ESTIMATION OF INSTANTANEOUS FREQUENCY OF SIGNALS
USING THE CONTINUOUS WAVELET TRANSFORM

by

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ABSTRACT

This report reviews methods and algorithms for estimation of instantaneous frequency of signals based on the Continuous Wavelet Transform (CWT), with emphasis on their accuracy, and for the purpose of use in analysis of nonlinear response of soil-structure systems from recorded earthquake response. The methods reviewed are the Hilbert Transform method, the “Marseille” method (using the phase of the CWT), and the “Carmona” method (using the amplitude of the CWT), which works also for signals with considerable amount of additive noise, and the “simple” method (using the amplitude of the CWT). The theory behind these methods and algorithms for implementation is presented in depth. These methods are applied to signals with well-defined frequency changes, amplitude and frequency modulated so that they resemble pulses in the response of structures to strong earthquake ground motion. Signals with additive noise are also considered.

The revision dated January 2004 has corrected typographical mistakes and error in $\sigma_{r,0}$ (was 0.63 $\sigma$ and now is 0.71 $\sigma$) and $\sigma_{w,0}$ (was 0.63/$\sigma$ and now is 0.71/$\sigma$) for the Morlet wavelet, which implies minimal changes in Chapter 2 (eqns. (2.27)–(2.30), (2.40)–(2.42), and (2.44)–(2.46)).
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1. INTRODUCTION

Wavelet analysis has its roots in isolated work of mathematicians carried out mostly around 1930s. The first wavelet basis, the Haar’s basis, was introduced even earlier (in 1910), but the modern wavelet analysis, the way we refer to it now, was introduced not so long ago, in the 1980s, by French geophysicists (Morlet et al., 1982a,b; Goupillaud et al., 1984/85; Grossmann and Morlet, 1984). For a historical review of its roots, rediscoveries and more recent developments, the reader is referred to Meyer (1993). Since then, wavelet analysis is being used in many fields of science and engineering, probably most extensively in digital signal and image processing and communication, due to their efficiency in data compression (Vetterli and Kovacević, 1995). It is also used in artificial intelligence for defining contours of objects in bitmap images and for pattern recognition (Tang et al., 1999), in statistics, for nonparametric function estimation (Antoniades and Oppenheim, 1995), and in many other fields. The wavelet transform is particularly suitable for analysis of transient signals and time varying systems, because it is localized both in time and frequency. Its widespread use is also due to the existence of orthogonal and bi-orthogonal bases, and the availability of fast and accurate computational algorithms for signal/image transformation and reconstruction.


This report reviews methods and algorithms for estimation of instantaneous frequency of signals based on the Continuous Wavelet Transform (CWT), with emphasis on their accuracy, and for the purpose of later use in analysis of nonlinear response of soil-structure systems from recorded earthquake response (Trifunac et al., 2001a,b). The methods reviewed are the Hilbert Transform method, the “Marseille” method (using the phase of the CWT; Deplart et al., 1992), and the “Carmona” method (using the amplitude of the CWT; Carmona et al., 1997), which works also for signals with considerable amount of additive noise, and the “simple” method (using the amplitude of the CWT). Chapter 2 presents a theoretical background on wavelets as related to and consistent with the conventions used in published work on these methods. For a general background on wavelets, the reader is referred to standard books on wavelets, e.g. by Daubechies (1992) and by Vetterli and Kovacević (1995). Chapter 3 presents in detail the theory these methods are based on, and algorithms for their implementation. Chapter 4 presents
numerical results for well-defined signals, amplitude and frequency modulated so that they resemble pulses in the response of structures to strong earthquake ground motion. Signals with additive noise are also considered. All in Fortran by the author (except for the Fast Fourier Transform, random number generator, and Gaussian noise generator subroutines which were taken from Numerical Recipes; Press et al., 1992), used to produce the results in Chapter 3. Electronic files of the Fortran programs can be downloaded from the USC Strong Motion Research Group web site at www.usc.edu/dept/civil_eng/Earthquake_eng/. This report is also intended to serve as educational material for civil engineering students on the subject of estimation of instantaneous frequency of signals and systems.
2. WAVELETS - THEORETICAL BACKGROUND

This chapter presents a theoretical background on wavelets and on the continuous wavelet transform, as related to and following the convention of the methods for determination of instantaneous frequency reviewed in this report.

