Exact traveling and non-traveling wave solutions of the time fractional reaction–diffusion equation

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HIGHLIGHTS

\begin{itemize}
  \item The time-fractional reaction–diffusion equation is considered.
  \item Non-traveling wave solutions are given.
  \item Traveling wave solutions are given.
  \item Phenomenon of anomalous diffusion is discussed.
\end{itemize}

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ABSTRACT

In this study, solutions to the nonlinear time-dependent fractional reaction–diffusion equation with conformal fractional derivative is considered. In the first part of the manuscript, we reduce the fractional equation to a traditional differential equation using the fractional complex transformation. Two cases where the exact solution exists are then presented. One of these solutions is used to model experimental data showing anomalous diffusion in freestanding graphene. In the second part, we study the traveling solutions and present two cases: Canonical-like transformation method and complete discrimination system for polynomial method.

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1. Introduction

Constructing solutions to well-known differential equation is an important task for many fields of science. Finding exact solutions has resulted in major scientific breakthroughs. Nevertheless, there are still many unsolved problems, especially for nonlinear differential equations. Due to their complexity, many different methods have been developed to aid the process of finding exact and approximate solutions to nonlinear problems. For example, there is the direct method [1], inverse scattering method [2], renormalization group method [3–7], complete discrimination system for polynomial method [8–13]. Other results about exact solutions of nonlinear wave equations can be seen in [14–19].

Nowadays, fractional calculus plays an important role in many fields of science, and there are many ways to define the fractional derivative. For example, the Riemann–Liouville derivative, Jumarie’s modified Riemann–Liouville derivative [20], conformal fractional derivative [21] and so on. However, Liu has already proved that the formulae proposed by Jumarie about the modified Riemann–Liouville derivative are wrong [22,23]. On the other hand, since the usual derivative rules
hold for conformal fractional derivative, the corresponding fractional differential equations can be reduced to usual differential equations. So in this paper, we consider the (1+1)-dimensional general fractional nonlinear reaction–diffusion equation by using the definition of the conformal fractional derivative, namely

\[\frac{\partial^\alpha u}{\partial t^\alpha} = a(u^{n+1})_x + b(u^{m+1})_{xx} + \lambda u(1 - u^k),\]

where \(\frac{\partial^\alpha u}{\partial t^\alpha}\) represents the conformal fractional derivative, \(n, m\) and \(k\) are nature numbers, \(\lambda\) is an arbitrary constant and \(u\) is the function to be determined.

The conformal fractional derivative is defined by

\[D_t^\alpha(u(t)) = \lim_{h \to 0} \frac{u(t + ht^{1-\alpha}) - u(t)}{h}.\]

From Ref. [21], if the limitation (2) exists, then we have the following basic properties

(i) \(D_t^\alpha(u(t) \pm v(t)) = D_t^\alpha(u(t)) \pm D_t^\alpha(v(t))\),

(ii) \(D_t^\alpha(u(t)v(t)) = D_t^\alpha(u(t)v(t)) + D_t^\alpha(v(t))u(t)\),

(iii) \(D_t^\alpha(u(t)/v(t)) = D_t^\alpha(u(t)v(t))^{-1}D_t^\alpha(v(t))u(t)\),

(v) \(D_t^\alpha(u(t)) = t^{1-\alpha}D_t^\alpha(u(t))\).

The detailed proofs can be found in [21]. Other results about fractional calculus and conformable derivatives can be find in [24–30].

Eq. (1) contains many famous equations, for example, if \(a = \lambda = m = 0\) and \(\alpha = 1\), then it becomes the well-known linear diffusion equation

\[\frac{\partial u}{\partial t} = u_{xx}.\]

Alternatively, if \(n = m = 0\) and \(a = \lambda = 1\) then Eq. (1) reduces to a Fisher-type equation. If we also set \(k = 1\) then we get the fractional Fisher equation

\[\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + u(1 - u).\]

Moreover, if we take \(\lambda = 0\) and \(a = b = 1\), Eq. (1) just becomes the fractional Fokker–Planck (FP) equation

\[\frac{\partial^\mu u}{\partial t^\mu} = u^{n+1} + u^{m+1}\]

considered in Ref. [26]. The fractional Fokker–Planck equation can successfully model anomalous diffusion in stochastic motion [31,32]. Also, using this equation, many other physical phenomena can be modeled, and it has found numerous applications in engineering and business (see Ref. [32] and references therein). In general, fractional calculus is now considered one of the best approaches for modeling anomalous diffusion, which has been observed in plasmas [33], freestanding graphene [34], chaotic Hamiltonian systems, turbulence, etc. (see Ref. [35] and references therein). Studying the time fractional reaction–diffusion Eq. (1) is significant as it may shed light on many different fields of science.

