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Diameter variability of cycles and tori

Jeng-Jung Wang [a,*], Tung-Yang Ho [b], Daniela Ferrero [c], Ting-Yi Sung [d]

[a] Department of Information Engineering, I-Shou University, Kaohsiung 84008, Taiwan, ROC
[b] Department of Industrial Engineering and Management, Ta Hwa Institute of Technology, Hsinchu 30740, Taiwan, ROC
[c] Department of Mathematics, Texas State University, San Marcos, TX 78666, USA
[d] Institute of Information Science, Academia Sinica, Taipei 115, Taiwan, ROC

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ABSTRACT

The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. Graham and Harary [N. Graham, F. Harary, Changing and unchanging the diameter of a hypercube, Discrete Applied Mathematics 37/38 (1992) 265–274] studied how the diameter of hypercubes can be affected by increasing and decreasing edges. They concerned whether the diameter is changed or remains unchanged when the edges are increased or decreased. In this paper, we modify three measures proposed in Graham and Harary (1992) to include the extent of the change of the diameter. Let $D^{-k}(G)$ is the least number of edges whose addition to G decreases the diameter by (at least) k, $D^{+0}(G)$ is the maximum number of edges whose deletion from G does not change the diameter, and $D^{+k}(G)$ is the least number of edges whose deletion from G increases the diameter by (at least) k. In this paper, we find the values of $D^{-k}(C_m)$, $D^{-1}(T_{m,n})$, $D^{-2}(T_{m,n})$, $D^{+1}(T_{m,n})$, and a lower bound for $D^{+0}(T_{m,n})$ where C_m be a cycle with m vertices, $T_{m,n}$ be a torus of size m by n.

1. Introduction

1.1. Basis

Let $G = (V, E)$ be a graph. $V(G)$ is the vertex set of G and $E(G) \subseteq V(G) \times V(G)$ is the edge set of G. Let u and v be two different vertices in a graph G. We say that u and v are *adjacent* if $(u, v) \in E(G)$. A path from u to v, delimited by $\langle u = x_0, x_1, x_2, \ldots, x_k = v\rangle$, is a sequence of distinct vertices such that x_i and x_{i+1} are adjacent for $0 \leqslant i \leqslant k-1$. The length of a path is the number of edges in it. The *distance* between u and v in G, denoted as $d_G(u, v)$, is the length of a shortest path joining them. The *diameter* of a graph G, denoted as $D(G)$, is the maximum distance between any two vertices.

An interconnection network connects the processors of a parallel and distributed system. The topology of an interconnection network for a parallel and distributed system can always be represented by a graph, where each vertex represents a processor and each edge represents a vertex-to-vertex communication link. Communication is a critical issue in the design of a parallel and distributed system. The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. To expedite communication, the smaller diameter is preferred. Besides, in order to increase the transmission rate and enhance the transmission reliability, it is also important to construct vertex-disjoint paths between any two vertices in a network [8,9].

* Corresponding author. Fax: +886 7 6578944.
E-mail address: jjwang@isu.edu.tw (J.-J. Wang).

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1.2. Definitions and properties

In fact, the diameter of a graph can be affected by adding or deleting edges [1,4,5]. For example, let m-cycle C_m be a graph with vertex set $\{0,1,2,\ldots,m-1\}$ and edge set $\{(i,i+1) \mid 0 \leqslant i \leqslant m-1\}$, where addition is in integer modulo m. It is known that $D(C_m) = \lfloor m/2 \rfloor$ [10]. Let P_m be a graph with vertex set $\{0,1,2,\ldots,m-1\}$ and edge set $\{(i,i+1) \mid 0 \leqslant i \leqslant m-2\}$, it is known that $D(P_m) = m-1$ [10]. It is easy to check that deleting any edge renders C_m to a path of m vertices and then the diameter increases to $m-1$. On the other hand, a cycle C_m can be obtained by adding the edge $(0,m)$ to path P_m. The diameter decrease to $\lfloor m/2 \rfloor$.

Let k be an arbitrary positive integer. The diameter variability arising from change of edges of graph G is defined as follows:

$D^{-k}(G)$: the least number of edges whose addition to G decreases the diameter by (at least) k;
$D^{+0}(G)$: the maximum number of edges whose deletion from G does not change the diameter;
$D^{+k}(G)$: the least number of edges whose deletion from G increases the diameter by (at least) k.

For example, $D^{-1}(P_m) = D^{-2}(P_m) = \cdots = D^{-(m-1-\lfloor m/2 \rfloor)}(P_m) = 1$ and $D^{+1}(C_m) = D^{+2}(C_m) = \cdots = D^{+(m-1-\lfloor m/2 \rfloor)}(C_m) = 1$. The n-dimensional hypercube, Q_n, consists of all n-bit binary strings as its vertices. Two vertices are adjacent if they differ only in one bit position. Graham and Harary [4] considered changing the diameter without considering the extent of the change, i.e., they considered $D^{-1}(G)$ and $D^{+1}(G)$. They showed that $D^{-1}(Q_n) = 2$, $D^{+1}(Q_n) = n-1$ and $D^{+0}(Q_n) \geqslant (n-3)2^{n-1}+2$. Bouabdallah et al. [1] improved the lower bound of $D^{+0}(Q_n)$ and furthermore gave an upper bound, $(n-2)2^{n-1} - \binom{n}{\lfloor n/2 \rfloor} + 2 \leqslant D^{+0}(Q_n) \leqslant (n-2)2^{n-1} - \lceil (2^n-1)/(2n-1) \rceil + 1$.