2.1 Wavelets and Wavelet Families

A wavelet is a zero mean wiggle (real or complex), localized both in time and in frequency (in other words, the wavelet is nonzero only near time \( t=t_0 \) and its Fourier transform is also nonzero near some frequency \( \omega=\omega_0 \)). For the zero mean condition (also called admissibility condition) to be satisfied, it must be oscillatory - hence the name wavelet (Vetterli and Kovacevic, 1995).

Given a prototype wavelet \( \psi(t) \in L_2(R) \), a family \( \psi_{a,b}(t) \) can be constructed by elementary operations consisting of time shifts and scaling (i.e. dilation or contraction). The family of wavelets is defined by

\[
\psi_{b,a}(t) = \frac{1}{a} \psi\left(\frac{t-b}{a}\right), \quad b \in R, \quad a > 0
\]  

(2.1)

where \( a \) is the scaling factor and \( b \) is the time shift. The prototype wavelet is called “the mother wavelet”, and it is the member of the family corresponding to \( b=0 \) and \( a=1 \). Scale factor \( a > 1 \) corresponds to dilation and \( a < 1 \) to contraction of the “mother wavelet”. In eqn (2.1), \( 1/a \) in front of \( \psi \) is a normalizing factor for the amplitude of the wavelet, chosen so that all wavelets in the family have same \( L_1 \) norm

\[
\int_{-\infty}^{\infty} |\psi_{b,a}(t)| dt = \int_{-\infty}^{\infty} |\psi(t)| dt
\]  

(2.2)

This convention is same as the one used by Carmona et al. (1995; 1997; 1998) and Deplart et al. (1992), and differs from the convention in most books and papers on wavelets where all wavelets in a family have same \( L_2 \) norm.

2.2 Localization in Time and Frequency

The time localization of a wavelet in a family of wavelets, i.e. its central time and spread around the central time, are defined via the average and variance of the time in the expression for \( \psi_{b,a}(t) \), as follows (Carmona et al., 1998; Flandrin, 1999)
\[
\overline{t}_{(b,a)} = \frac{\int_{-\infty}^{\infty} |\psi_{(b,a)}(t)|^2 \, dt}{\int_{-\infty}^{\infty} |\psi_{(b,a)}(t)|^2 \, dt}
\]  
(2.3)

\[
\sigma^2_{\tau_{(b,a)}} = \frac{\int_{-\infty}^{\infty} (t - \overline{t}_{(b,a)})^2 |\psi_{(b,a)}(t)|^2 \, dt}{\int_{-\infty}^{\infty} |\psi_{(b,a)}(t)|^2 \, dt}
\]  
(2.4)

Similarly, the central frequency and the spread around the central frequency are defined as the average and standard deviation of the circular frequency, \(\omega\), in the Fourier Transform of the wavelet, \(\hat{\psi}_{(b,a)}(\omega)\), as follows

\[
\overline{\omega}_{(b,a)} = \frac{\int_{-\infty}^{\infty} \omega |\hat{\psi}_{(b,a)}(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |\hat{\psi}_{(b,a)}(\omega)|^2 \, d\omega}
\]  
(2.5)

\[
\sigma^2_{\omega_{(b,a)}} = \frac{\int_{-\infty}^{\infty} (\omega - \overline{\omega}_{(b,a)})^2 |\hat{\psi}_{(b,a)}(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |\hat{\psi}_{(b,a)}(\omega)|^2 \, d\omega}
\]  
(2.6)

Let \(t_0\) and \(\omega_0\) be the central time and frequency of the “mother wavelet”, and \(\sigma^2_{\tau,0}\) and \(\sigma^2_{\omega,0}\) be the corresponding variances. Then eqns (2.3) through (2.6) imply the following relationships

\[
\overline{t}_{(b,a)} = t_0 + b
\]  
(2.7)

\[
\overline{\omega}_{(b,a)} = \frac{\omega_0}{a}
\]  
(2.8)

\[
\sigma^2_{\tau_{(b,a)}} = a^2 \sigma^2_{\tau,0}
\]  
(2.9)

and
\[ \sigma^2_{\omega, (b,a)} = \frac{\sigma^2_{\omega, 0}}{a^2} \]  

(2.10)

Equations (7) through (10) imply that a dilated wavelet \((a > 1)\) has \(a\) times lower frequency than the mother wavelet, and is better localized in frequency but more poorly in time.

