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On eccentricity sequences of connected graphs*

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Abstract

We survey the literature on the eccentricity sequence of a connected graph and make the following contribution. The eccentricity sequence of a graph G is the list of its eccentricities in non-increasing order. Two graphs G_1 and G_2 are co-eccentric when they have the same eccentricity sequence. Then, we say that G_1 and G_2 are co-eccentric mates. We characterize the eccentricity sequence of almost all graphs.

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1. Introduction

A nontrivial connected *graph* G consists of a finite non-empty set V of n *nodes* together with a set E of m unordered pairs of distinct nodes of G called *edges*. The *distance* $d(u, v)$ between two nodes u and v of G is the smallest length of a path connecting u and v. The *eccentricity* $e(v)$ of a node v is the maximum distance to another node. The *diameter* $diam(G)$ is the largest of its eccentricities. In general, we follow the notation and terminology of [5].

The *eccentricity sequence* $e\text{-}seq(G)$ of G is the list of its eccentricities in non-increasing order. Figure 1 shows four graphs of order 4 and their respective eccentricity sequences.

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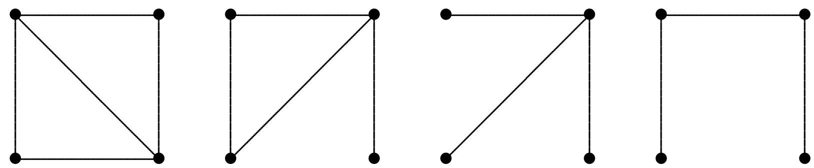

(a) 2211 (b) 2221 (c) 2221 (d) 3322

Figure 1: Some graphs and their eccentricity sequences

The discovery of the probabilistic method and its usefulness in solving graph problems lead to the theory of random graphs. For a positive integer n and a real number p between 0 and 1, the *random graph* $G(n,p)$ denotes the probability space whose elements are the $2^{\binom{n}{2}}$ different graphs on n nodes. The probabilities are determined under the assumption that the probability of an edge between any two nodes is an independent event with probability p. In general, authors take $p = 1/2$. Basic properties of random graphs can be found in [4] and earlier in Bollobás [1]and Palmer[10].

2. Survey

A sequence of integers is *eccentric* if it is the eccentricity sequence of some graph. Eccentricity sequences of graphs were first studied by Lesniak [7] who also worked on the characterization of eccentric sequences. Among other properties, in [7] it was presented the following characterization of eccentric sequences.

Theorem 2.1. [7] *A nondecreasing sequence of positive integers* $S : a_1, a_2, \ldots, a_p$ *with* m *distinct values is eccentric if and only if some subsequence of* S *with* m *distinct values is eccentric.*

Since a subsequence of S may be S itself, this result leads to the concept of *minimal eccentric sequence* presented by Nandakumar [9]. An eccentric sequence $S : a_1, a_2, \ldots, a_p$ with m distinct values is *minimal* if it has no proper eccentric subsequence with m distinct values. Nandakumar [9] also determined all minimal eccentric sequences with smallest eccentricity 1 or 2. His result was later extended by Haviar, Hrnciar and Monoszová [6], who found all the eccentric sequences with smallest eccentricity 3.

Theorem 2.2. [6] *The minimal eccentric sequences with minimum eccentricity* 3 *are exactly:* 3^6; $3^5, 4^2$; $3^4, 4^4$; $3^3, 4^6$; $3^2, 4^8$; $3, 4^{10}$; $3, 4^2, 5^{12}$; $3, 4^3, 5^9$; $3, 4^4, 5^7$; $3, 4^5, 5^4$; $3, 4^7, 5^2$;$3^2, 4^2, 5^2$ *and* $3, 4^2, 5^2, 6^2$.

For graphs with minimum eccentricity greater than 3 there is a result by Haviar, Hrnciar and Monoszová [6], which characterizes a class of eccentric sequences.