The *edge connectivity* of a graph G, denoted by $\kappa\prime(G)$, is the least number of edges whose deletion disconnects G. Clearly, $D^{+i}(G) \leqslant \kappa\prime(G)$ for all i. The diameter of a complete graph equals one. Given a graph G, for $1 \leqslant i \leqslant D(G)-1$, $D^{-i}(G)$ is no more than the number of edges needed to be added to G to make G be a complete graph. Note that $0 \leqslant D^{-i}(G) \leqslant D^{-j}(G)$ and $0 \leqslant D^{+i}(G) \leqslant D^{+j}(G)$ if $i \leqslant j$. For convenience, we write $D^{-1}(G)$ and $D^{+1}(G)$ as $D^{-}(G)$ and $D^{+}(G)$, respectively, throughout the paper.

In this paper, we study the change of diameter arising from the change of edges in cycles and tori. A cycle is the topological structure of a ring network. It is one of the most common, simple and useful interconnection networks [10]. More properties, performances, and details about cycles can be found in [2,7,10]. A torus, denoted as $T_{m,n}$, is a graph obtained by the Cartesian product of cycles C_m and C_n. It is a 2-dimension array with wraparound wires in the rows and columns. The number of edges of $T_{m,n}$ is $2mn$ and it is known that the diameter of $T_{m,n}$ is $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$ [7]. For more details on properties and performances, such as throughput, latency, and path diversity, see [2].

2. Changing the diameter of cycles

Since deleting any edge renders C_m to a path P_m of m vertices, the diameter increases to $m-1$. It follows that

$$D^{+0}(C_m) = 0 \quad \text{and} \quad D^{+k}(C_m) = 1 \text{ for } 1 \leqslant k \leqslant m-1-\lfloor m/2 \rfloor.$$

To find $D^{-k}(G)$, it suffices to consider adding edges to reduce the distance of all of farthest neighbors. Given a vertex v in graph G, vertex u is called a *farthest neighbor* of v, denoted as v^f, if $d_G(u,v) = D(G)$. Given a vertex i in C_m with $0 \leqslant i \leqslant \lfloor m/2 \rfloor$, farthest neighbor of i is $i + \lfloor m/2 \rfloor$ if m is even; $i + \lceil m/2 \rceil$ or $i + \lfloor m/2 \rfloor$ if m is odd.

Lemma 1. $D^{-}(C_m) \geqslant 2$.

Proof. Suppose $D^{-}(C_m) = 1$. We can assume without loss of generality that adding an edge $e = (0,l)$ with $2 \leqslant l \leqslant \lfloor m/2 \rfloor$ reduces the diameter. Let $v = \lfloor l/2 \rfloor$ and $\bar{v} = v + \lfloor m/2 \rfloor$. Since the distance from v to $\bar{v}$ is reduced, the edge e must be used in the shortest path from v to $\bar{v}$. It follows that

$$\begin{aligned} d_{C_m}(v,\bar{v}) &= \min\{d_{C_m}(v,0) + 1 + d_{C_m}(l,\bar{v}), d_{C_m}(v,l) + 1 + d_{C_m}(0,\bar{v})\} \\ &= \min\{\lfloor l/2 \rfloor + 1 + (\lfloor l/2 \rfloor + \lfloor m/2 \rfloor - l), (l - \lfloor l/2 \rfloor) + 1 + (m - \lfloor l/2 \rfloor - \lfloor m/2 \rfloor)\} \geqslant \lfloor m/2 \rfloor, \end{aligned}$$

which is a contradiction. Therefore $D^{-}(C_m) \geqslant 2$. □

Now, we consider the case of adding two edges to cycle C_m. Let $D^{*}(C_m)$ denote the minimum diameter among those graphs obtained by adding two edges to C_m.

Lemma 2. *Let* $m \geqslant 5$. *Then,* $D^{*}(C_m) = \begin{cases} \lfloor m/4 \rfloor + 1 & \text{if } m \equiv 0,1,2 \textbf{ mod } 4, \\ \lfloor m/4 \rfloor + 2 & \text{if } m \equiv 3 \textbf{ mod } 4. \end{cases}$

Proof. Assume that we are adding two intersecting edges $e_1 = (0, l_1 + l_2)$ and $e_2 = (l_1, l_1 + l_2 + l_3)$ to the cycle. Let G denote the resulting graph. The four endpoints of these two edges divide the cycle into four paths Q_1, Q_2, Q_3 and Q_4 of length l_1, l_2, l_3 and l_4, respectively, where $\sum_{i=1}^{4} l_i = m$. The four paths Q_1, Q_2, Q_3 and Q_4 are depicted in Fig. 1.

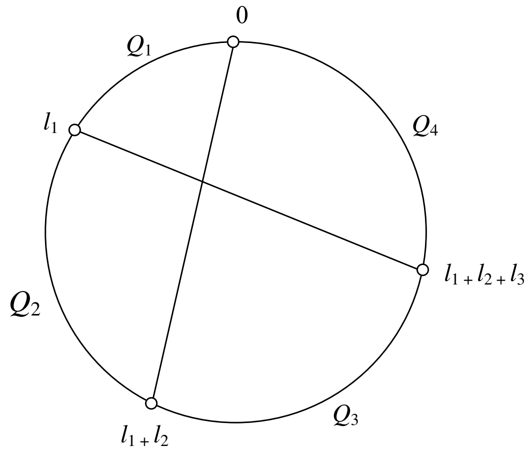

Fig. 1. The two intersecting edges $e_1=(0,l_1+l_2)$ and $e_2=(l_1,l_1+l_2+l_3)$ divide the cycle into four paths Q_1,Q_2,Q_3 and Q_4 of length l_1,l_2,l_3 and l_4, respectively, where $\sum_{i=1}^4 l_i=m$.

