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Algebraic properties of a digraph and its line digraph

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Abstract

Let G be a digraph, LG its line digraph and $A(G)$ and $A(LG)$ their adjacency matrices. We present relations between the Jordan Normal Form of these two matrices. In addition, we study the spectra of those matrices and obtain a relationship between their characteristic polynomials that allows us to relate properties of G and LG, specifically the number of cycles of a given length.

1 Introduction

A digraph can be uniquely assigned a $(0,1)$-adjacency matrix, and reciprocally, every $(0,1)$-matrix represents a digraph. Therefore, many topological properties of digraphs can be studied by using algebraic methods. The spectrum, the characteristic and the minimal polynomials of a matrix are very closely related to the graph topology [2]. As it is expected, it is not possible, to reconstruct the adjacency matrix of a digraph from the spectrum or the characteristic polynomial only. However, we shall prove that this is possible for the case of line digraphs. Line digraphs are very important models for interconnection networks due to their small degree and diameter in relation to their large order [6]. We will explore the relationship between the adjacency matrices of a digraph and its line digraph and obtain results regarding the reconstruction of the line digraph from the spectrum of a given digraph. Our work uses the Jordan Normal Form of the adjacency matrix. This is a useful approach when reconstructing a digraph from its spectrum. In the case of line digraphs, the relation obtained between the Jordan Normal Form of the adjacency matrices of a digraph and its line digraph leads to a solution for a problem presented by Schwenk and Wilson [13] by showing a relation between the characteristic polynomials of a digraph and its line digraph. As a consequence, we derive some new properties regarding the relationship between the number of cycles of certain length in a digraph and its line digraph. This is a very important result because it is well-known that cycles are fixed points under the line digraph, but there are also certain circuits and trails which produce cycles in the line digraph.

2 Definitions and notation

In this paper $G = (V, E)$ stands for a simple digraph, i.e. without loops or multiple edges, with set of vertices $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. The cardinality of V is the *order* of G and the cardinality of E is the *size* of G. By reverting the direction of every edge we obtain the *converse* digraph of G, denoted by G^*.

For a vertex v, the *out-degree* $\delta^+(v)$ is the number of vertices that are adjacent from v, and the *in-degree* $\delta^-(v)$ is the number of vertices adjacent to v. The minimum out-degree and minimum in-degree will be denoted by $\delta^+ = \delta^+(G)$ and $\delta^- = \delta^-(G)$, respectively. The *minimum degree* of G is $\delta = \delta(G) = \min\{\delta^-, \delta^+\}$. If $\delta^+(v) = \delta^-(v) = \delta$ for every vertex v, the digraph is said to be *δ-regular*. A vertex is a *source* if its in-degree is 0, and a *sink* if its out-degree is 0. The number of sources and sinks of G are denoted by $\alpha = \alpha(G)$ and $\beta = \beta(G)$, respectively. Note that in the converse digraph G^*, $\alpha(G^*) = \beta$ and $\beta(G^*) = \alpha$.

The adjacency matrix of a digraph G of order n is the $n \times n$ dimensional $(0,1)$-matrix $A = A(G)$, whose (i,j) entry is 1 if and only if (v_i, v_j) is an edge of G. Since G has no loops, the trace of A is 0.

We assume the reader is familiar with eigenvalues, eigenvectors, characteristic polynomial, minimal polynomial and Jordan Normal Form of matrices. The characteristic polynomial of the graph G is $p_G(x)$, the characteristic polynomial of its adjacency matrix A. The eigenvalues of G are the eigenvalues of A. The algebraic multiplicity of λ, $m_A(\lambda)$, is the multiplicity of λ considered as a zero of the corresponding characteristic polynomial. Always, $\dim Ker(A - \lambda I_n) \leq m_A(\lambda)$ and

$$Ker(A-\lambda I) \subset Ker(A-\lambda I)^2 \subset \ldots Ker(A-\lambda I)^{r_\lambda} = Ker(A-\lambda I)^{r_\lambda+1} = \ldots$$

where $r_\lambda \leq m_A(\lambda)$ is the smallest integer such that $\dim Ker(A-\lambda I)^{r_\lambda} = m_A(\lambda)$. The minimal polynomial of G can be obtained as

$$\mu_G(x) = \prod_{\lambda} (x-\lambda)^{r_\lambda}.$$

A *Jordan block matrix* associated with λ, denoted as $J(\lambda)$, is a direct sum of *elementary Jordan matrices* associated with λ. These elementary Jordan matrices have $\lambda' s$ in the main diagonal and 1's in the above diagonal. The *Jordan normal form* or *canonical form* of a matrix is the direct sum of its Jordan blocks. At least one of those elementary Jordan matrices has size r_λ, which is the greatest order of the elementary matrices in the corresponding Jordan block. Therefore, if $r_\lambda = 1$, each elementary Jordan matrix has size 1 and the corresponding Jordan block is diagonal. Through this paper we assume the sizes of elementary Jordan matrices in each Jordan block in decreasing order. The number of elementary Jordan matrices of order $p \geq 1$ in the $J(\lambda)$ is denoted by N_p. Then, if $B = A - \lambda I$,

$$N_p = 2\dim KerB^p - \dim KerB^{p-1} - \dim KerB^{p+1} \tag{1}$$

For details about the above formula see [8].

