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EDGE SUMS OF DEBRUIJN INTERCONNECTION NETWORKS

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An interconnection network is a highly symmetrical connected graph of order n nodes, size m edges, connectivity κ and diameter d, where n and κ are large but m and d are small. Many interconnection networks are defined algebraically in such a way that each node has an integer value. Then every edge can be assigned the sum of the two nodes it joins. These numbers are called the edge sums of the graph. The edge sum problem of a graph is to characterize the set of edge sums. This problem was introduced by Graham and Harary who presented the solution for hypercubes. Our object is to characterize the edge sums for another family of interconnection networks, namely, deBruijn graphs.

Keywords: deBruijn graphs; deBruijn sequences; Edge sums

C.R. Categories: G.2.2

1 INTRODUCTION

We begin by describing the solution [2] to the edge sum problem for hypercubes. The hypercube Q_n has for its nodes the set V_n of all the binary sequences with n terms, two of which are adjacent whenever the sequences disagree in exactly one place. The number assigned to an edge of Q_n is the sum of the numbers of its two nodes (see Figure 1). The following characterization was established:

THEOREM 1A [2] *A positive integer x is an edge sum of some hypercube if and only if $x \not\equiv 3 (\operatorname{mod} 4)$.*

Given a digraph D, we write $\mathbf{G}D$ for the unary operation on D called *the graph of G*. Now $\mathbf{G}D$ is the graph obtained from digraph D by keeping all the nodes of D, but replacing each single arc joining two nodes of D by an undirected edge, and also replacing each symmetric pair of arcs in D by a single edge. If there are loops, they are removed.

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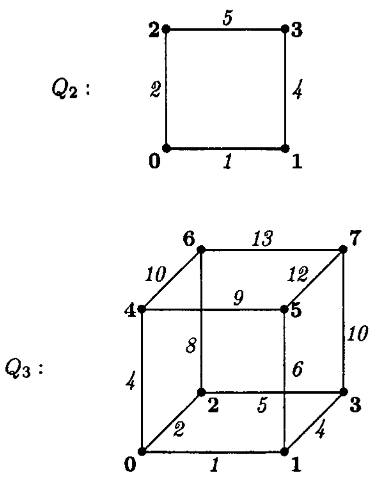

FIGURE 1 Graphs Q_2 and Q_3 with their edge sums.

2 deBRUIJN DIGRAPHS AND GRAPHS

We must first define the deBruijn digraph B_n, $n \geq 2$. Again, B_n has the same node set V_n as for the hypercube Q_n. But now, two nodes α and β are adjacent whenever the last $n-1$ terms of α are identical with the first $n-1$ terms of β. Now we can define the well-known family of interconnection networks, the deBruijn graphs $\mathbf{G}B_n$ (see Figures 2 and 3 for examples). For conciseness, we write $\mathbf{B}_n = \mathbf{G}B_n$.

Given a numerical labeling of the nodes of a graph G, the network $N(G)$ is constructed by assigning an integer weight to the edges of G as follows. The weight w_{ij} or *edge sum* of edge $ij \in E(G)$ is defined by $w_{ij} = i + j$.

With each graph G_n we can associate a set L_n of the weights on the edges of $N(G_n)$. The *edge sum problem* consists of the characterization of the set L_n of all integers that are the weights on the edges of $N(G_n)$. When restricted to a family of graphs, the problem consists

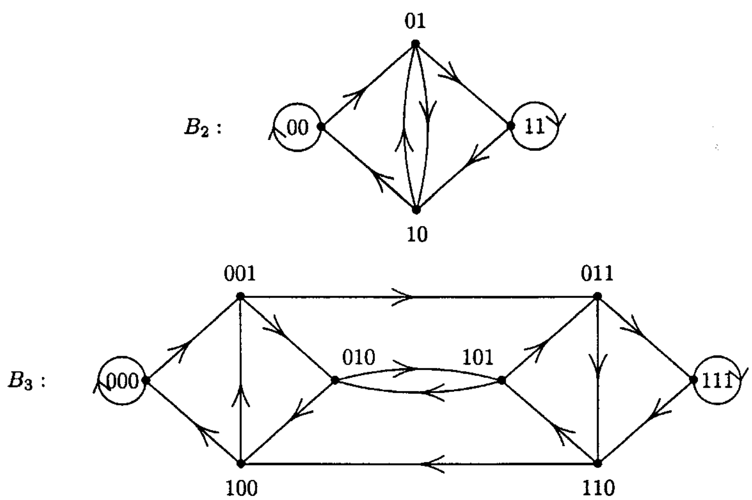

FIGURE 2 The two smallest deBruijn digraphs.