The longest shortest paths among all pairs of vertices are contained in the following six cycles Y_i for $1\leqslant i\leqslant 6$ given by

$$Q_1\cup Q_2\cup e_1,\quad Q_2\cup Q_3\cup e_2,\quad Q_3\cup Q_4\cup e_1,$$
$$Q_4\cup Q_1\cup e_2,\quad Q_1\cup Q_3\cup\{e_1,e_2\},\quad Q_2\cup Q_4\cup\{e_1,e_2\},$$

respectively. Then, $D(G)=\max_{1\leqslant i\leqslant 6}\{\lfloor\frac{|Y_i|}{2}\rfloor\}$, where $|Y_i|$ denotes the number of edges in cycle Y_i. It follows that $D(G)$ can be minimized when $|l_i-l_j|\leqslant 1$ for all $1\leqslant i\neq j\leqslant 4$. To be specific, when $m=2$ **mod** 4, it follows that two longer paths of length $\lceil m/4\rceil$ are adjacent, e.g., $l_1=l_4=\lfloor m/4\rfloor$ (or $\lceil m/4\rceil$) and $l_2=l_3=\lceil m/4\rceil$ (or $\lfloor m/4\rfloor$) so that the longest cycle among Y_j has a length of $\lfloor m/2\rfloor+2$. Therefore, for $m\equiv 2$ **mod** 4, $D^*(C_m)=\lfloor m/4\rfloor+1$ which can be achieved by adding edges $(0,\lfloor m/2\rfloor)$ and $(\lfloor m/4\rfloor,\lceil 3m/4\rceil)$ into C_m. When $m\equiv 0,1,3$ **mod** 4, we can arbitrarily assign $\lfloor m/4\rfloor$ and $\lceil m/4\rceil$ to l_i, say $(0,\lfloor m/2\rfloor)$ and $(\lfloor m/4\rfloor,\lfloor 3m/4\rfloor)$. Thus,

$$D^*(C_m)=\begin{cases}\lfloor m/4\rfloor+1 & \text{if } m\equiv 0,1,2 \textbf{ mod } 4,\\ \lfloor m/4\rfloor+2 & \text{if } m\equiv 3 \textbf{ mod } 4.\end{cases}$$

Suppose that we can also reduce the diameter to $\lfloor m/4\rfloor+1$ or $\lfloor m/4\rfloor+2$, depending on m, by adding two non-intersecting edges. These two non-intersecting edges partition C_m into four paths of length h_1,h_2,h_3, and h_4, where $\sum_{i=1}^4 h_i=m$. That is, the two non-intersecting edges are given by $(0,h_1)$ and $(h_1+h_2,h_1+h_2+h_3)$. We can assume without loss of generality that where $2\leqslant h_1\leqslant h_3$ and $0\leqslant h_2\leqslant h_4$. Let $v_1=\lfloor h_1/2\rfloor$, $v_2=h_1+h_2+\lfloor h_3/2\rfloor$ and $v_3=h_1+h_2+h_3+\lfloor h_4/2\rfloor$. It follows that

$$d_{C_m}(v_1,v_2)=\lceil h_1/2\rceil+h_2+\lfloor h_3/2\rfloor,\, d_{C_m}(v_1,v_3)=\lfloor h_1/2\rfloor+\lceil h_4/2\rceil,\, d_{C_m}(v_2,v_3)=\lceil h_3/2\rceil+\lfloor h_4/2\rfloor$$

and moreover, $d_{C_m}(v_1,v_2)+d_{C_m}(v_1,v_3)+d_{C_m}(v_2,v_3)=m$. Therefore, $\max\{d_{C_m}(v_1,v_2),\, d_{C_m}(v_1,v_3),d_{C_m}(v_2,v_3)\}\geqslant\lceil m/3\rceil$, which is a contradiction. Hence, the lemma follows. □

Based on the above proof, the two edges e_1 and e_2 added to C_m achieving $D^*(C_m)$ are given by

$$\begin{aligned}&e_1=(0,\lfloor m/2\rfloor)\quad\text{and}\quad e_2=(\lfloor m/4\rfloor,\lceil 3m/4\rceil)\text{ for } m\equiv 2 \textbf{ mod } 4,\\ &e_1=(0,\lfloor m/2\rfloor)\quad\text{and}\quad e_2=(\lfloor m/4\rfloor,\lfloor 3m/4\rfloor)\quad\text{for } m\equiv 0,1,3 \textbf{ mod } 4.\end{aligned}\tag{1}$$

By Lemmas 1 and 2 we have the following theorem.

Theorem 1. *$D^{-k}(C_m)=2$ for all $m\geqslant 8$ and $1\leqslant k\leqslant\lfloor m/2\rfloor-D^*(C_m)$.*

When $m=7$, adding two edges to C_m does not decrease the diameter, hence, $D^-(C_7)\geqslant 3$. In fact, it is easy to show that $D^-(C_7)=3$. It can be verified that $D^*(C_6)=2$ and then $D^-(C_6)=2$. To find $D^-(C_m)$ for $m\leqslant 5$ and $D^{-2}(C_m)$ for $m=6,7$, we need to add edges for C_m to become a complete graph.

One may ask whether the diameter can be further reduced by adding few more edges. However, we cannot reduce the diameter by using similar idea as in the proof of Lemma 2 to add three or four intersecting edges which equi-partition a cycle into six or eight paths, i.e., the length of these paths differing at most one.

