Ill-Posed Problems, Parabolic PDEs

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Book: Equations of Mathematical Physics

A.N. Tikhonov, A.A. Samarskii
What is an ill-posed problem?

A problem is ill-posed if it does not satisfy the 3 conditions of a well-posed problem:

- **Existence**: There exists a solution.
- **Uniqueness**: The solution is unique.
- **Stability**: The solution depends continuously on initial conditions.

The inverse of a well-posed problem is generally ill-posed.
Classification of PDEs

Given a PDE of the form:

\[ Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G \]

Where \( A \ldots G \) are functions of \((x, t)\), the PDE is classified by the value of the discriminant \( B^2 - 4AC \)

\[ B^2 - 4AC < 0 \quad \text{Elliptic} \quad \text{ex: Laplace’s equation} \quad u_{xx} + u_{tt} = 0 \]

\[ B^2 - 4AC = 0 \quad \text{Parabolic} \quad \text{ex: Heat equation} \quad u_t - u_{xx} = 0 \]

\[ B^2 - 4AC > 0 \quad \text{Hyperbolic} \quad \text{ex: Wave equation} \quad u_{xx} - u_{tt} = 0 \]
Solving a PDE - Separation of Variables

\[ u_t - u_{xx} = 0 \]

Assume the solution is of the form \( u(x, t) = X(x)T(t) \) then, 
\[ u_t = XT' \quad \text{and} \quad u_{xx} = X''T \]

\[ XT' - X''T = 0 \quad \rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda \]

Solving for \( X(x) \) and \( T(t) \) gives us the following solution sets:

<table>
<thead>
<tr>
<th>( \lambda &lt; 0 )</th>
<th>( \lambda = 0 )</th>
<th>( \lambda &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 e^{-\lambda t}(c_2 e^{\sqrt{-\lambda}x} + c_3 e^{-\sqrt{-\lambda}x}) )</td>
<td>( c_1(c_2x + c_3) )</td>
<td>( c_1 e^{-\lambda t}(c_2 \cos(\sqrt{\lambda}x) + c_3 \sin(\sqrt{\lambda}x)) )</td>
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Given conditions, we can narrow down to one form and solve w/ Fourier series.
Solving a PDE - Fourier series

Big idea: we can approximate any function of $x$ on an interval as a series of sine/cosine waves.

$$f(x) = -x, [-1, 1] \text{ can be approximated by } \frac{2}{\pi} \sum_{n=1}^{k} \frac{(-1)^n}{n} \sin(\pi nx)$$

We must represent $u(x, 0) = f(x)$ in this form in order to satisfy the PDE.
Parabolic PDEs: well-posed vs. ill-posed

<table>
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<th>Heat equation</th>
<th>(Reversed time)</th>
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<td>( u_t - u_{xx} = 0 )</td>
<td>( u_t + u_{xx} = 0 )</td>
</tr>
<tr>
<td>( u(x, 0) = f(x) )</td>
<td>( u(x, 0) = f(x) )</td>
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<tr>
<td>( u(0, t) = u(\pi, t) = 0 )</td>
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<tr>
<td>[ \sum_{n=1}^{\infty} f_n \sin(nx) e^{-n^2t} ]</td>
<td>[ \sum_{n=1}^{\infty} f_n \sin(nx) e^{n^2t} ]</td>
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Existence? ✓  Uniqueness? ✓  Stability? ✓

[Graph showing exponential decay for different values of \( n \)]

Existence? ×  Uniqueness? ✓  Stability? ×

[Graph showing exponential growth for different values of \( n \)]
Given the Fourier series we found for $f(x) = -x, [-1, 1]$ at $k = 3$, here's what the graph of $u(x, t)$ looks like:

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A reasonable approximation: Huh…?
Why heat equation with reverse time is ill-posed

As we increase $k$, one graph becomes more accurate, while the other becomes more and more chaotic. $k = 15$:

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More accurate

Chaos!
Thank you!

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