The reaction–diffusion equation has been studied for many years and in a large number of studies. For example, E. V. Krishnan used the hyperbolic function method to obtain exact traveling wave solutions to the reaction–diffusion equation [36]. Deng [37] used the finite-element method to solve the space and time fractional FP equation. He proved that the convergence order is \(O(k^{2-\alpha} + h^\mu)\), where \(k\) is the time step size and \(h\) is the space step size. Lie symmetry analysis of the fractional FP equation was conducted by M. S. Hashemi, and he found exact analytical solutions using the reduction method [26]. Mao used the canonical-like transformation method and the trail equation method to investigate Chaffee–Infante equation [38], which is another famous reaction–diffusion equation. Some new solutions in the form of the elliptic functions were shown in that study, which are very difficult to be obtained by other methods. In this paper, we focus on constructing exact solutions to Eq. (1). Of course, there is no exact solution to the general equation, so we present several situations where an exact solution exists and show how to obtain them by various methods. Discussions about anomalous diffusion are presented and in order to better understand the dynamical properties of the solutions, specific examples are given and plotted.

2. Non-traveling wave solutions

One motivation for this work is the recent discovery of the anomalous diffusion properties in the dynamics of freestanding graphene (see Ref. [34]). Using point-mode scanning tunneling microscopy, researchers were able to precisely track the out-of-plane fluctuations in time. In fact, \(8 \times 106\) out-of-plane, height-time data points were continuously measured per run, providing a large and statistically significant data set spanning nearly seven decades. This data was collect using feedback-on, constant current (0.1 nA), and constant voltage (0.1 V) tunneling conditions. Additional details
are presented elsewhere [34]. From the time series data, \( u(t) \), we computed its mean-squared displacement \( \text{MSD}(\tau) = \langle [u(t + \tau) - u(t)]^2 \rangle \), which is shown in Fig. 1. This data is characterized by a power-law dependence of \( \text{MSD} \sim \tau^\alpha \) with \( \alpha \) being the anomalous diffusion exponent. The motion at short times is characterized by \( \alpha = 1.4 \) (superdiffusive motion) followed by a range for which \( \alpha = 0.3 \) (subdiffusive motion). In this section, we solve the fractional differential equation and predict this experimental outcome.

To begin, we first transform the fractional differential Eq. (1) into a traditional partial differential equation by taking a special kind of traveling wave transformation [21,25]. This equation is then solved for special conditions where the exact non-traveling wave solutions exist. After which, the phenomenon of anomalous diffusion can be revealed and discussed.

The fractional transformation is given by:

\[
\tau = \frac{t^\alpha}{\alpha}.
\]  

(7)

Using this, Eq. (1) is reduced to the following traditional partial differential equation:

\[
\frac{\partial u}{\partial \tau} = a(u^{n+1})u_x + b(u^{m+1})u_{xx} + \lambda u(1 - u^k).
\]  

(8)

Next, by constructing exact solutions to Eq. (8), we can obtain the corresponding solutions to Eq. (1). We do this for two cases.

**Case 2. 1.** If we set \( a = \lambda = m = 0 \), then Eq. (8) reduces to the traditional diffusion equation:

\[
\frac{\partial u}{\partial \tau} = bu_{xx},
\]  

(9)

where \( b \) is the diffusion coefficient. The solution starting from a point distribution is known to be:

\[
u(\tau, x) = \frac{M}{\sqrt{4\pi bt}} e^{-\frac{x^2}{4bt}},
\]  

(10)

where \( M \) is the total amount of solute, whose initial value is given by

\[f(x, 0) = M \delta(x),\]

(11)

The mean-square displacement from (10) is:

\[\langle x^2 \rangle = 2bt,\]

(12)

Undoing the fractional transformation yields:

\[\langle x^2 \rangle \sim \tau \sim t^\alpha,\]