Nonetheless, we can further reduce the diameter of C_m to approximately $\lfloor m/5\rfloor$, $\lfloor m/6\rfloor$ and $\lfloor m/8\rfloor$ by adding five edges, six edges and twelve edges, that equi-partition the cycle into five paths, six paths and eight paths. Let the resultant graphs be called $C_{m;5}, C_{m;6}$, and $C_{m;8}$. To be specific, let $r=5,6,8$. For $m\equiv 2$ **mod** r we particularly put two longer paths of length $\lceil m/r\rceil$ to be adjacent. It follows that

$$D(C_{m;r}) = \begin{cases} \lfloor m/r \rfloor + 1 & m \equiv 0,1,2 \ \mathbf{mod}\ r, \\ \lfloor m/r \rfloor + 2 & m \equiv 3,4,\ldots,r-1 \ \mathbf{mod}\ r \end{cases}$$

for $r = 5, 6, 8$. And yet it is unknown whether these constructions are optimal.

3. Changing the diameter of tori

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G = (V, E)$ of G_1 and G_2, denoted by $G = G_1 \times G_2$, is given by $V = V_1 \times V_2$, and $E = \{(u_1u_2, v_1v_2) \mid u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1\}$. Let $T_{m,n}$ be a 2-dimensional torus of size m by n which can be treated as the cartesian product of C_m and C_n, i.e., $C_m \times C_n$. We assume without loss of generality that $m \geqslant n \geqslant 3$ throughout this section. Each vertex v in $T_{m,n}$ is represented by $v = (v_1, v_2)$ where $0 \leqslant v_1 \leqslant m-1$ and $0 \leqslant v_2 \leqslant n-1$. We also use $v(1)$ and $v(2)$ to denote the first and the second coordinates of v, respectively, i.e., $v(1) = v_1$ and $v(2) = v_2$. For ease of exposition in this section, each edge of $T_{m,n}$ or path are delimited by $\langle$ and $\rangle$. We assume $m \geqslant n$ throughout this section.

Let $V^f(v)$ denote the set of the farthest neighbors of v, then

$$V^f(v) = \{(v(1) + \lfloor m/2 \rfloor, v(2) + \lfloor n/2 \rfloor), (v(1) + \lfloor m/2 \rfloor, v(2) + \lceil n/2 \rceil), (v(1) + \lceil m/2 \rceil, v(2) + \lfloor n/2 \rfloor), (v(1) + \lceil m/2 \rceil, v(2) + \lceil n/2 \rceil)\}.$$

Let $P(i, C_n)$ and $P(C_m, j)$ denote paths in $T_{m,n}$ defined as follows:

$$P(i, C_n) = \langle (i, 0), (i, 1), \ldots, (i, n-1) \rangle,$$
$$P(C_m, j) = \langle (0, j), (1, j), \ldots, (m-1, j) \rangle.$$

We use $C(i, C_n)$ and $C(C_m, j)$ to denote cycles given as follows:

$$C(i, C_n) = \langle (i, 0), (i, 1), \ldots, (i, n-1), (i, 0) \rangle,$$
$$C(C_m, j) = \langle (0, j), (1, j), \ldots, (m-1, j), (0, j) \rangle.$$

Note that $C(i, C_n)$ and $C(C_m, j)$ are cycles of length n and m, respectively. Let u and v be two distinct vertices in $T_{m,n}$. We use $d_{C_m}(u, v)$ to denote the distance from u_1 to v_1 in a cycle $C(C_m, j)$ with or without some additional edges as specified from the context without ambiguity. Similarly, we denote $d_{C_n}(u, v)$.

3.1. Finding $D^{-k}(T_{m,n})$

To find $D^-(T_{m,n})$, we first find a lower bound for it.

Lemma 3. $D^-(T_{m,n}) \geqslant 2$.

Proof. We show that the diameter of $T_{m,n}$ can not be reduced by adding one edge. It suffices to show the case of m, n even. Suppose that the diameter is reduced by adding an edge e. We can assume without loss of generality that $e = \langle (0,0), (y_1, y_2) \rangle$. We consider three possibilities of the edge e. First, let $y_2 = 0$. Let $v = (\lfloor y_1/2 \rfloor, 0)$ and v^f be a farthest neighbor of v. Then $d_{T_{m,n}}(v, v^f) = d_{C_m}(v(1), v^f(1)) + n/2$. It follows from Lemma 1 that $d_{C_m}(v(1), v^f(1))$ is unchanged, i.e., $d_{C_m}(v(1), v^f(1)) = m/2$. Therefore, $d_{T_{m,n}}(v, v^f) = \lfloor n/2 \rfloor + \lfloor m/2 \rfloor$, which is a contradiction. Consequently, e can not have $y_2 = 0$. Similarly, we can prove that e cannot have $y_1 = 0$.

Finally, let $y_1 \neq 0$ and $y_2 \neq 0$. Furthermore, we can assume without loss of generality that $0 \leqslant y_1 \leqslant m/2$ and $0 \leqslant y_2 \leqslant n/2$. For vertices v satisfying $v(1) > m/2$ and $v(2) \leqslant n/2$, then e is not in any shortest path from v to v^f, i.e., $d_{T_{m,n}}(v, v^f) = \lfloor n/2 \rfloor + \lfloor m/2 \rfloor$, which is a contradiction. Thus, the lemma follows. □

Theorem 2. $D^-(T_{m,n}) = 2$ *for* $m \geqslant 12$.