In the *line digraph* LG of a digraph G each vertex represents an edge of G, that is, $V(LG) = \{uv : (u,v) \in E(G)\}$. A vertex uv is adjacent to a vertex wz if $v = w$, that is, whenever the edge (u,v) of G is adjacent to the edge (w,z) [9]. From the definition, it follows that if G has size m then LG has order m and size $\sum \delta^+(v)\delta^-(v)$. Furthermore, if G is d-regular ($d > 1$), has diameter D and order n, then L^kG is d-regular, has diameter $D+k$ and order d^kn. Therefore, the iteration of the line digraph technique is a good method to obtain large digraphs with fixed degree and diameter. See, for example, [1, 6, 12, 7] for proofs and more information.

We first show the relationship between the adjacency matrices of a digraph and its line digraph. For this purpose we introduce two new matrices associated with a digraph, whose properties allow us to obtain interesting results by applying Flanders Theorem [5]. As a first consequence, a simple relation between the characteristic polynomials of a digraph and its line digraph is obtained. This solves in a new way a problem introduced by Schwenk and Wilson [13], which has been previously solved by Montserrat [11] and later by Zhang and Lin [14]. Applying Sachs's "coefficients theorem for digraphs" [4], we derive some new properties regarding the relationship between the number of cycles of certain length in G and LG .

3 Adjacency matrices of a digraph and its line digraph

We begin by defining two matrices related to G, which play a fundamental role throughout this paper.

Definition 3.1 *Let G denote a digraph with vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ and edges $E(G) = \{e_1, e_2, \dots, e_m\}$.*

1. *Incidence matrix of heads of G, $H = H(G)$, is the $n \times m$ matrix defined by*
$$H_{ij} = \begin{cases} 1, & \text{if } e_j = (v_i, -); \\ 0, & \text{otherwise.} \end{cases}$$

2. *Incidence matrix of tails of G, $T = T(G)$, is the $n \times m$ matrix defined by*
$$T_{ij} = \begin{cases} 1, & \text{if } e_j = (-, v_i); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $H - T$ is the standard $(0, 1, -1)$-incidence matrix of G [4]. Besides, if $e_j = (v_r, v_s)$ then v_r is the head of the arc e_j and v_s is its tail. This means $H_{rj} = 1$ and $H_{kj} = 0$ for any $k \neq r$; also $T_{sj} = 1$ and $T_{kj} = 0$ for any $k \neq s$.

The incidence matrices $H(G)$ and $T(G)$ and the adjacency matrices $A(G)$ and $A(LG)$ are related in the following way.

Lemma 3.1 *Let $A(G)$ and $A(LG)$ be the adjacency matrices of G and LG, respectively. Let H and T be the incidence matrices of head and tails of G, respectively. Then,*

1. $HT^t = A(G)$.
2. $T^t H = A(LG)$.

Proof. *1.* If $A(G)_{ij} = 1$ then G contains an edge $e_l = (v_i, v_j)$, which means that $H_{il} = T_{jl} = 1$, and $H_{ik}T_{jk} = 0$ for any $k \neq l$, so $(HT^t)_{ij} = 1$. If $A(G)_{ij} = 0$, there is no edge (v_i, v_j), and hence, $H_{ik}T_{jk} = 0$. Thus, $(HT^t)_{ij} = 0$. *2.* If $A(LG)_{ij} = 1$, the edge e_i is adjacent to e_j in G through a common vertex, let us say v_l, which implies that $T_{li} = H_{lj} = 1$, and $T_{ki}H_{kj} = 0$ for any $k \neq l$. Then, $(T^t H)_{ij} = 1$. Moreover, $A(LG)_{ij} = 0$ means that the edge e_i is not adjacent to e_j in G, so $(T^t H)_{ij} = 0$. ∎

We recall next a result by Flanders [5] about the relationship between AB and BA for arbitrary matrices A and B, later proved by Bernau and Abian [3] and Johnson and Schreiner [10].

Let C and D be $n \times n$ and $m \times m$ complex matrices, respectively. Then C may be written as AB while D is written as BA if and only if (i) the Jordan structure associated with nonzero eigenvalues is identical in C and D and (ii) if $n_1 \geq n_2 \geq \ldots \cdots$ are the sizes of elementary Jordan matrices associated with 0 in C while $m_1 \geq m_2 \geq \ldots \cdots$ are the corresponding in D, then $|n_i - m_i| \leq 1$ for all i.