The above statement is true for any function, and is formally expressed by the Heisenberg’s Uncertainty Principle, which states that a function cannot be arbitrarily well localized both in time and in frequency, and that the product of the uncertainties in those localizations is constant. If the uncertainties are expressed via the variances in eqns (2.4) and (2.6), then the Heisenberg’s Uncertainty Principle can be expressed by

\[ \sigma_t \cdot \sigma_\omega = C \leq \frac{1}{2} \]  

(2.11)

where the constant \(C\) is the smallest for Gaussian functions.

Obviously, increased localization in time can be achieved only at the cost of decreased localization in frequency. Two extreme functions are the complex exponential and the Dirac Delta-function. The complex exponential is perfectly localized in frequency (i.e. at a point) but not localized at all in time, while the Dirac Delta-function is perfectly localized in time but is not localized at all in frequency (its Fourier Transform is that of white noise). A windowed complex exponential is localized in time but not perfectly, and the localization depends on the length of the window. For a larger window, the spread in time increases but the spread in frequency decreases.

2.3 The Continuous Wavelet Transform

The continuous wavelet transform of a function \(f(t) \in L^2(R)\) is defined as its inner product with a family of admissible wavelets \(\psi_{b,a}(t)\), i.e.

\[ T_f(b,a) = < f, \psi_{(b,a)} >_{L^2} = \int_{-\infty}^{\infty} f(t) \overline{\psi_{(b,a)}}(t) dt \]  

(2.12)

where \(b\) and \(a\) are the time and scale variables, and the bar over \(\psi\) indicates complex conjugate. The inverse transform is (Carmona et al., 1998)

\[ f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_f(b,a) \psi_{(b,a)}(t) \frac{dadb}{a} \]  

(2.13)
where

\[ C_{\psi} = \int_{0}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega \quad (2.14) \]

We note that the reconstruction formula, eqs (2.13), and the expression for \( C_{\psi} \) in eqn (2.13) are such that they are consistent with the definition of the wavelet family in eqn (2.1).

In the Wavelet Transform, the scaling factor \( a \) for the wavelet family is used as a variable called *scale*. The scale is related to frequency. The frequency corresponding to scale \( a \) is the central frequency of the wavelet divided by \( a \), i.e.

\[ \omega = \omega_0 / a \quad (2.15) \]

Hence, *scale* is inversely proportional to frequency.

### 2.4 The Wavelet Transform as a Time-Frequency Energy Distribution

Because the wavelet transform \( T_f(b, a) \) is essentially the projection of \( f(t) \) onto wavelet \( \psi_{(b,a)} \), which is nonzero only near the central time of \( \psi_{(b,a)} \) and the central frequency of \( \tilde{\psi}_{(b,a)} \), it describes the properties of the function near these localities, and is a form of a time-frequency energy distribution of \( f(t) \). The localization properties of the transform are those of the wavelet family, and \( T_f(b, a) \) represents the properties of \( f(t) \) in the cell

\[ \left( t_{(b,a)} \pm \sigma_{t, (b,a)} \right) \times \left( \omega_{(b,a)} \pm \sigma_{\omega, (b,a)} \right) = \left( (t_0 + b) \pm a \sigma_{t, 0} \right) \times \left( \omega_0 \pm a \sigma_{\omega, 0} / a \right) \quad (2.16) \]

of the time-phase plane. Figure 2.1 shows three such beans, for \( a=1, a<1 \) and \( a>1 \). It can be seen that for frequencies \( \omega > \omega_0 \) (\( a < 1 \)) the bean is stretched along the \( \omega \)-axis, but compressed along the \( t \)-axis, and the opposite is true for \( \omega < \omega_0 \) (\( a > 1 \)). This means that the lower frequency components are better localized in frequency than higher frequency components.