(13)

which is the desired anomalous diffusion. In our situation, the parameter \( \alpha \) is not fixed for all time. There is a critical time \( t_0 \) when \( 0 \leq t < t_0, \alpha = \alpha_1 \) and when \( t \geq t_0, \alpha = \alpha_2 \). Specifically, for the diffusion of the carbon atoms in the freestanding graphene membranes [34] (see Fig. 1) we have

\[\langle x^2 \rangle \sim \begin{cases} t^{\alpha_1}, & 0 \leq t < t_0 \\ t^{\alpha_2}, & t \geq t_0. \end{cases}\]

(14)

To model this problem, we establish the following piecewise equations:

\[
\frac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}} = bu_{xx}, \quad 0 \leq t < t_0
\]

\[
\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}} = bu_{xx}, \quad t \geq t_0.
\]  

(15)

From Fig. 1 we can see that our results agree well with the experimental data.

**Case 2. 2.** If we set \( b = \lambda = 0 \), then Eq. (8) reduces to:

\[
\frac{\partial u}{\partial \tau} = a(n + 1)u^n u_x.
\]  

(16)

The differential equation can be solved using the characteristic method [39]. To do this, we first introduce the parameter, \( s \) and assume that \( u, \tau \) and \( x \) are all functions of \( s \), namely

\[u = u(s), \quad x = x(s), \quad \tau = \tau(s).\]

(17)

After combining Eq. (16) with (17), we obtain:

\[
\begin{align*}
\frac{du}{ds} &= 0, \\
\frac{dx}{ds} &= a(n + 1)u^n, \\
\frac{d\tau}{ds} &= -1.
\end{align*}
\]  

(18)
which yields
\[
\begin{align*}
\tau &= -s + c_0, \\
x &= a(n + 1)c_2^n s + c_1, \\
u &= c_2,
\end{align*}
\]
where \(c_0, c_1\) and \(c_2\) are all integral constants. From this we can see that the implicit function
\[
x = a(n + 1)u^n(c_0 - \tau) + F(u),
\]
is the solution of \((16)\), where \(F(u)\) is an arbitrary differentiable function. We will verify this conclusion in the following.

**Theorem 2.1.** The implicit function \((20)\) solves Eq. \((16)\).

**Proof.** Taking the derivative of the both sides of \((20)\) with respect of \(\tau\) and \(x\), respectively, we have
\[
\begin{align*}
0 &= an(n + 1)(c_0 - \tau)u^{n-1}x_x - a(n + 1)u^n + F'(u)x_x, \\
1 &= an(n + 1)(c_0 - \tau)u^{n-1}x_x + F'(u)x_x,
\end{align*}
\]
which yields
\[
\begin{align*}
x_x &= \frac{au^n}{an(n + 1)(c_0 - \tau)u^{n-1} + F'(u)}, \\
\end{align*}
\]
and
\[
\begin{align*}
x_x &= \frac{1}{an(n + 1)(c_0 - \tau)u^{n-1} + F'(u)}.
\end{align*}
\]
It is easy to see that formulae \((22)–(23)\) satisfy Eq. \((16)\), so the proof is completed.

By use of the original variables, we can get the solution of the original equation as
\[
x = a(n + 1)(c_0 + \frac{t^\alpha}{\alpha} u^n) + F(u).
\]

To show the nature of the solution of \((24)\), two examples of the comparison of general nonlinear reaction–diffusion equation\((\alpha = 1)\) and the general nonlinear fractional reaction–diffusion equation\((\alpha = 0.2)\) are presented in Figs. 2–3.

We can see from the figures that the results of the fractional equation and the traditional nonlinear equation are truly different.

**Remark 1.** Given different kinds of function \(F(u)\), we can get the solutions of different forms. We only show two particular cases here.
3. Traveling wave solutions

In this section, some special cases where the exact traveling wave solution exists are discussed. We will only focus on the complex nonlinear cases, other simple cases such as $\lambda = 0$, $a = 0$ are not shown here.

Case 3.1. $a = m = 0$, $\lambda = 1$, then Eq. (8) becomes the Fisher-type equation

$$u_t = bu_{xx} + \lambda u(1-u^k).$$

(25)

This equation contains the Fisher equation, Newell–Whitehead equation and others. We will use the traveling wave transformation:

$$\xi = A_1 \tau + lx,$$

(26)

then Eq. (25) can be changed into

$$A_1 u_\prime - b^2 u'' = \lambda u + \lambda u^{k+1},$$

(27)

where the superscript $'$ represents the derivation with respect to $\xi$. We will solve this equation via Canonical-like transformation method [40].