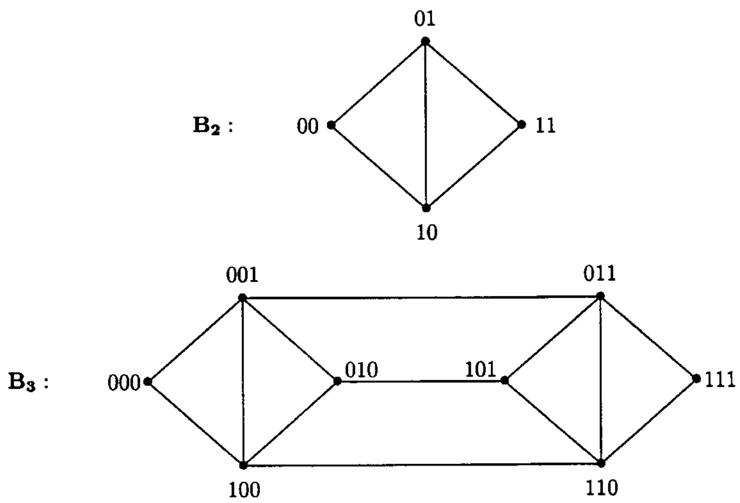

FIGURE 3 The two smallest deBruijn graphs.

of determining the set of all integers that are the weight of an edge in some graph of the family.

3 EDGE SUMS OF DEBRUIJN GRAPHS

The deBruijn graphs can also be defined in the following way. Given a positive integer n, the deBruijn graph $\mathbf{B}_n$ has 2^n nodes that can be labeled with $\mathbf{Z}_{2^n}$, the integers module 2^n. Its edges are in the form ij_k, where $j_k = 2i + k$ for some $k = 0$ or 1.

With this labeling on $\mathbf{B}_n$ we consider the network $N(\mathbf{B}_n)$ in which the weight of the edge ij_k is $w_{ij_k} = i + j_k$. Figure 4 shows an example.

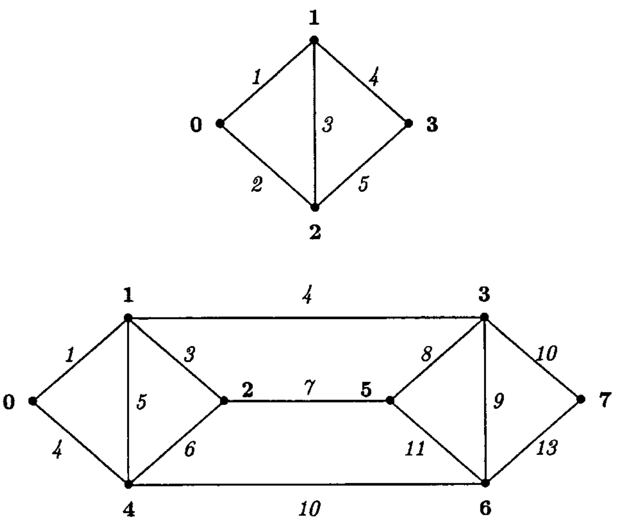

FIGURE 4 Graphs $\mathbf{B}_2$ and $\mathbf{B}_3$ with their edge sums.

To study the edge sums for the network $N(\mathbf{B}_n)$ we distinguish the following four classes of edges in $\mathbf{B}_n$:

(I) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$, and at least one term x_i differs from 0.
(II) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$.
(III) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$.
(IV) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$, where $x_i \in \{0, 1\}$ for all $i = 1, \dots, n-1$, and at least one term x_i differs from 1.

Let x be the integer whose binary representation is given by the sequence $x_1 \dots x_{n-1}$, then $0 \le x \le 2^{n-1} - 1$ and we can state:

(I) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$ has edge sum $3x$.
(II) $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$ has edge sum $3x + 1$.
(III) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$ has edge sum $3(x + 2^{n-1}) - 2^n$.
(IV) $\{1x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$ has edge sum $3(x + 2^{n-1}) + 1 - 2^n$.

Observe that in case (I) and (IV) we must ask $x \neq 0$ and $x \neq 2^{n-1} - 1$ respectively, because the graph has no loops.

