Simple modeling with polyominoes raises intriguing questions about evolution.

On the Evolution of Square-cell Animals

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Combinations of regular shapes into patterns by means of a set of predefined rules can be observed in many everyday situations, from the tiles on a bathroom floor, the hexagonal patterns of honeycombs, the games of dominoes and Tetris, or even chemical structures of molecules such as hydrocarbons. In all these examples the predefined rules consist of shapes, which abut one another by the contact of respective edges, leaving no interior holes. Interestingly, only three regular polygons can construct these two-dimensional patterns. When the polygons are squares the possible patterns have been called (square-cell) animals by Martin Gardner (1956) and polyominoes by Solomon Golomb (1965). The animals with four cells are the same as the tiles in the Tetris. As shown in Figure 1, the five animals with four cells have been called the names below them by Frank Harary (1982, 1983).

Combinatorial Explosion

Let a_n be the number of animals with n cells. As n increases, the numbers a_n increase exponentially, despite the fact that mirror images are considered identical. Table 1 shows the number a_n as the number of cells, n, grows from 1 to 12.

Figure 1. The square-cell animals with four cells.

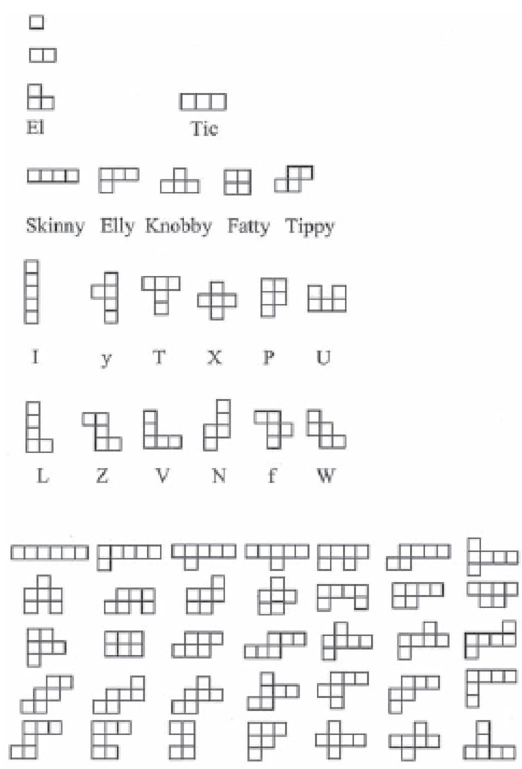


Figure 2. The variety of animals with one to six cells.

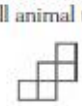
Table 1—The Numbers of Square-cell Animals with up to 12 Cells

n	1	2	3	4	5	6	7	8	9	10	11
a_n	1	1	2	5	15	35	108	369	1,285	4,655	17,073

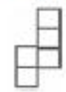
This phenomenon, called combinatorial explosion, is intrinsic to structures built on increasing complexity. The effect of combinatorial explosion indicates that organized objects would have great difficulty in originating and developing into an effective network without some form of control. Each animal produces other animals by adding one cell to the perimeter of its original structure. Figure 2 shows all the animals with $n \leq 6$.

We are interested in analyzing the slow evolution of small animals. We relate the shape of an animal with that of those that can produce it as well as with the animals that descend from it (see Figure 3). We can apply a series of selection rules to our evolution series and then use a probabilistic approach to study the impact on development of the different animals. Our probabilistic model is based on the premise that a parent is equally likely to produce any one of its possible offspring. For example, the two possible three-cell animals produced by the two-cell parent are equally likely ($p = 1/2$). Repeating this process, we obtain the evolutionary tree shown in Figure 3.

Therefore, the five-cell animal that Golomb calls W

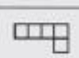
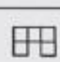


has probability $1/32 = (1/8)(1/4)$ because it has only one possible parent, Tippy, which appears with probability 1/8, and Tippy can generate four possible five-cell animals, each with probability of 1/4. Analogously, the animal called N



has probability $55/864 = (7/24)(1/9) + (1/8)(1/4)$ because it has two possible parents, Elly and Tippy. Elly appears with probability of 7/24 in its generation and has nine possible offspring, each with probability of 1/9. Tippy appears with probability of 1/8 in its generation and has four possible offspring, each with probability of 1/4.