3.1. Canonical-like transformation method

We present the main idea and procedure of Canonical-like transformation method as follows. First, we consider the following ordinary differential equation

$$u''(\xi) - Au'(\xi) = Bu^\alpha(\xi) + Du(\xi).$$

(28)

By choosing $\omega = u'$, Eq. (28) becomes

$$\frac{d\omega}{du} = \frac{Bu^\alpha + Du + A\omega}{\omega}.$$

(29)

To obtain the solutions to Eq. (8), we assume that the functions $u$ and $\omega$ are dependent on the parameter $s$, this is just the Canonical-like transformation

$$u = a_{11}(s)\nu(s) + a_{12}(s)\nu'(s),$$

(30)
& can be changed into Eq. (28), thus we can get the corresponding solutions. In Fig. 4, the 3-D graph of the exact solution

Now we can get the exact solutions to Eq. (27). By taking

Thus Eq. (28) is reduced into

and is determined by

where

where

and is determined by

where

We can get the general solution of Eq. (35) in the integral form, namely

where is an integral constant. For example, if , we have

Upon obtaining , we can get the solution of the original equation , which is given by

Now we can get the exact solutions to Eq. (27). By taking , , and , then Eq. (27) can be changed into Eq. (28), thus we can get the corresponding solutions. In Fig. 4, the 3-D graph of the exact solution given by (39)-(40) and the corresponding 2-D graph for are plotted when

And when , , , then we can get the elliptic functions solutions, which are given by

where

and

where

and


Fig. 4. (a) The 3-D graph of solution (39)–(40). (b) the corresponding 2-D graph for $u(x, t)$, when $t = 1$.

where $c_1$ and $s_0$ are two arbitrary constants.

3.2. Complete discrimination system for polynomial method

In this section, the classification of the single traveling wave solutions is given to Eq. (1) under a special parameter condition, namely $m = 0, n = 0$ and $al = A_1$. Then by taking the traveling wave transformation (26), the original equation is changed into

$$b^2 u'' + \lambda u(1 - u^k) = 0,$$

which yields

$$(u')^2 = a_1 \left( \frac{u^{k+2}}{k+2} - \frac{1}{2} u^2 + c_0 \right),$$

where $c_0$ is an integral constant and $a_1 = \frac{2b^2}{al}$. Though this integration could not be solved in the general case, we can still get the solution if the parameters $a, k$ and $c_0$ are given. We present the case of $k = 2$ here and use the fourth-order complete discrimination system for polynomial method to obtain the classification of the single traveling wave solutions to Eq. (45). Other cases of $k$, for example, $k = 3$, can be handled similarly. From (46), we can see that the integral form of Eq. (45) is given by

$$\pm (\xi_1 - \xi_0) = \int \frac{du}{\sqrt{u^4 - 2u^2 + c}},$$

where $c = \frac{4c_0}{a}$ and $\xi_1 = \sqrt{\frac{a_1}{4} \xi}$. According to Refs. [8–13], by setting $F(u) = u^4 - 2u^2 + c$, the complete discrimination system is given by

$$D_1 = 4, D_2 = 2, D_3 = 16 - 16c, D_4 = 64c - 128c^2 + 64c^3, E_2 = 64c.$$  

**Case 3.2.1** When $D_2 > 0$, $D_3 = 0$, $D_4 = 0$ and $E_2 > 0$, $F(u)$ has two distinct real roots of multiplicities two, such that $F(u) = (u - \mu)^2 (u - \nu)^2$,

$$u = \frac{\nu - \mu}{e^{(\mu - \nu)(\xi_1 - \xi_0)} - 1} + \nu = \frac{\nu - \mu}{2 \coth \left( \frac{(\mu - \nu)(\xi_1 - \xi_0)}{2} \right)} - 1 + \nu,$$

and when $\nu < u < \mu$, we can get

$$u = \frac{\nu - \mu}{e^{(\mu - \nu)(\xi_1 - \xi_0)} - 1} + \nu = \frac{\nu - \mu}{2 \tanh \left( \frac{(\mu - \nu)(\xi_1 - \xi_0)}{2} \right)} - 1 + \nu,$$

where the expressions (51) and (52) are solitary wave solutions. For example, when $c = 1$, we have $\mu = -\nu = 1$, then we can get the solutions of Eq. (47) as

$$\pm (\xi_1 - \xi_0) = \ln \left| \frac{u - 1}{u + 1} \right|,$$
\[ u = \coth(\xi_1 - \xi_0), \]  
\[ \text{and} \quad u = \tanh(\xi_1 - \xi_0). \]  

We present graphs of (55) under the particular parameters in Fig. 5.