An important property of the Wavelet Transform is that while the spread in frequency \( \Delta \omega \) is variable, the *relative spread* \( \Delta \omega / \omega \) is constant.
Wavelet Transform

\[
\frac{\Delta \omega}{\omega} = \frac{\sigma_{\omega,0}/a}{\omega_0/a} = \frac{\sigma_{\omega,0}}{\omega_0} = \text{const.}
\]

Fig. 2.1 Time-frequency resolution for the Wavelet Transform.
\[ \frac{\Delta \omega}{\omega} = \frac{\sigma_{\omega_0}}{a} = \frac{\sigma_{\omega}}{\omega_0} \]  

This property implies that the Wavelet Transform is better suited for time frequency analysis of signals with large variations in frequency (represented, e.g. on a logarithmic scale) than time-frequency distributions with constant spread.

### 2.5 The Morlet Wavelet

For analyses of phase related properties of real functions (e.g. determination of instantaneous frequency), the complex Continuous Wavelet Transform is more suitable than the real Wavelet Transform or the discrete Wavelet Transform. The most commonly used mother wavelet for such applications is the complex Morlet wavelet

\[ \hat{\psi}(t) = \exp(i\omega t) \exp(-t^2/2\sigma^2) \]  

which is essentially a complex exponential modulated by a Gaussian envelope. The amplitude is normalized so that \( \hat{\psi}(0) = 1 \), following the convention of Carmona et al. (1995; 1997), and \( \sigma \) is a measure of the spread in time. The Morlet wavelet has Fourier transform

\[ \hat{\psi}(\omega) = \sqrt{2\pi} \ \sigma \exp \left[ -\frac{1}{2} (\omega - \omega_0)^2 \sigma^2 \right] \]  

where the Fourier Transform of a function \( f(t) \) and the inverse transform are defined by

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \]  

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \]

The Fourier Transform of the Morlet wavelet is nonzero only for \( \omega > 0 \). Such wavelets are called **progressive**, and are particularly convenient for analyses of phase of real valued functions, the Fourier Transform of which for negative frequencies is the complex conjugate of those for positive frequencies (this simplifies the analysis without loss of information). The subspace of \( L_2 \) containing functions with such a property are called Hardy spaces.
\[ H(R) = \left\{ f \in L^2(R) : \hat{f} = 0, \omega < 0 \right\} \] (2.22)

The popularity of the Morlet wavelet as an analysis tool is due to the fact that it is described by an analytic function, and so is its Fourier Transform. A not so good quality is that it has infinite support, but this is not a practical problem as its envelope rapidly decreases away from \( t=0 \). Also, the Morlet wavelet is not strictly speaking an admissible wavelet, but for \( \omega_0 > 5 \) (if \( \sigma=1 \), the admissibility condition is practically satisfied
\[ (\int_{-\infty}^{\infty} \psi(t) dt \approx 0) \). The rigorous definition of the Morlet wavelet that satisfies literally the admissibility condition is (Foufoula-Georgiou and Kumar, 1994)

\[ \psi(t) = \exp\left[i \omega_0 t - \exp\left(-\omega_0^2 / 2\right)\right] \exp(-t^2 / 2\sigma^2) \] (2.23)

but this form is almost never used.

Figure 2.2 shows the Morlet wavelet and its Fourier Transform amplitude at scales \( a=1 \) and 2, computed for \( \sigma=1 \) and central frequency \( \omega_0=2\pi \). Commonly used value in literature are also \( \sigma=1 \) and \( \omega_0=5.6 \), which is frequency for which the amplitude of the second largest lobe of the wavelet is half of the amplitude of the largest lobe. In this report, \( \omega_0=2\pi \) will be used.

The \( L_1 \) norm of the mother wavelet (and of the entire family, according to the convention adopted in this report) is

\[ \|\psi(t)\|_{L_1} = \int_{-\infty}^{\infty} |\psi(t)| dt = \sqrt{2\pi\sigma} \] (2.24)

The scaled wavelet and its Fourier Transform are

\[ \psi_{(b,\omega)}(t) = \frac{1}{a} \exp\left[i \omega_0 \left( \frac{t-b}{a} \right) \right] \exp\left[-\left(\frac{t-b}{a}\right)^2 / 2\sigma^2 \right] \] (2.25)

\[ \bar{\psi}_{(b,\omega)}(\omega) = \exp(-i\omega b) \bar{\psi}(a\omega) \] (2.26)

