the edge sum of a generic edge $\{ax_1 \dots x_{n-1}, x_1 \dots x_{n-1} b\}$ is $(d+1)x + ad^{n-1} + b$, where $0 \le a$, $b \le d-1$, $0 \le x_i \le d-1$ for all $i = 1, \dots, n-1$ and not all the terms equal, in order to avoid loops. Then, considering all the possible values for x_i and b when $a = 0$, the edge sums obtained are the integers in the intervals $[1, d-1]$ and $[x(d+1), x(d+1)+d-1]$, for all integer x, $1 \le x \le d^{n-1}-1$. Observe that the maximum integer in those intervals is $d^n + d^{n-1} - 2 > d^n - 1$. Besides, the integers between 1 and $d^n + d^{n-1} - 2$ that do not belong to any of those intervals are those congruent to $d(\text{mod } d+1)$. Then, we only need to prove that the integers between $d^{n-1} - d + 1$ and $d^n - 1$ that are congruent to $d(\text{mod } d+1)$ are edge sums of some edges where $a \ne 0$. Indeed, if we proceed as before but with $a = 1$, considering all the edge sums for any integer x, $0 \le x \le d^{n-1} - 1$ and any b, $0 \le b \le d-1$, we obtain all the integer intervals $[x(d+1) + d^{n-1}, x(d+1) + d^{n-1} + d - 1]$. Since m is even, $d^{n-1} \equiv_{(d+1)} d$ so $x(d+1) + d^{n-1} \equiv_{(d+1)} d$ and all the integers congruent to d (mod $d+1$) between $d^{n-1} - d + 1$ and $d^n - 1$ are in those intervals. □

THEOREM 2.10. *Let d, n be two positive integers, n odd and $d \ge 2$. The edge sums set of* **B(d, n)** *contains all the integer between* 1 *and* $2d^n - 3$, *except the* $2\left(\dfrac{d^{n-1}-1}{d+1}\right)$ *integers in the set*

$$\{m : m \equiv_{(d+1)} 3d,\ 0 \le m \le d^{n-1} - 1\} \cup \{2d^n - 2 - m : m \equiv_{(d+1)} 3d,\ 0 \le m \le d^{n-1} - 1\}.$$

PROOF. By Proposition 2.8 it is enough to prove that the only integers that are not edge sums between 0 and $d^{n-1}-1$ are exactly the set $\{m : m \equiv_{(d+1)} d,\ 0 \le m \le d^{n-1} - d\}$. Now, the edge sum of a generic edge $\{ax_1 \dots x_{n-1}, x_1 \dots x_{n-1} b\}$ is $(d+1)x + ad^{n-1} + b$, where $0 \le a$, $b \le d-1$, $0 \le x_i \le d-1$ for all $i = 1, \dots, n-1$ and not all the terms equal because the graph has no loops. Then, considering all the possible values for x_i and b when $a = 0$, the edge sums obtained are the integers in the intervals $[1, d-1]$ and $[x(d+1), x(d+1)+d-1]$, for all integer x, $0 \le x \le d^{n-1} - 1$. Observe that the maximum integer in those intervals is $d^n + d^{n-1} - 2 > d^n - 1$. Also the integers between 1 and $d^n + d^{n-1} - 2$ that do not belong to any of those intervals are congruent to d (mod $d+1$). Then, it remains to prove that the integers between $d^{n-1} - d + 1$ and $d^n - 1$ that are congruent to d (mod $d+1$) are edge sum of some edge with $a \ne 0$. Proceeding as before but with $a = 1$ we obtain all the integers intervals $[x(d+1)+d^{n-1}, x(d+1)+d^{n-1}+d-1]$. Since m is odd $d^{n-1} \equiv_{(d+1)} 1$ so $x(d+1)+d^{n-1}+d-1 \equiv_{(d+1)} d$ and all the integers congruent to $d(\text{mod } d+1)$ between $d^{n-1}-d+1$ and d^n-1 are in those intervals. □

COROLLARY 2.11. *Let n be a positive even integer, then the edge sum set of* **B(d, n)** *is*

$$L_{d,n} = \{m : m \not\equiv_{(d+1)} d,\ 1 \le m \le d^{n-1} - d\}$$

$$\cup \{m : m \not\equiv_{(d+1)} 1,\ 2d^n - 2 \le m \le 2d^n - d - d^{n-1}\}$$

$$\cup \{m : d^{n-1} - d + 1 \le m \le 2d^n - d\}.$$

PROOF. By Theorem 2.9 we know that the only integers between 1 and $2d^n - 3$ that are not edge sums are in $\{m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - d\}$ or $\{2d^n - 2 - m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - d\}$. Then, it is obvious that the set $\{m : m \equiv_{(d+1)} d, 1 \le m \le d^{n-1} - d\}$ is included in $L_{d,n}$. It is also simple to check that $\{m : m \not\equiv_{(d+1)} 1, 2d^n - 2 \le m \le 2d^n - d + 1 - d^{n-1}\}$ reflects that the integers in the set $\{2d^n - 2 - m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - d\}$ are not edge sums. Again, by Theorem 2.9, the set $\{m : d^{n-1} - \le m \le 2d^n - d\}$ contains only edge sums. □