In addition, when studying the relation between the characteristic polynomials $p_{AB}(x)$ and $p_{BA}(x)$, Horn and C.R. Johnson [8] proved that

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

from which it immediately follows that the Jordan structures associated with the nonzero eigenvalues of AB and BA are the same.

The next result is a consequence of Flanders's theorem and Lemma 3.1.

Corollary 3.1 *Let $A(G)$ and $A(LG)$ be the adjacency matrices of a digraph G on n vertices and size m, and its line digraph LG, respectively.*

1. *The matrices $A(G)$ and $A(LG)$ have the same nonzero eigenvalues, counting their multiplicities. Moreover, their eigenvalue sets differ in $|m - n|$ zeros and the following relations hold for characteristic polynomials and algebraic multiplicities of* 0

$$x^n p_{LG}(x) = x^m p_G(x)$$

$$m_{A(LG)}(0) = m - n + m_{A(G)}(0)$$

2. *For each $\lambda \neq 0$ eigenvalue of $A(G)$ and $A(LG)$ the Jordan block matrix associated with λ is the same in the Jordan normal form of both, $A(G)$ and $A(LG)$.*

3. *The sizes of the corresponding elementary Jordan matrices associated with 0 in $A(G)$ and $A(LG)$ differ at most in one unit.* ■

The relationship obtained in the above Corollary between characteristic polynomials, together with the information they provide about the cycles and their lengths, permits us to obtain the following relations.

A *linear directed graph* is a digraph in which each in-degree and each out-degree is equal to 1, i.e., it consists of non-intersecting cycles.

The following result called the "coefficients theorem for digraphs" was given by Sachs [4] and later by others authors. If $\mathcal{L}_i$ denotes the set of all linear directed subgraphs L of G with exactly i vertices, $c(L)$ denotes the number of components of L, and $p_G(x) = x^n + a_1x^{n-1} + \ldots + a_n$ is the characteristic polynomial of G, it has been observed [4] that

$$a_i = \sum_{L \in \mathcal{L}_i} (-1)^{c(L)} \ (i = 1, 2, \ldots, n) \tag{2}$$

Theorem 3.1 *[4] Let G be a digraph with girth g and characteristic polynomial $p_G(x) = x^n + a_1x^{n-1} + \ldots + a_n$. Then, if $i \leq \min\{2g-1, n\}$, the number of cycles of length i contained in G is equal to $-a_i$, and g is equal to the smallest index i for which $a_i \neq 0$.* ∎

As a consequence, using Corollary 3.1 we obtain the following results in relation to the number of cycles in line digraphs.

Corollary 3.2 *Let G be a weakly connected digraph of order n and girt g, with the characteristic polynomial $p_G(x) = x^n + a_1x^{n-1} + \ldots + a_n$.*

1. *For $i \leq \min\{2g-1, n\}$ the number of cycles of length i contained in LG is equal to $-a_i$, that is, the number of cycles of length i contained in G and in LG coincides.*

2. *If $a_g = -1$ then for $i \leq \min\{2g, n\}$ the number of cycles of length i contained in G is equal to $-a_i$, and also this number represents the number of cycles of length i contained in LG.*

Proof. The result is obvious if G is acyclic. If G has cycles and its size is m then $m \geq n$, because G is weakly connected. So, *1.* is a direct consequence of Theorem 3.1 and of Corollary 3.1 as $p_{LG}(x) = x^{m-n}p_G(x)$. To prove*2.*, notice that if $a_g = -1$, the digraph G has only one cycle of length g. Hence, each linear directed subgraph of G with i vertices, $i \leq 2g$, is necessarily a cycle, since there are not two cycles with length g. From result (2) it follows that $a_i = \sum_{L \in \mathcal{L}_i} (-1)^{c(L)} = \sum_{\vec{C}_i \in G} (-1)$ $(i \leq 2g)$. Thus, $-a_i$ is the number of cycles of length i contained in G and hence in LG. ∎

Theorem 3.2 *Let G be a strongly connected digraph which is not a cycle with order n and size m, and such that $\delta^+(u) = \delta^-(u)$ for each vertex u. Then LG can be partitioned into an even number of disjoint cycles. Furthermore, the number of those partitions is at least the number of Hamiltonian cycles of LG.*

Proof. Since G is strongly connected, $m \geq n$. In fact, $m > n$ since G is not a cycle. If the out-degree of every vertex u equals its in-degree, then G must be Eulerian and hence, LG is Hamiltonian. Then, we can consider a cycle of length m in LG, which contributes a -1 to the coefficient a_m of the characteristic polynomial. Since $p_{LG}(x) = x^{m-n}p_G(x)$ implies $a_m = 0$, there must exist a linear directed subgraph $L \in \mathcal{L}_m$ of m vertices such that $(-1)^{c(L)} = 1$. This is equivalent to say that L consists of an even number of disjoint cycles, and hence, the result holds. Furthermore, the number of partitions into an even number of disjoint cycles coincides with the number of partitions into an odd number of disjoint cycles because $a_m = 0$. Thus, the number of partitions into an even number of disjoint cycles is at least the number of Hamiltonian cycles of LG. ∎