The localization properties of the Morlet wavelet are as follows. The central time of \( \psi(t) \) is \( t_0 = 0 \), the central frequency is \( \omega_0 \), the variances in time and frequency are
Fig. 2.2  Three wavelets of the Morlet family with $\omega_0 = 2\pi$ and $\sigma = 1$, in the time domain (top) and in the frequency domain (bottom).
\[ \sigma_{t,0}^2 = \frac{1}{2} \sigma_t^2 \quad \Leftrightarrow \quad \sigma_{t,0} = 0.71 \sigma \] (2.27)

\[ \sigma_{\omega,0}^2 = \frac{1}{2} \sigma_{\omega}^2 \quad \Leftrightarrow \quad \sigma_{\omega,0} = 0.71 \frac{1}{\sigma} \] (2.28)

the Heisenberg inequality becomes

\[ \sigma_{t,0} \cdot \sigma_{\omega,0} = \frac{1}{2} \] (2.29)

and the relative localization in frequency is

\[ \Delta \omega = \frac{\sigma_{\omega,0}}{a} = \frac{\sigma_{\omega,0}}{\omega_0} = \frac{1}{\sqrt{2} \sigma \omega_0} \] (2.30)

2.6 The Parseval Equality and Computation of Wavelet Transform via FFT

The Parseval equality for the inner product of two functions \( f \) and \( g \) is

\[ \int_{-\infty}^{\infty} f(t)\overline{g}(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}}(\omega)d\omega \] (2.31)

and it implies

\[ T_j(b,a) = \int_{-\infty}^{\infty} f(t)\overline{\psi}_{(b,a)}(t)dt \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\psi}_{(b,a)}(\omega)d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\psi}(a\omega)e^{i\omega b}d\omega \]

\[ = FT^{-1}\left\{ \hat{f}(\omega)\overline{\psi}(a\omega) \right\} \] (2.32)

Then \( T_j(b,a) \) can be computed efficiently using Fast Fourier Transform, by first computing \( \hat{f}(\omega) \), and then computing the inverse transform of the product \( \hat{f}(\omega)\overline{\psi}(a\omega) \), where \( \overline{\psi}(a\omega) \) is computed via eqn (2.19).
2.7 The Windowed Fourier Transform and the Gabor Transform

A usual tool for time frequency analysis has been the Windowed Fourier Transform. The Windowed Fourier Transform of a function $f$, $WFT_f$, consists of multiplying $f(t)$ by a (usually real) window function $w$ shifted in time. If $w(t)$ is a prototype window, symmetric about $t=0$, then

$$WFT_f(b, \omega) = \int_{-\infty}^{\infty} f(t)w(t-b)e^{-i\omega t} dt$$ (2.33)

The window is some tapering function, e.g. a rectangular box, box with linear ramps at the ends, triangular (Barlett), Hanning, Hamming, Blackman and other windows centered at $t=0$.

The Gabor Transform is essentially a Windowed Fourier Transform with a Gaussian window. It is defined as the inner product between $f(t)$ and a family of functions $g_{(b, \omega)}(t)$ called Gabor functions

$$G_f(b, \omega) = \int_{-\infty}^{\infty} f(t)\overline{g_{(b, \omega)}}(t)dt$$ (2.34)

where

$$g_{(b, \omega)}(t) = g(t-b)e^{i\omega(t-b)}$$ (2.35)

and $g(t-b)$ is a shifted Gaussian window. A comparison of eqns (2.33), (2.34) and (2.35) shows that $WFT_f(b, \omega)$ is essentially equal to $G_f(b, \omega)$ except for a phase shift in $w(t-b)$. Sometimes, the Gabor Transform is defined without this phase shift. In this report, the definition in eqns (2.34) and (2.35) is adopted (Carmona et al., 1998).

Substituting in $g_{(b, \omega)}(t)$ for a Gaussian window implies

$$g_{(b, \omega)}(t) = \exp\left[-\frac{(t-b)^2}{2\sigma^2}\right]\exp[i\omega(t-b)]$$ (2.36)

which has Fourier Transform

$$\hat{g}_{(b, \omega)}(\xi) = \exp[-i\xi b]\hat{g}_{(0, \omega)}(\xi)$$

$$= \exp[-i\xi b]\sqrt{2\pi}\sigma\exp\left[-\frac{1}{2}(\xi - \omega)^2\sigma^2\right]$$ (2.37)
Numerically, the Gabor Transform can also be computed using the Parseval equality and Fast Fourier Transform, as follows

\[ G_f(b, \omega) = \int_{-\infty}^{\infty} f(t) \overline{g_{(b,\omega)}(t)} dt \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}_{(b,\omega)}(\xi)} d\xi \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}_{(b,\omega)}(\xi)} e^{i\omega \xi} d\xi \]

\[ = \mathcal{F}^{-1}\left\{ \hat{f}(\xi) \overline{\hat{g}_{(b,\omega)}(\xi)} \right\} \]  \hspace{1cm} (2.38)