Theorem 2.3. [6] *All minimal eccentric sequences of the type*

$$r^{\alpha}, (r+1)^{\beta} \text{ for } r \geq 3 \text{ and } \alpha + \beta \leq \min\left\{3r-2, \frac{8r+5}{3}\right\}$$

are: $r^{2r-1}(r+1)^2$; $r^{2r-2}(r+1)^4$; $r^{2r-2i+1}(r+1)^{3i}$ *and* $r^{2r-2i}(r+1)^{3i+2}$ *for* $i = 2, 3, \ldots \frac{2r+1}{3}$.

The study of eccentricity sequences can be simplified when it is restricted to trees. Since trees have a unique shortest path between any two distinct nodes, the determination of the eccentricities, and more generally, of distances, becomes simpler. A sequence of integers is *t-eccentric* if it is the eccentricity sequence of some tree. A characterization of t-eccentric sequences was given by Lesniak in [7].

Theorem 2.4. [7] *Let* $S : a_1, a_2, \ldots, a_p$, $p \geq 3$, *be a nondecreasing sequence of positive integers. Then* S *is* t*-eccentric if and only if*

(i) for each integer k *with* $a_1 < k \leq a_p$, $a_i = a_{i+1} = k$ *for some* i, $2 \leq 2 \leq n-1$;

(ii) either $a_1 = \frac{a_n}{2}$ *and* $a_1 \neq a_2$ *or* $a_1 = a_2 = \frac{a_n+1}{2}$ *and* $a_2 \neq a_3$.

Caterpillars are a particular class of trees with the property that is all the leaves are pruned, the resulting tree is a (possibly degenerate) path. Skurnik [12] studied eccentric sequences corresponding to caterpillars. A sequence of integers is *c-eccentric* if it is the eccentricity sequence of some caterpillar.

Theorem 2.5. [12] *Let* $S : a_1, a_2, \ldots, a_p$, $p \geq 3$, *be a nondecreasing sequence of positive integers. If* S *is* t*-eccentric, then* S *is also* c*-eccentric.*

Besides, Skurnik was able to establish the exact number of non-isomorphic caterpillars having the same eccentric sequence.

Theorem 2.6. [12] *Let* $S : a_1, a_2, \ldots, a_p$, $p \geq 3$, *be a nondecreasing sequence of positive integers such that it is* t*-eccentric. Let* χ_k *be the number of times that the positive integer* k *appears in* S *and let* N *be the number of non-isomorphic caterpillars whose eccentricity sequence is* S.

(i) If $a_n = 0$, *then* $N = 1$;

(ii) If $a_n > 2$ *and even, then* $N = \left\lfloor \frac{1}{2}\left(1 + \Pi_{k=a_1+2}^{a_n}(\chi_k - 1)\right)\right\rfloor$.

(iii) If $a_n > 2$ *and odd, then* $N = \left\lfloor \frac{1}{2}\left(1 + \Pi_{k=a_1+1}^{a_n}(\chi_k - 1)\right)\right\rfloor$.

3. Co-eccentric graphs

Given a graph, it is easy to determine its eccentricity sequence from its distance matrix. Our object is to approach the converse problem by finding which sequences are eccentric, and in which cases it is possible to reconstruct a graph from its eccentricity sequence.

For $n = 2$, there is a unique connected graph on n nodes with eccentricity sequence 11. For $n = 3$, there are only two connected graphs, the complete graph K_3 and the path P_3, which have eccentricity sequences 111 and 122, respectively. However for $n = 4$, there are at least two graphs, the graphs (b) and (c) shown in Figure 1, which have the same eccentricity sequence.

We shall show that for every integer $n \geq 4$, there are at least two graphs on n nodes that have the same eccentricity sequence.

Two graphs G_1 and G_2 are *co-eccentric* if and only if $e\text{-}seq(G_1){=}e\text{-}seq(G_2)$. In that case, we will also say that G_1 and G_2 are co-eccentric mates.

A consequence of the use of random graphs is that many results in graph theory assert that almost all graphs or almost no graphs have a certain property. In order to obtain a theorem that will imply almost all of those results as corollaries, Blass and Harary [2] developed a first-order theory of graphs. Certain adjacency axioms were given and shown to be satisfied by almost all graphs, and it is then proved, that any first-order sentence about graphs holds in almost all graphs or in almost none.