4 The Jordan Normal Form of a digraph and its line digraph

¿From Corollary 3.1, the Jordan block matrices associated with nonzero eigenvalues of $A(G)$ and $A(LG)$ coincide. Moreover, we know that the sizes of the elementary Jordan blocks associated with 0 in $A(G)$ and $A(LG)$ differ at most in one unit. In order to obtain a complete description of the Jordan normal form of $A(LG)$ from $A(G)$, we need to study the sizes of the elementary Jordan matrices associated with the eigenvalue zero. In this section we find the exact relation between these nilpotent Jordan blocks.

Lemma 4.1 *Let G be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let T, H be the incidence matrices of G, and α and β the number of sources and sinks of G, respectively. Then,*

1. $TT^t = Diag(\delta^-(v_1), \delta^-(v_2), \ldots, \delta^-(v_n))$.
2. $HH^t = Diag(\delta^+(v_1), \delta^+(v_2), \ldots, \delta^+(v_n))$.
3. $rank(T) = n - \alpha$, $rank(H) = n - \beta$.

Proof. Note that $(TT^t)_{ij} = 0$ whenever $i \neq j$, since if $e_k = (-\,, v_i)$ then $T_{ik} = 1$ and $T_{jk} = 0$. Besides, $(TT^t)_{ii} = \delta^-(v_i)$. Analogously can be proved the result for HH^t. Consider now, for instance, that $\{v_1, v_2, \ldots, v_\alpha\}$ is the set of sources of G. Then, the rows $v_1, v_2, \ldots, v_\alpha$ of T are zero, and the other rows define an orthogonal set of non-zero vectors. Therefore these $n-\alpha$ rows are linearly independent, so $rank(T) = n - \alpha \leq m$. Analogously, can be proved that $rank(H) = n - \beta \leq m$. ∎

Notice that if n and m are the order and size of a digraph G with positive minimum degree, it has to be $m \geq n$, since $m = \sum_{v \in V(G)} \delta^+(v) = \sum_{v \in V(G)} \delta^-(v)$. In addition, if $m = n$, every vertex v has $\delta^+(v) = 1$, and from the above Lemma, H is an identity matrix. In this case, $A(LG) = H^t A(G) H$, which implies that the Jordan normal forms of G and LG are the same. In fact, this result was noted by Harary and Norman [9], namely that a weakly connected digraph is isomorphic to its line digraph if and only if each vertex has in-degree 1 or each vertex has out-degree 1.

As a first consequence of the above Lemma we obtain a sufficient condition for diagonalizing both Jordan block matrices associated with zero for $A(G)$ and $A(LG)$.

Theorem 4.1 *Let G be a digraph with α sources and β sinks. If $m_{A(G)}(0) = \max\{\alpha, \beta\}$, then $\dim KerA(G) = m_{A(G)}(0)$ and $\dim KerA(LG) = m_{A(LG)}(0)$.*

Proof. By Lemma 4.1, $rank(A(G)), rank(A(LG)) \leq \min\{rank(H), rank(T)\} = \min\{n-\beta, n-\alpha\} = n - \max\{\alpha, \beta\}$. Besides, $\dim KerA(G) = n - rank(A(G))$ so we obtain $\max\{\alpha, \beta\} \leq \dim KerA(G) \leq m_{A(G)}(0)$. By hypothesis, $m_{A(G)}(0) = \max\{\alpha, \beta\}$ so $\dim KerA(G) = m_{A(G)}(0)$. Also, by Lemma 4.1, $\dim KerA(LG) = m - rank(A(LG))$, so we have $m - n + \max\{\alpha, \beta\} \leq \dim KerA(LG) \leq m_{A(LG)}(0)$. By Corollary 3.1, $m_{A(LG)}(0) = m - n + m_{A(G)}(0)$, and it follows $\dim KerA(LG) = m_{A(LG)}(0)$. ∎

Notice that when $A(G)$ is a nonsingular matrix there are neither sources nor sinks in G and $m \geq n$. Hence, the condition of the above Theorem is directly satisfied and together with Corollary 3.1 we can readily obtain the following result.

Corollary 4.1 *Let G be a digraph on n vertices and size m, and assume that $A(G)$ is nonsingular. Then, the Jordan normal form $J_{A(G)}$ of $A(G)$ and the Jordan normal form $J_{A(LG)}$ of $A(LG)$ are related as follows:*

$$J_{A(LG)} = \begin{pmatrix} J_{A(G)} & \\ & O \end{pmatrix}$$

where O is a square zero matrix of order $m - n$. ■

Thus, the general case which remains to be solved is when $A(G)$ is a singular matrix. We start with the study of digraphs with minimum degree $\delta \geq 1$. We distinguish two cases, depending on $\delta(G)$ positive or zero.