Proof. Let G be the graph obtained by adding the following two edges to $T_{m,n}$:

$$e_1 = \langle (0, \lfloor n/2 \rfloor),\ (\lfloor m/2 \rfloor, \lfloor n/2 \rfloor) \rangle,$$
$$e_2 = \begin{cases} \langle\ (\lfloor m/4 \rfloor, \lfloor n/2 \rfloor),\ (\lceil 3m/4 \rceil, \lfloor n/2 \rfloor)\ \rangle & \text{for } m \equiv 2 \ \mathbf{mod}\ 4, \\ \langle\ (\lfloor m/4 \rfloor, \lfloor n/2 \rfloor),\ (\lfloor 3m/4 \rfloor, \lfloor n/2 \rfloor)\ \rangle & \text{for } m \equiv 0,1,3 \ \mathbf{mod}\ 4. \end{cases} \quad (2)$$

Note that e_1 and e_2 are obtained from projecting the two edges specified in (1) to $C(C_m, \lfloor n/2 \rfloor)$. Let v be a vertex of G and v^f be a farthest neighbor of v. It suffices to show that in G we have $d_{T_{m,n}}(v, v^f) \leqslant \lfloor n/2 \rfloor + \lfloor m/2 \rfloor - 1$ for all vertices v.

Note that the shortest paths from v to any v^f takes at most $\lceil n/2 \rceil$ steps to change $v(2)$ to $v^f(2)$. We construct a path from v to v^f via e_1 or e_2, depending on $d_{C_m}(v(1), v^f(1))$. It follows that

$$d_{T_{m,n}}(v, v^f) \ \leqslant\ d_{C_m}(v(1), v^f(1)) + \lceil n/2 \rceil. \quad (3)$$

It follows from Lemma 2 that $d_{C_m}(v(1), v^f(1)) \leqslant \lfloor m/2 \rfloor - 2$ when $m \geqslant 12$. Thus $d_{T_{m,n}}(v, v^f) \leqslant \lfloor m/2 \rfloor + \lfloor n/2 \rfloor - 1$, and the theorem follows. □

It follows from (3) that $D^-(T_{m,n}) = 2$ also holds for $m = 10$ and n odd and for $m = 9, 11$ and n even.

Adding the two edges specified in (2) to $T_{m,n}$, we have $d_{C_m}(v(1), v^f(1)) \leqslant \lfloor m/2 \rfloor - 3$ for all vertices v when $m \geqslant 14$ and $m \neq 15$. Therefore, following from (3), adding these two edges can reduce the diameter by two for $m \geqslant 14$ and $m \neq 15$. Thus we have the following theorem.

Theorem 3. $D^{-2}(T_{m,n}) = 2$ *for* $m \geqslant 14$ *and* $m \neq 15$.

For m, n even and $m \geqslant 4$, we can reduce the diameter by one by adding two edges, $\langle (m/2-1, n/2-1),\ (m/2, n/2) \rangle$ and $\langle (m/2-1, n/2),\ (m/2, n/2-1) \rangle$, as shown in Fig. 2.

Note that the subgraph in $T_{m,n}$ induced by the vertex set $\{(v(1)+i, v(2)+j) \mid 0 \leqslant i \leqslant m/2, 0 \leqslant j \leqslant n/2\}$ is a mesh of size $m/2+1$ by $n/2+1$. The diameter of a mesh of size $m/2+1$ by $n/2+1$ is $m/2+n/2$. For any vertex v in $T_{m,n}$, v and its farthest neighbor v^f are contained in a induced mesh of size $m/2+1$ by $n/2+1$. Besides, any induced mesh of size $m/2+1$ by $n/2+1$ in $T_{m,n}$ contains the vertices $(m/2-1, n/2-1)$, $(m/2, n/2)$, $(m/2-1, n/2)$, and $(m/2, n/2-1)\}$, thus the distance between v and v^f is reduced by one when we add the edges $\langle (m/2-1, n/2-1),\ (m/2, n/2) \rangle$ and $\langle (m/2-1, n/2),\ (m/2, n/2-1) \rangle$ to $T_{m,n}$.

3.2. Finding $D^{+0}(T_{m,n})$

By the definition of $D^{+0}(T_{m,n})$, we construct a spanning subgraph $S_{m,n}$ of $T_{m,n}$ with diameter $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$ instead of deleting edges from $T_{m,n}$. We first construct a spanning tree $\Gamma_{m,n}$ of $T_{m,n}$ and then add pertinent edges to $\Gamma_{m,n}$ to generate $S_{m,n}$. The construction is described as follows:

1. Start with a spanning tree of $T_{m,n}$ given by $\Gamma_{m,n} = \bigcup_{j=0}^{n-1} P(C_m, j) \cup P(\lfloor m/2 \rfloor, C_n)$, as shown in Fig. 3(a), which has a diameter $m+n-2$ for m odd, and $m+n-1$ for m even.
2. Add the edge $\langle (\lfloor m/2 \rfloor, 0), (\lfloor m/2 \rfloor, n-1) \rangle$ to $P(\lfloor m/2 \rfloor, C_n)$ to form a cycle $C(\lfloor m/2 \rfloor, C_n)$, since otherwise the distance from $(0, n-1)$ to $(\lfloor m/2 \rfloor, 0)$ is $\lfloor m/2 \rfloor + n - 1 > \lfloor m/2 \rfloor + \lfloor n/2 \rfloor$.
3. Add an edge $\langle (0,j), (0,j+1) \rangle$ for all $0 \leqslant j \leqslant n-2$, since otherwise the distance from $(0,j)$ to $(0,j+1)$ is $2\lfloor m/2 \rfloor + 1 > \lfloor m/2 \rfloor + \lfloor n/2 \rfloor$. Thus, the path $P(0, C_n)$ is added.
4. Add an edge $\langle (0,j), (m-1,j) \rangle$ to $P(C_m, j)$ to form a cycle $C(C_m, j)$ for all $2 \leqslant j \leqslant n-1$, since otherwise the distance from $(0, j-2)$ to $(m-1, j)$ is $m+1 > \lfloor m/2 \rfloor + \lfloor n/2 \rfloor$. Since the distance from $(0, j+2)$ to $(m-1, j)$ for $j = 0, 1$ is $m+1 > \lfloor m/2 \rfloor + \lfloor n/2 \rfloor$, we add two edges $\langle (0,0), (m-1,0) \rangle$ and $\langle (0,1), (m-1,1) \rangle$. That is, $C(C_m, j)$ for all $0 \leqslant j \leqslant n-1$ are formed.