**Case 3.2** $D_2 > 0, D_3 > 0, \text{and} D_4 > 0.$ $F(u)$ has four distinct real roots, namely

\[ F(u) = (u - \alpha_1)(u - \alpha_2)(u - \alpha_3)(u - \alpha_4), \]  
where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4.$ Then we have

\[ \xi_1 - \xi_0 = \int \frac{du}{\sqrt{(u - \alpha_1)(u - \alpha_2)(u - \alpha_3)(u - \alpha_4)}}. \]  

When $\alpha_4 > 0,$ if $u > \alpha_1 \text{ or } u < \alpha_4,$ By Eq. (56) and the definition of Jacobian elliptic sine function, we have

\[ u = \alpha_2(\alpha_2 - \alpha_3) \text{s}n^2\left(\frac{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}{2}(\xi_1 - \xi_0), m\right) - \alpha_1(\alpha_2 - \alpha_4), \]  
and if $\alpha_3 < u < \alpha_2,$ then we can get

\[ u = \alpha_2(\alpha_2 - \alpha_3) \text{s}n^2\left(\frac{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}{2}(\xi_1 - \xi_0), m\right) - \alpha_3(\alpha_2 - \alpha_4), \]  
where $m^2 = \frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4)}.$

For example, when $c = \frac{3}{4},$ we have $\alpha_1 = -\alpha_4 = \sqrt{\frac{3}{2}}, \alpha_2 = -\alpha_3 = \frac{\sqrt{3}}{2},$ then we can get the solutions of Eq. (47) as

\[ u = \frac{\sqrt{3} \text{s}n^2\left(\frac{\sqrt{3}}{2}(\xi_1 - \xi_0), m\right) - \frac{\sqrt{3}}{2} - \frac{1}{2}}{\sqrt{6} \text{s}n^2\left(\frac{\sqrt{3}}{2}(\xi_1 - \xi_0), m\right) - \frac{\sqrt{6}}{2} - \frac{\sqrt{3}}{2}}, \]  
and

\[ u = \frac{-\sqrt{3} \text{s}n^2\left(\frac{\sqrt{3}}{2}(\xi_1 - \xi_0), m\right) + \sqrt{6}}{\sqrt{2} \text{s}n^2\left(\frac{\sqrt{6}}{2}(\xi_1 - \xi_0), m\right) - \frac{\sqrt{6}}{2} - \frac{\sqrt{3}}{2}}, \]  
where $m^2 = \frac{8\sqrt{3}}{8 + 8\sqrt{3}}.$ We give the corresponding 3-D and 2-D graphs in Fig. 6.

**Case 3.2** $D_2D_3 \geq 0 \text{ and } D_4 < 0.$ $F(u)$ has two distinct real roots and a pair of conjugate complex roots, i.e.

\[ F(u) = (u - \mu)(u - v)((u - h)^2 + s^2), \]  
where $\mu > v$ and $s > 0.$ From Eq. (47), we have

\[ \xi_1 - \xi_0 = \int \frac{du}{\sqrt{(u - \mu)(u - v)((u - h)^2 + s^2)}}. \]
Let
\[ a = \frac{1}{2}(\mu + \nu)c - \frac{1}{2}(\mu - \nu)d, \]  
\[ b = \frac{1}{2}(\mu + \nu)d - \frac{1}{2}(\mu - \nu)c, \]  
\[ c = \mu - h - \frac{s}{m_1}, \]  
\[ d = \mu - h - sm_1, \]  
\[ E = s^2 + (\mu - h)(\nu - h), \]  
\[ m_1 = E \pm \sqrt{E^2 + 1}, \]

then we can get the solutions to Eq. (47)