4.1 Digraphs with positive minimum degree

We deal with digraphs G with $\delta \geq 1$, so there are no sources and no sinks, i.e. $\alpha = \beta = 0$, and $m \geq n$. In the following we use a result by Frobenius which states that if ABC exists, then

$$rank(ABC) \geq rank(AB) + rank(BC) - rank(B). \tag{3}$$

If B is the identity matrix I_n, the above expression gives $rank(AC) \geq rank(A) + rank(C) - n$.

First of all, we obtain the following lemma.

Lemma 4.2 *Let G be a digraph of order n and size m, with minimum degree $\delta \geq 1$. Then,*

$$\dim KerA(G) \leq m - n, \text{ and } \dim KerA(LG) = m - n$$

Proof. Since $A(G) = HT^t$, applying Frobenius Theorem(3), $rankA(G) \geq rank(H) + rank(T^t) - m = 2n - m$ due to Lemma 4.1. The substitution of this bound for $rank\,A(G)$ in $\dim KerA(G) = n - rankA(G)$ gives the first result. Furthermore, if $\delta \geq 1$, then $KerT^t = \{0\}$ because of Lemma 4.1. Let U a nonzero vector of $KerA(LG)$. Since $T^tHU = 0$, then $HU \in KerT^t$, that is, $HU = 0$. Hence $KerA(LG) = KerH$, which implies that $\dim KerA(LG) = m - n$. ■

Theorem 4.2 *Let G be a digraph of order n and size m, with minimum degree $\delta \geq 1$, and let $p \geq 0$ any integer. Then,* $\dim KerA(LG)^{p+1} = \dim KerA(G)^p + m - n$.

Proof. When $p = 0$, the result follows from Lemma 4.2. So assume that $p \geq 1$. Since $A(LG)^{p+1} = T^tA(G)^pH$, we have that $rank\,A(LG)^{p+1} \leq rank\,A(G)^p$. That is,

$$m - \dim KerA(LG)^{p+1} \leq n - \dim KerA(G)^p.$$

On the other hand, from Lemma 4.1 with $\alpha = \beta = 0$, and from expression (3) it follows that

$rank\, A(LG)^{p+1} = rank(T^t A(G)^p H) \geq rank(T^t A(G)^p) + rank(A(G)^p H) - rank\, A(G)^p \geq$

$rank(T^t) + rank\, A(G)^p - n + rank\, A(G)^p + rank(H) - n - rank\, A(G)^p = n + rank\, A(G)^p \; - n + n - n = rank\, A(G)^p$. Hence,

$$m - \dim Ker A(LG)^{p+1} \geq n - \dim Ker A(G)^p.$$

These two inequalities complete the proof. ■

Now, in order to find the Jordan block associated with zero of $A(LG)$ from that of $A(G)$ when the minimum degree of G is $\delta \geq 1$, we need the formula (1) given in the Introduction to compute the number of elementary Jordan matrices of a given order.

Theorem 4.3 *Let G be a digraph of order n and size m, with minimum degree $\delta \geq 1$. If $A(G)$ is singular with $J_{A(G)}(0)$ as Jordan block associated with zero, then the Jordan block associated with zero for $A(LG)$ can be written as*

$$\begin{pmatrix} \tilde{J}_{A(G)}(0) & \\ & O \end{pmatrix}$$

where $\tilde{J}_{A(G)}(0)$ is obtained by increasing one unit the order of each of the elementary Jordan matrices in $J_{A(G)}(0)$, and $O = [0]$ is a square zero-matrix of order $m - n - \dim Ker A(G)$.

Proof. As $\delta \geq 1$, Lemma 4.2 allows us to assure that $\dim Ker A(LG) = m - n$. Notice that $m > n$, because otherwise, from Corollary 3.1 it follows that $0 = m_{A(LG)}(0) = m_{A(G)}(0)$ contradicting the hypothesis. Let us denote by $N_p(G)$ and $N_p(LG)$ the number of elementary Jordan matrices of order p associated with zero in the Jordan normal forms $J_{A(G)}$ and $J_{A(LG)}$, respectively. Hence, by Theorem 4.2 we can write now for each integer $p \geq 1$ that

$$\dim Ker A(LG)^p = \dim Ker A(G)^{p-1} + m - n.$$

Property (1) enable us to compute $N_p(LG)$ for $p \geq 1$. First, when $p = 1$:

$$N_1(LG) = 2 \dim Ker A(LG) - 0 - \dim Ker A(LG)^2 = m - n - \dim Ker A(G)$$

Notice that $N_1(LG) \geq 0$ by Lemma 4.2, and that this result is the responsible for the square zero-matrix O.