Theorem A. [2] *Let H_1 be an induced subgraph of H_2. Almost all graphs G have the property that every isomorphism f_1 from H_1 onto an induced subgraph of G can be extended to an isomorphism f_2 from H_2 onto an induced subgraph of G.*

Corollary A. [2] *Almost all graphs are connected with diameter* 2.

Theorem 3.1. *Almost all graphs have eccentric sequence $2^i 1^{n-i}$, for some integer i, $2 \leq i \leq n$.*

Proof. Let G be a graph in $G(n,p)$. From Corollary A it follows that $diam(G) = 2$. Therefore, for every node v, either $e(v) = 1$ or $e(v) = 2$, but G must have, at least, two nodes at distance 2, so $e - seq(G) = 2^i 1^{n-i}$ for some integer i, $2 \leq i \leq n$. □

Conversely, for any integer $n \geq 3$, and for every integer i, $2 \leq i \leq n$, the sequence $2^i 1^{n-i}$ is eccentric. To do this, we start by proving the following proposition.

Proposition 3.2. *Let G be a graph on n nodes with maximum degree $\Delta \leq n-2$. Let $G+v$ be the graph obtained by adding a node v to G, and an edge between v and every node of G. Then $e - seq(G+v) = 2^n 1$.*

Proof. Obviously $e(v) = 1$. Since the maximum degree of G is at most $n-2$, every node x of G has $e(x) \geq 2$ in G. Since $\Delta \leq n-2$, for any node x in G there exists at least one node y at distance at least 2 in G. However, since xvy is a path in G', $e(x) = 2$ in G'. □

This construction can be extended in order to obtain graphs with eccentricity sequence $2^i 1^{n-i}$, for any integer $n \geq 3$ and for any i, $2 \leq i \leq n-1$.

Proposition 3.3. *Let n be an integer, $n \geq 3$. For every integer i, $2 \leq i \leq n-1$, there exists a graph G such that $e - seq(G) = 2^i 1^{n-i}$.*

Proof. Let G_1 be a graph on i nodes with maximum degree not greater than $i-2$. Add a node v_1 and an edge between v_1 and every node of G_1 obtaining a new graph $G_1 + v_1$. If $n > i+1$, add another node v_2 and an edge between v_2 and every node of G_1+v_1, obtaining a graph $G_1 + v_1 + v_2$. Repeat this procedure until nodes $v_1, \ldots, v_{n-i}$ have been added to G_1. Let $G = G_{n-i} + v_1, \ldots, v_{n-i}$. Then, $e(v_j) = 1$, for every $j = 1, \ldots, n-i$. Also note that since the maximum degree of G is at most $i-2$, for any node x in G_{n-i} which was originally in G, there exists at least one node y in G_1 at distance at least 2, so $e(x) \geq 2$. Besides, since xvy is a path in G, it must be $e(x) = 2$. Then, $e - seq(G) = 2^i 1^{n-i}$. □

Finally, we need to study the sequences 2^n, for some integer n.

Proposition 3.4. *For every integer $n \geq 4$, there exists a graph G such that $e - seq(G) = 2^n$.*

Proof. It is enough to observe that $e - seq(K_{2,n-2}) = 2^n$. □

In the next part of this paper we deal with the characterization of those eccentric sequences corresponding to more than one graphs.

Proposition 3.5. *Let n be an integer, $n \geq 3$. For every integer i, $3 \leq i \leq n-1$, there exist at least two graphs G_1 and G_2 such that $e - seq(G_1) = 2^i 1^{n-i}$ and $e - seq(G_1) = 2^i 1^{n-i}$.*

Proof. Let G be a graph on i nodes with maximum degree not greater than $i-2$. As it was done in the proof of Proposition 3.3, we add a nodes and edges in G, obtaining a new graph $G_1 = G + v_1 \ldots v_{n-i}$, such that $e - seq(G_1) = 2^i 1^{n-i}$. Notice that since $3 \leq i$, there exist at least two different graphs on i nodes with maximum degree not greater than $i-2$. If we repeat the procedure with a graph on i nodes different from G, we will obtain a graph G_2 different from G_1, but also with $e - seq(G_2) = 2^i 1^{n-i}$. □

Observe that in the previous result we need the condition $i \geq 3$, because if $i = 2$ there is only one graph on i nodes with degree at most $i-2$.