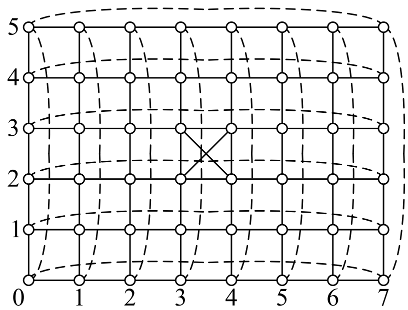

Fig. 2. Adding two edges in $T_{8,6}$ to reduce the diameter by one.

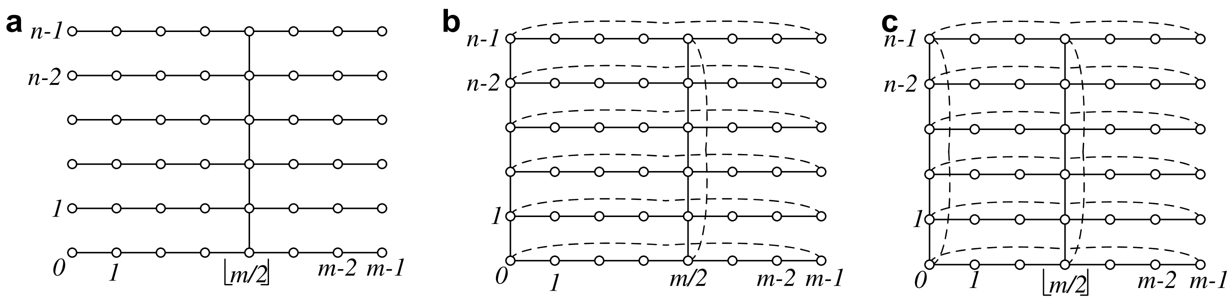

Fig. 3. (a) $\Gamma_{m,n}$, (b) $S_{m,n}$ for m even, and (c) $S_{m,n}$ for m odd.

5. When m is odd, we add the edge $\langle(0,0),(0,n-1)\rangle$ to $P(0,C_n)$ to form $C(0,C_n)$, since otherwise the distance from $(i,0)$ to $(i-\lfloor m/2\rfloor-1,\lfloor n/2\rfloor+1)$ for $i \geqslant \lfloor m/2\rfloor+1$ is $\lfloor m/2\rfloor+\lfloor n/2\rfloor+1$.

In summary, the spanning subgraph $S_{m,n}$ is given by

$$S_{m,n} = \begin{cases} \bigcup_{j=0}^{n-1} C(C_m,j) \cup C(\lfloor m/2\rfloor, C_n) \cup C(0,C_n) & \text{for } m \text{ odd}, \\ \bigcup_{j=0}^{n-1} C(C_m,j) \cup C(\lfloor m/2\rfloor, C_n) \cup P(0,C_n) & \text{for } m \text{ even}, \end{cases}$$

as shown in Fig. 3b and c. It is observed that the number of edges in $S_{m,n}$ is $mn+2n-1$ for m even, and $mn+2n$ for m odd.

Lemma 4. $D(S_{m,n}) = \lfloor m/2\rfloor + \lfloor n/2\rfloor$.

Proof. First consider m even. Let $x=(x_1,x_2)$ and $y=(y_1,y_2)$ be two distinct vertices. We distinguish the following cases of x and y.

Case (1): $x_1, y_1 \leqslant m/2$.
We can assume without loss of generality that $x_2 \leqslant y_2$ after relabeling of the second coordinate. Then, $d_{S_{m,n}}(x,y) = \min\{x_1+y_1+y_2-x_2,\ m-(x_1+y_1)+d_{C_n}(x_2,y_2)\}$, where $d_{C_n}(x_2,y_2) \leqslant \lfloor n/2\rfloor$ since the cycle $C(m/2,C_n)$ can be traversed.
If $x_1+y_1 \geqslant m/2$, then $d_{S_{m,n}}(x,y) = m-(x_1+y_1)+d_{C_n}(x_2,y_2) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.
If $x_1+y_1 < m/2$ and $x_1+y_1+y_2-x_2 \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$, then $d_{S_{m,n}}(x,y) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.
If $x_1+y_1 < m/2$ and $x_1+y_1+y_2-x_2 > \lfloor m/2\rfloor+\lfloor n/2\rfloor$, then $y_2-x_2 > \lfloor n/2\rfloor$, i.e., $d_{C_n}(x_2,y_2) = n-(y_2-x_2)$. It follows that $d_{S_{m,n}}(x,y) = m-(x_1+y_1)+ d_{C_n}(x_2,y_2) < m+n- \lfloor m/2\rfloor+\lfloor n/2\rfloor = m/2+\lceil n/2\rceil$, i.e., $d_{S_{m,n}}(x,y) \leqslant \lfloor m/2\rfloor+ \lfloor n/2\rfloor$.