PROPOSITION 3.1 *Let n be a positive integer and y an integer, $0 \le y \le 2^n - 2$. Then, $1 + y$ is an edge sum of $\mathbf{B}_n$ if and only if $2^{n+1} - 3 - y$ is an edge sum of $\mathbf{B}_n$.*

Proof We start proving that $1 + y$ is the edge sum of a type (I) edge, if and only if $2^{n+1} - 3 - y$ is the edge sum of a type (IV) edge. Indeed, $1 + y$ is the edge sum of an edge $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}0\}$ if and only if $1 + y = 3x$, where x is the integer represented by $x_1 \dots x_{n-1}$. Then, $y = 3x - 1$ and so $2^{n+1} - 3 - y = 2^{n+1} - 3 - (3x - 1)$, which is exactly the edge sum of the type (IV) edge $\{1(1 - x_1) \dots (1 - x_{n-1}), (1 - x_1) \dots (1 - x_{n-1})1\}$. In the same way we can show that $1 + y$ is the edge sum of a type (II) edge $\{0x_1 \dots x_{n-1}, x_1 \dots x_{n-1}1\}$ if and only if $2^{n+1} - 3 - y$ is the edge sum of a type (III) edge $\{1(1 - x_1) \dots (1 - x_{n-1}), (1 - x_1) \dots (1 - x_{n-1})0\}$, which is sufficient to conclude the proof. ■

If we look at the edge sums module 3,

- The sums for edges of type (I) are all integers congruent to $0 \pmod 3$, between 3 and $3(2^{n-1} - 1)$.
- The sums for edges of type (II) are all integers congruent to $1 \pmod 3$, between 1 and $3(2^{n-1} - 1) + 1$.

For the next two classes we need to consider two cases depending on n being even or odd, since $2^m \equiv_3 2$ if m is even and $2^m \equiv_3 1$ if m is odd.

– If n is even:
  - The sums for edges of type (III) are all integers congruent to $2 \pmod 3$ between 2^{n-1} and $3(2^{n-1} - 1) + 2^{n-1}$.
  - The sums for edges of type (IV) are all integers congruent to $0 \pmod 3$ between $1 + 2^{n-1}$ and $3(2^{n-1} - 1) + 2^{n-1} + 1$.
– If n is odd:
  - The sums for edges of type (III) are all integers congruent to $1 \pmod 3$ between 2^{n-1} and $3(2^{n-1} - 1) + 2^{n-1}$.
  - The sums for edges of type (IV) are all integers congruent to $2 \pmod 3$ between $1 + 2^{n-1}$ and $3(2^{n-1} - 1) + 2^{n-1} + 1$.

Now we are able to state the following two theorems that give a complete description of the edge sums of $\mathbf{B}_n$, the set L_n, depending on n even or odd.

THEOREM 3.2 *Let n be a positive even integer, then the edge sums set of $\mathbf{B}_n$ is*

$$L_n = \{m\colon m \equiv_3 0 \text{ or } m \equiv_3 1,\ 1 \le m \le 3(2^{n-1}-1)+1\}$$
$$\cup\ \{m\colon m \equiv_3 2 \text{ or } m \equiv_3 0,\ 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}.$$

Proof The edge sums of type (I) edges in $\mathbf{B}_n$ is the set $\{m\colon m \equiv_3 0, 1 \le m \le 3(2^{n-1}-1)+1\}$ and the edge sums of type (II) edges is the set $\{m\colon m \equiv_3 1, 1 \le m \le 3(2^{n-1}-1)+1\}$. Also if n is even the edge sum of type (III) edges in $\mathbf{B}_n$ is the set $\{m\colon m \equiv_3 2, 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}$ and the edge sums of type (IV) edges is the set $\{m\colon m \equiv_3 0, 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}$. The union of those four sets gives the set of edge sums of $\mathbf{B}_n$. ■

THEOREM 3.3 *Let n be a positive odd integer, then the edge sums set of $\mathbf{B}_n$ is*

$$L_n = \{m\colon m \equiv_3 0 \text{ or } m \equiv_3 1,\ 1 \le m \le 3(2^{n-1}-1)+1\}$$
$$\cup\ \{m\colon m \equiv_3 1 \text{ or } m \equiv_3 2,\ 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}.$$

Proof The edge sums of types (I) and (II) edges in $\mathbf{B}_n$, are respectively the sets $\{m\colon m \equiv_3 0, 1 \le m \le 3(2^{n-1}-1)+1\}$ and $\{m\colon m \equiv_3 1, 1 \le m \le 3(2^{n-1}-1)+1\}$. Moreover, since n is odd, the edge sum of types (III) and (IV) edges in $\mathbf{B}_n$ are the sets $\{m\colon m \equiv_3 1, 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}$ and $\{m\colon m \equiv_3 2, 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}$. The union of those four sets is the set of edge sums of $\mathbf{B}_n$. ■

Observe that the edge sums of $\mathbf{B}_n$ are integers in the interval $[1, 2^{n+1}-3]$. The next two corollaries provide a characterization of the integers in that interval that are not in the edge sums set of $\mathbf{B}_n$.