\[ u = \frac{a}{c} \text{cn}\left(\sqrt{\frac{2m_1(\mu - \nu)}{2m_1}}(\xi_1 - \xi_0), m_2\right) + b \]
\[ \frac{b}{c} \text{cn}\left(\sqrt{\frac{2m_1(\mu - \nu)}{2m_1}}(\xi_1 - \xi_0), m_2\right) + d \]

where \( m_2 = \frac{1}{1 + m_1^2} \), and the expression (70) is an elliptic function solution of double periodic. For example, when \( c = -8 \), we have \( \mu = -v = 2, h = 0, s = \sqrt{2} \), then the solution of Eq. (47) is given by

\[ u = -\text{cn}\left(\frac{\sqrt{2}sm_1(\mu - \nu)}{2m_1}(\xi_1 - \xi_0), m_2\right) + 1 \]

where \( m_1 = \frac{\sqrt{2}}{2} \) and \( m_2 = \frac{3}{2} \).

The corresponding 3-D and 2-D graphs are given in Fig. 7.

**Case 3. 2. 4**

\( D_2 > 0, D_3 > 0, \) and \( D_4 = 0. \) \( F(u) \) has two single real roots and a real root with multiplicities two, i.e.

\[ F(u) = (u - \alpha_1)(u - \alpha_2)^2, \]

where \( \alpha_1 > \alpha_2, \) and \( \alpha_1, \alpha_2, \alpha_3 \) are three different real numbers. Let

\[ r = \alpha_1 - \alpha_2, \frac{\alpha_1 + \alpha_2}{2} - \alpha_3. \]

From Eq. (47), we have

\[ (\xi_1 - \xi_0) = \int \frac{du}{(u - \alpha_3)\sqrt{(u - \alpha_1)(u - \alpha_2)}}. \]

Because of the relations between \( \alpha_1, \alpha_2 \) and \( \alpha_3, r \neq \pm 1. \) For \( r^2 - 1 > 0, \) we have

\[ (\xi_1 - \xi_0) = -\frac{1}{\sqrt{r^2 - 1}} \ln \left| \frac{y - r_1}{y + r_1} \right|. \]
Fig. 7. (a) The 3-D graph of (71), when \( a = 5, l = 1, \xi_0 = 0, A_1 = 1 \) and \( \alpha = 0.5 \). (b) the corresponding 2-D graph when \( t = 1 \).

Fig. 8. (a) The 3-D graph of (73), when \( a_1 = 5, l = 1, \xi_0 = -1, A_1 = 1 \) and \( \alpha = 0.5 \). (b) the corresponding 2-D graph when \( t = 1 \).

and when \( r^2 - 1 < 0 \), by using Eq. (72), we can obtain

\[
(\xi_1 - \xi_0) = -\sqrt{(1 - r^2)} \arctan \frac{r + 1}{1 - r} y, \tag{76}
\]

where \( r_1 = \sqrt{\frac{r + 1}{r - 1}} \) and \( y = \sqrt{1 - \frac{\alpha_1 - \alpha_2}{u - \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_1 - \alpha_2}{2}}} \).

For example, when \( c = 0 \), we have \( \alpha_1 = -\alpha_2 = \sqrt{2} \) and \( \alpha_3 = 0 \), then we can get the solution of Eq. (43) as

\[
u = \frac{2\sqrt{2}}{1 - \tan^2(\xi_1 - \xi_0)} - \sqrt{2}. \tag{77}\]

Corresponding graphs can be seen in Fig. 8.

4. Conclusion

In this paper, exact solutions of the generalized time fractional nonlinear reaction–diffusion equation are investigated. Non-traveling wave and traveling wave solutions were found. In Section 2, the fractional order differential equation is firstly transformed into the traditional differential equation and then two cases of non-traveling wave solutions were presented. One solution is related to anomalous diffusion, and we showed that by applying the fractional diffusion equation to this problem, the form of anomalous diffusion is naturally obtained. To model the anomalous diffusion data for carbon atoms in freestanding graphene membranes, we introduced the piecewise fractional equations. Traveling wave solutions were presented in Section 3. Specifically, the Canonical-like transformation method and the complete discrimination system for polynomial method were applied to the equation. Exact solutions were shown, including the elliptic function solution, which are normally very difficult to be obtained using other methods. Many specific solutions
and plots were presented as examples to show the nature of the solutions. Moreover, comparisons of the solution of the equation of fractional order and traditional nonlinear equation are also shown in the paper.

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