COROLLARY 2.12. *Let n be a positive odd integer, then the edge sum set of* **B(d, n)** *is*

$$L_{d,n} = \{m : m \not\equiv_{(d+1)} d,\ 1 \le m \le d^{n-1} - 1\}$$

$$\cup \{m : m \not\equiv_{(d+1)} 1,\ 2d^n - 2 \le m \le 2d^n - 1 - d^{n-1}\}$$

$$\cup \{m : d^{n-1} - \le m \le 2d^n - 1\}.$$

PROOF. By Theorem 2.10 we know that the only integers between 1 and $2d^n - 3$ that are not edge sums are in $\{m : m \equiv_{(d+1)} 3d, 0 \le m \le d^{n-1} - 1\}$ or $\{2d^n - 2 - m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - 1\}$. Then, it is obvious that the set $\{m : m \not\equiv_{(d+1)} d, 1 \le m \le d^{n-1} - 1\}$ is included in $L_{d,n}$. It is also simple to check that $\{m : m \not\equiv_{(d+1)} 1, 2d^n - 2 \le m \le 2d^n - 1 - d^{n-1}\}$ reflects that the integers in the set $\{2d^n - 2 - m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - 1\}$ are not edge sums. Again, by Theorem 2.10 the set $\{m : d^{n-1} - \le m \le 2d^n - 1\}$ contains only edge sums. □

Figure 7 shows the previous results. Note that from all the integers in the interval $[1, 23^2 - 3]$. As a consequence of Theorem 2.9, for every graph **B(d, 2)** we can state the following Corollary.

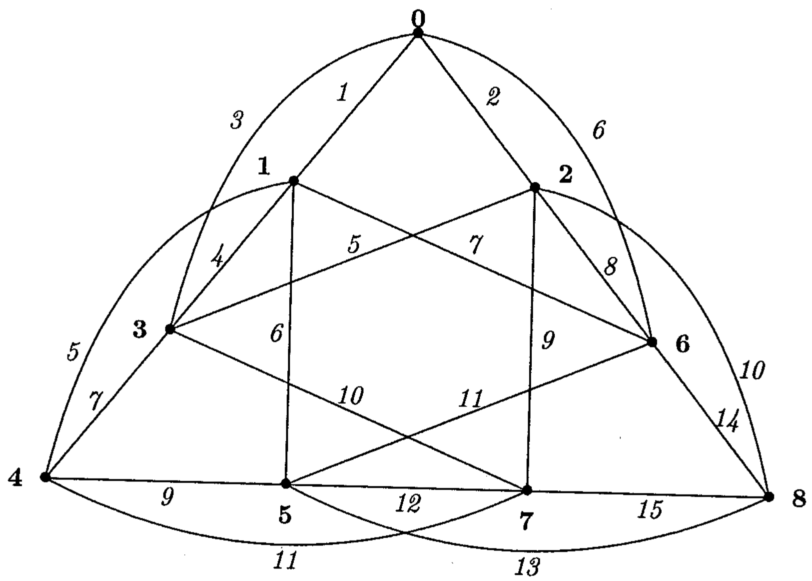

Figure 7. The deBruijn graph B(3, 2) with its edge sums

COROLLARY 2.13. *Let* d *be a integer. The edge sums set of* **B(d, 2)** *contains all the integer between* 1 *and* $2^{n+1} - 3$.

PROOF. By Theorem 2.9 we know that the edge sums of **B(d, 2)** are all the integers between 1 and $2^{n+1} - 3$ except by those integers in the set $\{m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - d\}$ or in the set $\{2d^n - 2 - m : m \equiv_{(d+1)} d, 0 \le m \le d^{n-1} - d\}$. However, since $n = 2$, those sets are both empty. □

3. CONCLUSION

The solution to the edge sum problem for a general deBruijn graph has been presented. Furthermore, those integer that appear more than once in the edge sums set have been identified. However, the multiplicity of an integer as an edge sum has been calculated only for deBruijn graphs in the form $B(2, n)$. The calculus of the multiplicity of an edge sum in the general case $B(d, n)$ where $d \ge 2$ is still open and it is an interesting extension of the work exposed. In addition, other related problems may provide research projects suitable for undergraduates. For example, to solve the edge sums problem for other interconnection networks, such as the Kautz graphs [4] which have very similar properties to the deBruijn graphs from the applied point of view. [redacted] Another family of graphs where the edge sum problem is also interesting is the cube-connected cycles [5], a generalization of hypercube networks. Investigation of this family would provide an opportunity to extend the earlier results of Graham and Harary [2].

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Received October, 2001