Next, for any $p \geq 2$,

$$\begin{aligned} N_p(LG) &= 2(\dim KerA(G)^{p-1} + m - n) - (\dim KerA(G)^{p-2} + m - n) \\ &\quad -(\dim KerA(G)^p + m - n) \\ &= 2\dim KerA(G)^{p-1} - \dim KerA(G)^{p-2} - \dim KerA(G)^p \\ &= N_{p-1}(G) \end{aligned}$$

So, for each elementary Jordan matrix of order $p-1$ in $J_{A(G)}(0)$ there exists one elementary Jordan matrix of order p in $\tilde{J}_{A(G)}(0)$, for every $p \geq 2$. ■

By applying recursively Corollary 4.1 or Theorem 4.3, and Corollary 3.1, the Jordan normal form of $A(L^kG)$ can be found from the Jordan normal form of $A(G)$, where G is any digraph with $\delta \geq 1$, $k \geq 1$ is any integer, and L^kG is the k-iterated line digraph of G. For the case of digraphs G that are d-regular, an explicit formula can be given for the Jordan block associated with zero for $A(L^kG)$, which, as known, represents the only difference respect to the Jordan normal form of $A(G)$. That formula is shown in the following Corollary, whose proof only involves simple calculations.

Corollary 4.2 *Let G be a d-regular digraph of order n, with $d > 1$. Let $k \geq 1$ be any integer, and L^kG be the k-iterated line digraph of G. If for any integer $q \geq 1$, $N_q(L^kG)$ and $N_q(G)$ stand for the number of elementary Jordan matrices of order q associated with zero for $A(L^kG)$ and $A(G)$ respectively, then for any integer $p \geq 1$*

$$N_p(L^kG) = \begin{cases} n(d-1)^2 d^{k-1-p} & \text{if } p \leq k-1; \\ n(d-1) - \dim KerA(G) & \text{if } p = k; \\ N_{p-k}(G) & \text{if } p \geq k+1. \end{cases} \quad ■$$

4.2 Digraphs with minimum degree zero

If G is a digraph with minimum degree $\delta = 0$, it has sources and/or sinks. Note that every isolated vertex of G is a source and a sink at the same time. For the following study of the line digraph of a digraph G with $\delta = 0$, we will use a set of sources, S_o, and a set of sinks, S_i, that are disjoint, that is to say, each isolated vertex of G, if one exists, is chosen arbitrarily to belong to the set of sources or to the set of sinks, but not to both simultaneously. For instance, if we consider the isolated vertices as sources we will have $|S_o| = \alpha$, $|S_i| = \beta - \gamma$, where γ is the number of isolated vertices.

Definition 4.1 *Let G be a digraph and let $S_o \subset V(G)$ and $S_i \subset V(G)$ be the sets of sources and sinks of G, respectively, $S_o \cap S_i = \emptyset$. If $V(G) \neq S_o \cup S_i$, the reduced digraph of G, G_R, is defined as $G_R = G - \{S_o \cup S_i\}$.*

Definition 4.2 *Let G be a digraph and let $S_o \subset V(G)$ and $S_i \subset V(G)$ be the sets of sources and sinks of G, respectively, $S_o \cap S_i = \emptyset$. Let C_o, C_i be two sets such that $C_o \cap C_i = \emptyset$, $|C_o| = |S_o|$, $|C_i| = |S_i|$, and let f_o, f_i be two one-to-one maps from S_o onto C_o and from S_i onto C_i, respectively. The increase digraph of G, $G_I = (V(G_I), E(G_I))$, is defined as follows:*

$$V(G_I) = V(G) \cup C_o \cup C_i$$

$$E(G_I) = E(G) \cup [\bigcup_{x \in S_o} ((x, f_o(x)) \cup (f_o(x), x))] \cup [\bigcup_{y \in S_i} ((y, f_i(y)) \cup (f_i(y), y))]$$

This definition has been given for the most general case. If, for instance, G has no sinks or all its sinks are also sources, we can set $S_i = \emptyset$, and then we will avoid using the set C_i and the map f_i. Analogously if G has no sources or all its sources are also sinks. Finally, if $S_o = S_i = \emptyset$, then $G_R = G_I = G$.