Proposition 3.6. *Let n be an integer, $n \geq 3$. Then, $e - seq(K_n - e) = 2^2 1^{n-2}$, and this is the only graph with eccentricity sequence $2^2 1^{n-2}$.*

Proof. It is easy to see that $e - seq(K_n - e) = 2^2 1^{n-2}$. Reciprocally, let G be a graph on n nodes with $e - seq(G) = 2^2 1^{n-2}$. Then, there exist exactly two nodes in G, let us say u and v, such that $d(u,v) = 2$. Therefore, if we add the edge $e = uv$ to G, we must obtain the complete graph. As a consequence, G must be $K_n - e$. □

The remaining eccentric sequence to be discussed is 2^n, for $n \geq 4$. It is simple to check that the only graph on 4 nodes with eccentricity sequence 2^4 is the cycle C_4. However, this is the only value of n for which the sequence 2^n is associated to a unique graph. In Figure 2 we show two graphs, G'_1 and G'_2, on 5 nodes with eccentricity sequence 2^5.

Proposition 3.7. *For every integer n, $n \geq 5$, there exist at least two graphs with eccentricity sequence 2^n.*

Proof. Notice that $e - seq(K_{2,n-2}) = 2^n$. If $n > 5$, then $K_{3,n-3} \neq K_{2,n-2}$ and $e - seq(K_{3,n-3}) = 2^n$. If $n = 5$, $e - seq(K_{2,3}) = 2^5$ and also $e - seq(K_{2,3} + e) = 2^5$. $\square$

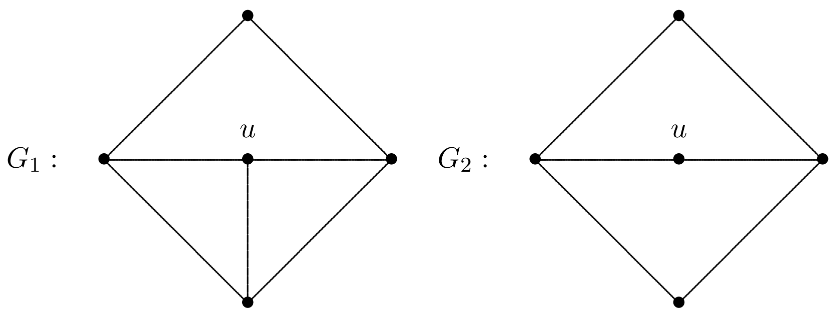

Figure 2: Two graphs with eccentric sequence 2^5.

4. Conclusions

We have shown that almost all graphs on n nodes have eccentricity sequence of the form $2^i 1^{n-i}$ for some integer i, $2 \leq i \leq n$, and we have also answered two questions in relation to the converse statement:

- For which values of the integers n and i is the sequence $2^i 1^{n-i}$ eccentric?
- For which values of the integers n and i there exist multiple graphs with eccentricity sequence $2^i 1^{n-i}$?

Summarizing our results, the values of the integers n and i for which the sequence $2^i 1^{n-i}$ is eccentric are:

$n \geq 3$	$2 \leq i \leq n-1$	Proposition 3.3
$n \geq 4$	$i = n$	Proposition 3.4

The values of the integers n and i for which there are at least two graphs with eccentricity sequence $2^i 1^{n-i}$ are:

$n \geq 3$	$3 \leq i \leq n-1$	Proposition 3.5
$n \geq 5$	$i = n$	Proposition 3.7

The values of the integers n and i for which there is a unique graph with eccentricity sequence $2^i 1^{n-i}$ are:

$n = 3$	$i = 2$	Path P_3
$n \geq 4$	$i = 2$	Proposition 3.6
$n \geq 4$	$i = 4$	Cycle C_4

5. Open Problems

Constructive characterization of eccentricity sequences.

Characterize the eccentric sequences corresponding to unique graphs with diameter greater than 2.

Related results and open problems were presented by Buckley [1, 3], Harary [3], McDougal [8] and Prisner [11].

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