Case (2): $x_1 \leqslant m/2$ and $y_1 \geqslant m/2$.
We can assume without loss of generality that $y_2 \geqslant x_2$. Then, $d_{S_{m,n}}(x,y) = \min\{d_{C_m}(x_1,y_1)+y_2-x_2,\ y_1-x_1+ d_{C_n}(x_2,y_2)\}$, where $d_{C_n}(x_1,y_1) \leqslant m/2$ and $d_{C_m}(x_2,y_2) \leqslant \lfloor n/2\rfloor$.
If $y_2-x_2 \leqslant \lfloor n/2\rfloor$, then $d_{S_{m,n}}(x,y) = d_{C_m}(x_1,y_1)+ y_2-x_2 \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.
If $y_2-x_2 > \lfloor n/2\rfloor$ and $y_1-x_1 \leqslant m/2$, then $d_{S_{m,n}}(x,y) = y_1-x_1+d_{C_n}(x_2,y_2) \leqslant \lfloor m/2\rfloor+ \lfloor n/2\rfloor$.
If $y_2-x_2 > \lfloor n/2\rfloor$, $y_1-x_1 > m/2$ and $y_1-x_1-m/2 \geqslant y_2-x_2-\lfloor n/2\rfloor$, then $y_1-x_1-(y_2-x_2) \geqslant m/2-\lfloor n/2\rfloor$. It follows that

$$d_{S_{m,n}}(x,y) = d_{C_m}(x_1,y_1)+y_2-x_2 = m-(y_1-x_1)+y_2-x_2 \leqslant \lfloor n/2\rfloor - m/2 + m = \lfloor m/2\rfloor+\lfloor n/2\rfloor.$$

If $y_2-x_2 > \lfloor n/2\rfloor$, $y_1-x_1 > m/2$ and $y_1-x_1-m/2 < y_2-x_2-\lfloor n/2\rfloor$, then $y_1-x_1-(y_2-x_2) < m/2-\lfloor n/2\rfloor$. It follows that

$$d_{S_{m,n}}(x,y) = (y_1-x_1)+d_{C_n}(x_2,y_2) = y_1-x_1+n-(y_2-x_2) < m/2-\lfloor n/2\rfloor+n,$$

i.e., $d_{S_{m,n}}(x,y) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.

Case (3): $x_1 \geqslant m/2$ and $y_1 \leqslant m/2$.
It can be similarly proved as Case (2).

Case (4): $x_1, y_1 \geqslant m/2$.
We can assume without loss of generality that $y_2 \geqslant x_2$. Then, $d_{S_{m,n}}(x,y) = \min\{2m-(x_1+y_1)+y_2-x_2,\ x_1+y_1- m+d_{C_n}(x_2,y_2)\}$, where $d_{C_n}(x_2,y_2) \leqslant \lfloor n/2\rfloor$.

If $x_1+y_1 \leqslant 3m/2$, then $d_{S_{m,n}}(x,y) = x_1+y_1-m+d_{C_n}(x_2,y_2) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.
If $x_1+y_1 > 3m/2$, and $y_2-x_2 \leqslant \lfloor n/2\rfloor$, then $d_{S_{m,n}}(x,y) = 2m-(x_1+y_1)+y_2-x_2 \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.
If $x_1+y_1 > 3m/2$, $y_2-x_2 > \lfloor n/2\rfloor$ and $x_1+y_1-3m/2 \geqslant y_2-x_2-\lfloor n/2\rfloor$, then $x_1+y_1-(y_2-x_2) \geqslant 3m/2-\lfloor n/2\rfloor$. It follows that

$$d_{S_{m,n}}(x,y) = 2m-(x_1+y_1)+y_2-x_2 \leqslant m/2+\lfloor n/2\rfloor = \lfloor m/2\rfloor+\lfloor n/2\rfloor.$$

If $x_1+y_1 > 3m/2$, $y_2-x_2 > \lfloor n/2\rfloor$ and $x_1+y_1-3m/2 < y_2-x_2-\lfloor n/2\rfloor$, then $x_1+y_1-(y_2-x_2) < 3m/2-\lfloor n/2\rfloor$. It follows that

$$d_{S_{m,n}}(x,y) = x_1+y_1-m+n-(y_2-x_2) < 3m/2-\lfloor n/2\rfloor-m+n = m/2+\lceil n/2\rceil,$$

i.e., $d_{S_{m,n}}(x,y) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$.

Hence, $D(S_{m,n}) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$ when m is even. Similarly, we can show that $D(S_{m,n}) \leqslant \lfloor m/2\rfloor+\lfloor n/2\rfloor$ for m odd, where $S_{m,n}$ contains one more edge $\langle(0,0),(0,n-1)\rangle$ than it for m even. Thus the lemma follows. □

A lower bound for $D^{+0}(T_{m,n})$ is immediately obtained.

Lemma 5. $D^{+0}(T_{m,n}) \geqslant \begin{cases} mn-2n+1 & \text{if } m \text{ is even}, \\ mn-2n & \text{if } m \text{ is odd}. \end{cases}$

Let $D^{*}(ST_{m,n}) = \min\{D(T) \mid T \text{ is any spanning tree of } T_{m,n}\}$.