Lemma 4.3 *Let G be a digraph and let $S_o \subset V(G)$ and $S_i \subset V(G)$ be the sets of sources and sinks of G, respectively, $S_o \cap S_i = \emptyset$. Let G_I be the increase digraph of G, with adjacency matrix $A(G_I)$. If $V(G) \neq S_o \cup S_i$, then the Jordan blocks associated with zero for $A(G_R)$ and for $A(G_I)$ coincide, where G_R is the reduced digraph of G and $A(G_R)$ is its adjacency matrix.*

Proof. We start by setting $|S_o| = \epsilon$, $|S_i| = \omega$. Let $S_o = \{a_1, a_2, \ldots, a_\epsilon\}$ and $S_i = \{b_1, b_2, \ldots, b_\omega\}$ be the sets of sources and sinks of G. Let $f_o : S_o \to C_o$ and $f_i : S_i \to C_i$ be the one-to-one maps defining the digraph G_I. Then, the vertex sets of G and G_I can be written as:

$V(G) = \{a_1, a_2, \ldots, a_\epsilon, \ldots, b_1, b_2, \ldots, b_\omega\}$

$V(G_I) = \{f_o(a_\epsilon), f_o(a_{\epsilon-1}), \ldots, f_o(a_1), a_1, a_2, \ldots, a_\epsilon, \ldots, b_1, b_2, \ldots, b_\omega, f_i(b_\omega), f_i(b_{\omega-1}), \ldots, f_i(b_1)\}$

This ordering of the vertices allows the following expression for $A(G_I)$:

$$A(G_I) = \begin{pmatrix} Z_\epsilon & M & N \\ 0 & A(G_R) & U \\ 0 & 0 & Z_\omega \end{pmatrix},$$

where $Z_q = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix}$, $2q \times 2q$ matrix $(q = \epsilon, \omega)$. Then, for any integer $p \geq 1$:

$$A(G_I)^p = \begin{pmatrix} (Z_\epsilon)^p & \tilde{M} & \tilde{N} \\ 0 & A(G_R)^p & \tilde{U} \\ 0 & 0 & (Z_\omega)^p \end{pmatrix}$$

As Z_ϵ and Z_ω are nonsingular, we obtain $rank\, A(G_I)^p = rank\, A(G_R)^p + 2(\epsilon + \omega)$, and so:

$$\dim Ker A(G_I)^p = \dim Ker A(G_R)^p, \quad \text{for each integer } p \geq 1.$$

■

Lemma 4.4 *Let G be a digraph and G_I be its increase digraph. If $A(LG)$ and $A(LG_I)$ are the adjacency matrices of the line digraphs of G and of G_I respectively, then the Jordan blocks associated with zero for $A(LG)$ and for $A(LG_I)$ coincide.*

Proof. Let $S_o = \{a_1, a_2, \dots, a_\epsilon\}$ and $S_i = \{b_1, b_2, \dots, b_\omega\}$ be the sets of sources and sinks of G, $S_o \cap S_i = \emptyset$. Let $f_o : S_o \to C_o$ and $f_i : S_i \to C_i$ be the one-to-one maps defining the digraph G_I. Let $\omega^+(S_o)$ and $\omega^-(S_i)$ be the subsets of arcs:

$\omega^+(S_o) = \{e \in E(G) : e = (a, x),\, a \in S_o\}$; $\quad \omega^-(S_i) = \{e \in E(G) : e = (x, b),\, b \in S_i\}$.

Hence, $\omega^+(S_o)$, $\omega^-(S_i)$ are subsets of vertices of LG and of LG_I. Notice that in LG_I, each vertex $f_o(a)a$, $a \in S_o$, is adjacent to any vertex of $\omega^+(S_o)$, and also each vertex $bf_i(b)$, $b \in S_i$, is adjacent from any vertex of $\omega^-(S_i)$. Now, assuming a natural ordering of the vertices from left to right, the vertex sets of the digraphs LG and LG_I can be written as

$V(LG) = \omega^+(S_o) \cup (E(G) \setminus [\omega^+(S_o) \cup \omega^-(S_i)]) \cup \omega^-(S_i)$

$V(LG_I) = \bigcup_{a \in S_o} \{f_o(a)a,\, af_o(a)\} \cup V(LG) \bigcup_{b \in S_i} \{f_i(b)b,\, bf_i(b)\}$,

Thus, the following expression for $A(LG_I)$ holds

$$A(LG_I) = \begin{pmatrix} Y_\epsilon & M & N \\ 0 & A(LG) & U \\ 0 & 0 & Y_\omega \end{pmatrix},$$

where $Y_q = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$, $2q \times 2q$ matrix $(q = \epsilon, \omega)$. Then, for any integer $p \geq 1$:

$$A(LG_I)^p = \begin{pmatrix} (Y_\epsilon)^p & \tilde{M} & \tilde{N} \\ 0 & A(LG)^p & \tilde{U} \\ 0 & 0 & (Y_\omega)^p \end{pmatrix}$$

As Y_ϵ and Y_ω are nonsingular, we obtain $rank\, A(LG_I)^p = rank\, A(LG)^p + 2(\epsilon + \omega)$, and so:

$$\dim KerA(LG_I)^p = \dim KerA(LG)^p, \quad \text{for each integer } p \geq 1. \quad \blacksquare$$

Since G_I has neither sources nor sinks, we have that $\delta(G_I) \geq 1$. So, using the above lemma and Lemma 4.2, we can characterize those digraphs whose line digraphs have a diagonal Jordan block associated with zero.