Table 2—Probability of the Unique Offspring of the Five Four-cell Animals within the Five-cell Generation

Parent	Offspring	Probability
Skinny		$^1/_{18}$
Knobby		$^7/_{120}$
Tippy		$^1/_{32}$
Elly		$^{19}/_{216}$
Elly		$^7/_{216}$
Elly		$^7/_{216}$

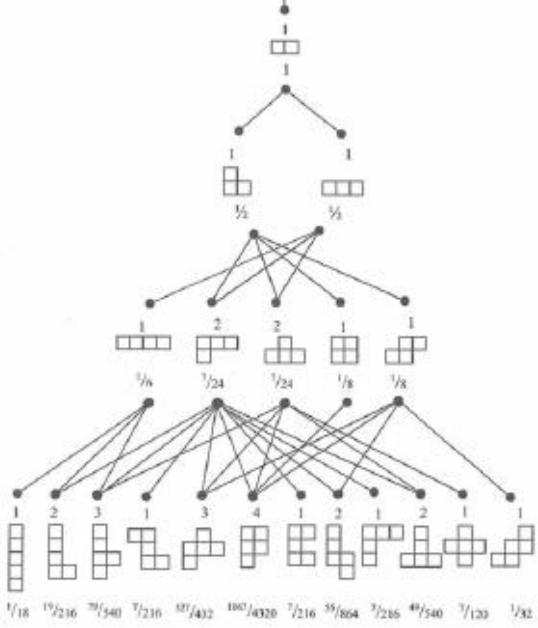


Figure 3. Evolutionary tree of animals increasing by a single-cell addition each generation, with each possible sibling equally likely of deriving from the parent. The number above each animal is the number of possible parents. The fraction below each animal is the probability of that offspring's presence in its generation.

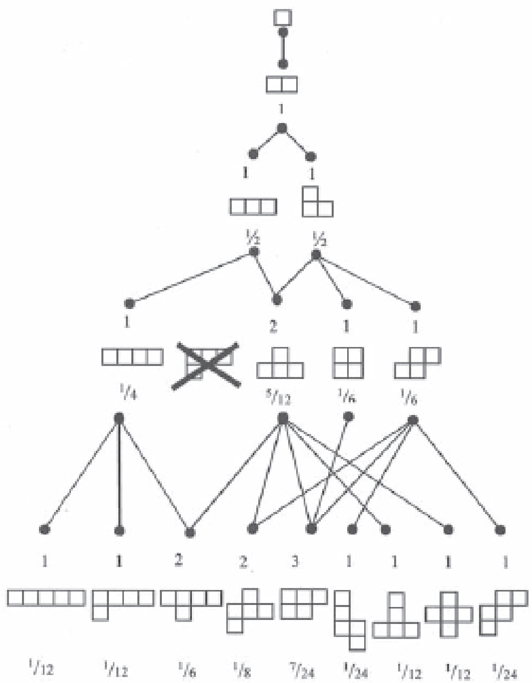


Figure 4. Effect of eliminating Elly on the formation of resulting four- and five-cell animals. The number above each animal is the number of possible parents. The fraction below each animal is the probability of that offspring appearing in its generation.

Some interesting facts can be concluded from Figure 3. As expected, a higher level of symmetry in an animal implies a lesser diversity of offspring. This explains how only Fatty can produce only one animal, while Elly is able to produce nine of the $a_5 = 12$ animals with five cells. Indeed, there are three animals uniquely created from Elly, whereas Skinny, Knobby, and Tippy each produce only one uniquely shaped offspring, and Fatty none. Conversely, the probability of any specific animal being produced is lower than an animal whose parents have a more restricted range of diversity for its offspring. This information is presented in Table 2.

What effect will deleting one of the animals have on the evolutionary tree? Eliminating Elly, as in Figure 4, will most greatly reduce diversity in the offspring. However, this deletion does not produce a dramatic impact on the proportions of the different diversities when compared to the evolutionary tree with no selection pressures (Figure 3). On the other hand, the deletion of Fatty (see Figure 5), which produces less diversity of offspring but with a higher probability of each, results in the diversity of the next generation remaining the same.

We have observed the impact of removing different four-cell animals on the offspring in later generations. So what effect will deletion of an offspring earlier in the evolutionary tree have on the tree's formation, that is, the levels of animal diversity and representation of the aforementioned in each subsequent generation? Indeed, it would seem more logical that in a living system selection pressures would occur from the onset rather than initiate after a certain number of generations. Therefore, we examined the effects of deleting one or another of the three-cell animals, as shown in Figures 6 and 7.

Only-child phenomenon

Diversity is reduced only when a parent that is the unique source of an offspring is deleted; otherwise another parent could still produce that offspring. It would seem logical to examine the proportion of animals in each generation uniquely derived from a sole parent; this is shown in Table 3.