Lemma 6. $D^{*}(ST_{m,n}) = \begin{cases} m+n-1 & \text{if } m \text{ and } n \text{ are even}, \\ m+n-2 & \text{otherwise}. \end{cases}$

Proof. We construct a spanning tree other than $\Gamma_{m,n}$ as follows:

$$\Gamma'_{m,n} = \bigcup_{i=0}^{m-1} P(i, C_n) \cup P(C_m, \lfloor n/2 \rfloor).$$

The upper bound of $D^{*}(ST_{m,n})$ is provided by the spanning trees $\Gamma_{m,n}$ for m odd and $\Gamma'_{m,n}$ for m even.

To show the lower bound, we first consider m, n even. Suppose that the diameter of a spanning tree is less than or equal to $m+n-2$. Then all of the vertices are at distance $m/2 + n/2 - 1$ or less from its center which is a contradiction since the eccentricity of each vertex in $T_{m,n}$ is $m/2 + n/2$. (The eccentricity of a vertex v in a graph G is defined as $\max_{u \in V(G)} d(v,u)$. A center of graph G is a vertex with smallest eccentricity.)

Second, consider that one of m and n is odd, say, m is odd. Suppose that the diameter of a spanning tree is less than or equal to $m+n-3$. Then all of the vertices are at distance $\lfloor m/2 \rfloor + n/2 - 1 = \lceil m/2 \rceil + n/2 - 2$ or less from its center which is a contradiction since the eccentricity of $T_{m,n}$ is $\lfloor m/2 \rfloor + n/2$. Similarly, we can show the case that n is odd.

Third, consider that both m and n are odd. Removing an edge from C_m and from C_n yield a path of even length $m-1$ and $n-1$, respectively. It follows that the center of any spanning tree has a radius at least $(m-1)/2 + (n-1)/2$. Therefore, the minimum diameter of any spanning tree is at least two times of the radius, i.e., $m+n-2$.

Hence, the lemma follows. □

Our constructed spanning trees $\Gamma_{m,n}$ and $\Gamma'_{m,n}$ can achieve the $D^{*}(ST_{m,n})$ for m odd and for m even, respectively. Note that $\Gamma_{m,n}$ also achieves the $D^{*}(ST_{m,n})$ when both m and n are even. Our construction of $S_{m,n}$ starts with a spanning tree with diameter $D^{*}(ST_{m,n})$ if m is odd. On the other hand, if m is even, we start with $\Gamma'_{m,n}$ to construct a spanning subgraph similar to $S_{m,n}$ as follows:

$$S'_{m,n} = \bigcup_{i=0}^{m-1} C(i, C_n) \cup C(C_m, \lfloor n/2 \rfloor) \cup P(C_m, 0),$$

which also has a diameter $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$. We note $\mid E(S'_{m,n}) \mid = mn + 2m - 1 \geqslant \mid E(S_{m,n}) \mid = mn + 2n - 1$.

3.3. Finding $D^{+1}(T_{m,n})$

We have the following lemma.

Theorem 4. $D^{+}(T_{m,n}) = \begin{cases} 2 & \text{when } m \text{ and } n \text{ are odd number}, \\ 4 & \text{when } m \text{ and } n \text{ are even number}, \\ 3 & \text{otherwise}. \end{cases}$

Proof. Each pair of farthest neighbors in $T_{m,n}$ can be connected by two, three and four internally vertex-disjoint shortest paths of length $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$ for m, n odd, one of m, n odd, and m, n even, respectively. Thus we obtain a lower bound of $D^{+}(T_{m,n})$. On the other hand, when m, n are odd, deleting the edges $\langle (0,0),(0,1) \rangle$ and $\langle (0,0),(1,0) \rangle$ increases the distance from $(0,0)$ to $(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor)$ by 1. When one of m and n is odd, say m, deleting these three edges $\langle (0,0),(1,0) \rangle$, $\langle (0,0),(0,1) \rangle$ and $\langle (0,0),(0,(n-1)) \rangle$ increases the distance from $(0,0)$ to $(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor)$ by 1. When m and n are even, we can always find four internally vertex-disjoint shortest paths between $(0,0)$ to its farthest neighbors. Hence the lemma follows. □

4. Summary

In this paper, we have studied the diameter variability arising from the change of edges for cycles and tori. The relation between the change of edges and diameters are listed in following table.

	P_m	C_m	Q_m	$T_{m,n(m \geqslant n)}$
$D^{+1}(*)$	1	1	$n-1$ [4]	2 when m and n are odd number 4 when m and n are even number 3 otherwise
$D^{-1}(*)$	1	2 when $m \geqslant 8$	2 [4]	2 when $m \geqslant 12$
$D^{-2}(*)$	1	2 when $m \geqslant 8$	Unknown	2 when $m \geqslant 12$ and $m \neq 15$
⋮	⋮	⋮	Unknown	Unknown
$D^{-(\lfloor m/2 \rfloor - D^{*}(C_m))}(*)$	1	2 when $m \geqslant 8$	Unknown	Unknown
⋮	⋮	Unknown	Unknown	Unknown
$D^{-(m-1-\lfloor m/2 \rfloor)}(*)$	1	Unknown	Unknown	Unknown

In particular, we construct supergraphs of these given graphs such that the diameter is reduced by a constant k. Some supergraphs presented in this paper are shown to be the smallest in terms of the number of edges.

In Section 3.2, we wonder whether $S_{m,n}$ is a smallest spanning subgraph having a diameter $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$. And we conjecture that $D^{+0}(T_{m,n}) = | E(T_{m,n}) | - | E(S_{m,n}) |$. Calculating the number of edges whose removal increases the diameter may help in finding the wide diameter of the underlying graph. We wonder whether we can apply similar proof techniques presented in this paper to find the wide diameter of graph products of two graphs. [3] and [6] are surveys of wide diameter and related things.

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