Corollary 4.3 *A digraph G with α sources, β sinks and γ isolated vertices satisfies $m_{A(G)}(0) \geq \alpha+\beta-\gamma$, and also $\dim KerA(LG) = m_{A(LG)}(0)$ if and only if $m_{A(G)}(0) = \alpha + \beta - \gamma$.*

Proof. Notice that the order of G_I is $n+\alpha+\beta-\gamma$, and its size is $m+2(\alpha+\beta-\gamma)$. From Lemma 4.4 follows that $\dim KerA(LG) = \dim KerA(LG_I) = m - n + \alpha + \beta - \gamma$, applying Lemma 4.2 to G_I since $\delta(G_I) \geq 1$. From Corollary 3.1 $m_{A(LG)}(0) = m - n + m_{A(G)}(0)$, so the results follow easily. $\blacksquare$

If G is a digraph such that $V(G) = S_o \cup S_i$, that is, $n = \alpha + \beta - \gamma$ then $m_{A(G)}(0) = \alpha + \beta - \gamma$, as a consequence of Corollary 4.3 and $n \geq m_{A(G)}(0)$. Furthermore, $\dim KerA(LG) = m_{A(LG)}(0) = m$, by means of Corollary 3.1, that is, $A(LG) = [0]$. So, the general case which remains to be solved is when $V(G) \neq S_o \cup S_i$.

The Jordan block associated with zero for $A(LG)$ can be obtained directly from that for $A(G_I)$ with the help of Corollary 4.1 and Theorem 4.3. But the order of the matrix $A(G_I)$ is greater than the order of $A(G)$, and the problem of predicting the Jordan block associated with zero for $A(LG)$ when $\delta(G) = 0$ seems to be computationally more complex that in the case $\delta(G) \geq 1$. In fact, this is the main reason for introducing the digraph G_R. Now, the following theorem is straightforward.

Theorem 4.4 *Let G be a digraph of order n and size m with α sources, β sinks and γ isolated vertices, with $n > \alpha + \beta - \gamma$. Let $S_o \subset V(G)$ and $S_i \subset V(G)$ be the sets of sources and sinks of G, respectively, $S_o \cap S_i = \emptyset$. Being $J_{A(LG)}(0)$ the Jordan block associated with zero for the adjacency matrix of the line digraph of G, and $A(G_R)$ the adjacency matrix of its reduced digraph, the following results hold:*

1. *If $A(G_R)$ is nonsingular, $J_{A(LG)}(0)$ is a square zero matrix of order $m - n + \alpha + \beta - \gamma$.*

2. *If $A(G_R)$ is singular with $J_{A(G_R)}(0)$ as Jordan block associated with zero, then*
$$J_{A(LG)}(0) = \begin{pmatrix} \tilde{J}_{A(G_R)}(0) & \\ & O \end{pmatrix},$$
where $\tilde{J}_{A(G_R)}(0)$ is the Jordan block whose elementary matrices result by increasing one unit the order of each of the elementary Jordan matrices in $J_{A(G_R)}(0)$, and $O = [0]$ is a square zero matrix of order $m - n + \alpha + \beta - \gamma - \dim Ker A(G_R)$.

Proof. Notice that $V(G) \neq S_o \cup S_i$ because $n > \alpha + \beta - \gamma$. If G_I is the increase digraph of G, Lemma 4.4 states that $J_{A(LG)}(0)$ is equal to the Jordan block associated with zero for $A(LG_I)$.

If $A(G_R)$ is nonsingular, from Lemma 4.3 $A(G_I)$ is also nonsingular, so the Jordan block associated with zero for $A(LG_I)$ is diagonal of order $m - n + \alpha + \beta - \gamma$ by applying Corollary 4.1.

If $A(G_R)$ is singular, its Jordan block associated with zero is the same as for $A(G_I)$, due to Lemma 4.3. Theorem 4.3 gives the result for the Jordan block associated with zero for $A(LG_I)$, that is to say, it gives $J_{A(LG)}(0)$. ■

It must be emphasized, that from a computational complexity point of view, no real difference exists between digraphs G with $\delta(G) \geq 1$ or with $\delta(G) = 0$ in order to predict the Jordan normal form of $A(LG)$ from the Jordan blocks for G and/or G_R, except for the fact that the order of $A(G)$ is greater than the order of the adjacency matrix of the digraph G_R when $G_R \neq G$. Actually, the Jordan blocks associated with nonzero eigenvalues for $A(G)$ and the Jordan block associated with zero for $A(G_R)$ are assumed to be known in both cases, since $G_R = G$ when $\delta(G) \geq 1$ and $G_R \neq G$ when $\delta(G) = 0$.

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