Thus, to prevent combinatorial explosion, selection should best occur at the earliest possible stage because the proportion of offspring which are uniquely derived from a single parent is highest in the earliest generations and declines exponentially with each following generation.

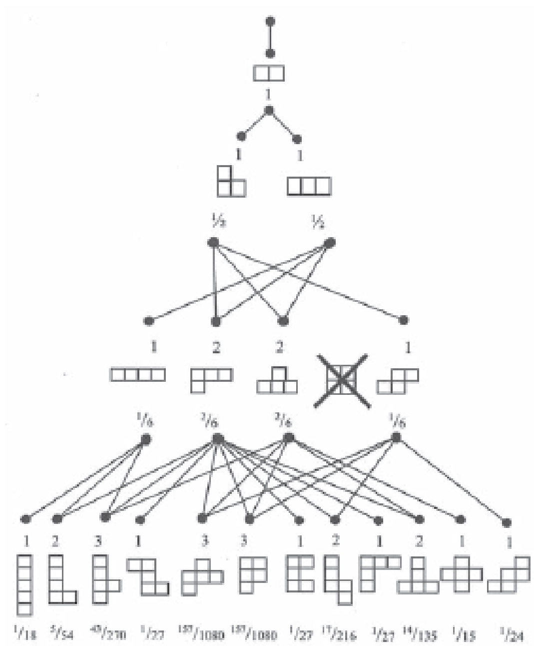


Figure 5. Effect of eliminating Fatty on the formation of resulting four- and five-cell animals. The number above each animal is the number of possible parents. The fraction below each animal is the probability of that offspring appearing in its generation.

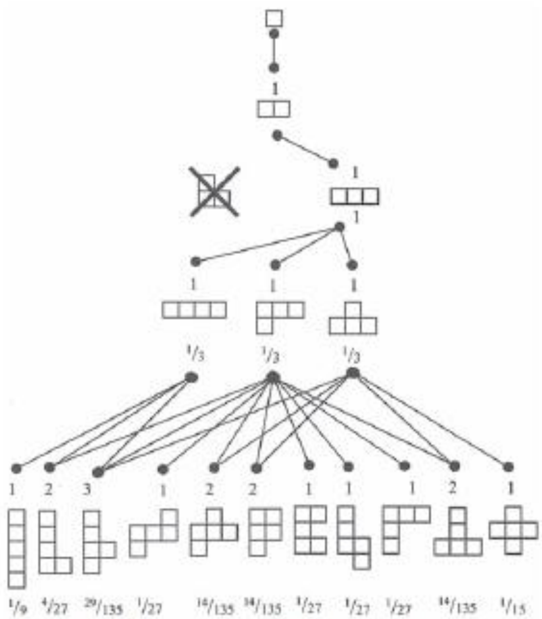


Figure 6. Effect of removing El. The number above each animal is the number of possible parents. The fraction below each animal is the probability of that offspring appearing in its generation.

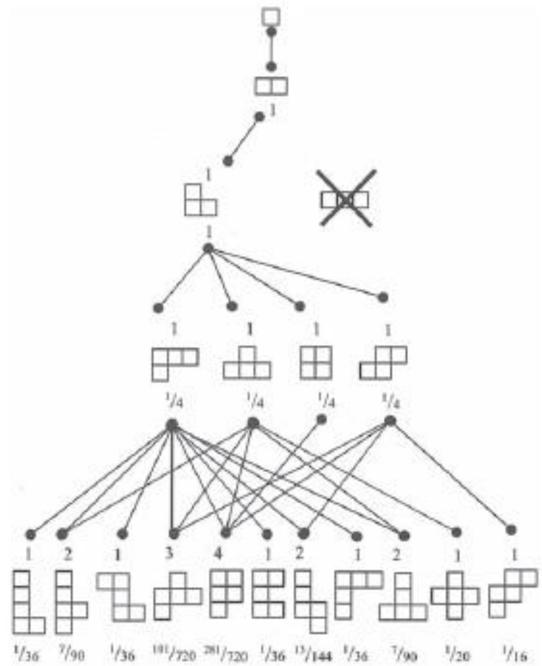


Figure 7. Effect of removing Tic. The number above each animal is the number of possible parents. The fraction below each animal is the probability of that offspring appearing in its generation.

References

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Table 3—The Percentage of the Population in a Given Generation Producible from Only One of the Parents in the Preceding Generation

Number of Cells	Animals with a Unique Parental Source
2	100%
3	100%
4	60%
5	50%
6	14.3% (approx.)