COROLLARY 3.4 *Let n be a positive even integer, the edge sums set of $\mathbf{B}(\mathbf{2}, \mathbf{n})$ contains all the integers between 1 and $2^{n+1}-3$, except the $2((2^{n-1}-2)/3)$ integers in the set:*

$$\{m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-2\} \cup \{2^n-2-m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-2\}.$$

Proof By Theorem 3.2 we know the set L_n of all the edge sums of $\mathbf{B}(\mathbf{2}, \mathbf{n})$. Therefore, $\{m\colon 1 \le m \le 2^{n+1}-3\} - L_n$ is the union of the sets $\{m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-2\}$ and $\{m\colon m \equiv_3 1, 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}$. This last set can be expressed as $\{2^n-2-m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-2\}$ using Proposition 3.1. ■

COROLLARY 3.5 *Let n be a positive odd integer, the edge sums set of $\mathbf{B}(\mathbf{2}, \mathbf{n})$ contains all integers between 1 and $2^{n+1}-3$, except the $2((2^{n-1}-1)/3)$ integers in the set:*

$$\{m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-1\} \cup \{2^n-3-m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-1\}.$$

Proof By Theorem 3.3 we know the set L_n of all the edge sums of $\mathbf{B}(\mathbf{2}, \mathbf{n})$. As in Corollary 3.4, $\{m\colon 1 \le m \le 2^{n+1}-3\} - L_n$ is union of the sets $\{m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-1\}$ and $\{2^n-3-m\colon m \equiv_3 2, 0 \le m \le 2^{n-1}-1\}$. ■

The *multiplicity* of an integer s as edge sum of a certain $\mathbf{B}_n$ is the number of edges whose edge sum is s.

THEOREM 3.6 *Let n be a positive even integer, the multiplicity of all the integers in the edge sum set of* $\mathbf{B}_n$ *is* 1, *except the* $((2^n-4)/3)+1$ *integers in the set* $\{m\colon m \equiv_3 0,\ 1+2^{n-1} \leq m \leq 3(2^{n-1}-1)\}$ *that have multiplicity* 2.

Proof The edge sums of type (II) edges have all multiplicity 1 because each integer in the set $\{m\colon m \equiv_3 1,\ 1 \leq m \leq 3(2^{n-1}-1)+1\}$ is the edge sum of one and only one type (II) edge, while other types of edges have their edge sums not congruent to $1 \pmod 3$. Similarly, the edge sums of type (III) edges also have multiplicity 1. However, types (II) and (IV) edges have edge sums congruent to $0 \pmod 3$ in the integer intervals $[1, 3(2^{n-1}-1)+1]$ and $[2^{n-1}, 3(2^{n-1}-1)+2^{n-1}]$, respectively. The intersection of those intervals, the set $\{m\colon m \equiv_3 0,\ 1+2^{n-1} \leq m \leq 3(2^{n-1}-1)\}$, gives the only edge sums with multiplicity 2. ■

THEOREM 3.7 *Let n be a positive odd integer, the multiplicity of all the integers in the edge sum set of* $\mathbf{B}_n$ *is* 1, *except the* $((2^n-2)/3)+1$ *integers in the set* $\{m\colon m \equiv_3 0,\ 2^{n-1} \leq m \leq 3(2^{n-1}-1)+1\}$ *that have multiplicity* 2.

Proof The edge sums of type (I) edges have multiplicity 1 because each integer in the set $\{m\colon m \equiv_3 0,\ 1 \leq m \leq 3(2^{n-1}-1)+1\}$ is the edge sum of exactly one type (I) edge, while other edges have their edge sums not congruent to $0 \pmod 3$. Analogously, the edge sums of type (IV) edges have multiplicity 1. Both, types (II) and (III) edges have edge sums congruent to $1 \pmod 3$ in the integer intervals $[1, 3(2^{n-1}-1)+1]$ and $[2^{n-1}, 3(2^{n-1}-1)+2^{n-1}]$, respectively. The intersection of those intervals, the set $\{m\colon m \equiv_3 0,\ 1+2^{n-1} \leq m \leq 3(2^{n-1}-1)\}$, contains exactly the edge sums with multiplicity 2. ■

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