The similarity between the Continuous Wavelet Transform with the Morlet wavelet and the Gabor Transform is striking. To compare these two transforms most directly, we express a scaled wavelet in terms of \( \omega \) instead of scale, recalling that \( \omega_{b} / a = \omega \)

\[ \psi_{(b,\omega)}(t) = \frac{1}{a} \exp\left[ -\frac{(t-b)^2}{2(a\sigma)^2} \right] \exp\left[ i\omega(t-b) \right] \]  \hspace{1cm} (2.39)

A comparison of eqns (2.36) and (2.39) shows that \( \psi_{(b,\omega)}(t) \) is essentially \( g_{(b,\omega)}(t) \) but with a variable window width, depending on the value of the scale \( a \). (There is also a normalization factor for the amplitude but this is not significant.) At scale \( a = 1 \), they are identical. At scales \( a > 1 \) (i.e. for smaller frequencies), the wavelet transform has a wider window, while the window width for the Gabor Transform is fixed for all frequencies. An implication of this is that the wavelet always contains the same number of wavelengths, while the Gabor function contains a smaller number of wavelengths for longer periods. As both transforms are meaningful at a particular period only if there is at least one wavelength contained within the width of the window (i.e. within part of the window with significant amplitudes), this implies greater flexibility for the Wavelet Transform, which can be used for a larger span of frequencies for a chosen “mother wavelet” and Gabor function.

The localization properties of the Gabor Transform are determined by those of the Gaussian window. As the width of the window is constant for all frequencies, the variance of the time localization will be the same for all frequencies, and, by the Heisenberg uncertainty principle, the variance of the frequency localization will also be the same for all frequencies. These variances are same as those for the Morlet wavelet in eqns (2.27) and (2.28) and are those for the entire family of the Gabor functions.
\[ \Delta \omega = \sigma_\omega = \text{const.} \]

Fig. 2.3  Time-frequency resolution for the Gabor Transform.
\[
\sigma_i^2 = \frac{1}{2} \sigma^2 \quad \Leftrightarrow \quad \sigma_i = 0.71 \sigma \quad (2.40)
\]

\[
\sigma_\omega^2 = \frac{1}{2} \frac{1}{\sigma^2} \quad \Leftrightarrow \quad \sigma_\omega = 0.71 \frac{1}{\sigma} \quad (2.41)
\]

and

\[
\sigma_i \cdot \sigma_\omega = \frac{1}{2} \quad (2.42)
\]

Figure 2.3 shows the constant size localization cells in the time-frequency plane for the Gabor Transform.

Again, the main difference between the Wavelet and Gabor Transforms is that the former offers same relative accuracy of localization in frequency (same $\Delta \omega/\omega$), while the latter offers same absolute accuracy of localization in frequency (same $\Delta \omega$). A comparison of lines 4 in eqns (2.32) and (2.38) shows that the computational efforts to compute both, following the Parseval equality and using FFT, are the same. We also note that for both transforms, inverse FFT needs to be evaluated for a chosen grid of scales (frequencies) within the range of interest, and each run of inverse FFT gives the transforms for all times $b$. Although an arbitrarily fine grid can be preselected, the accuracy of localization in frequency cannot be arbitrarily small. For the Gabor Transform, the absolute accuracy of frequency localization ($\Delta \omega$) is limited by $\sigma$ of the Gaussian window, and it can be increased only by increasing $\sigma$. For the Wavelet Transform, the relative accuracy of frequency localization ($\Delta \omega/\omega$) can be increased by decreasing $\sigma$, by decreasing $\omega_0$, or by decreasing both (see eqn (2.30)). If both $\sigma$ and $\omega_0$ change but their product remains the same, $\Delta \omega/\omega$ will not change.

2.8 Meaningful Scales and Frequencies for Computing the Wavelet Transform and the Gabor Transform

The maximum frequency, $\omega_{\text{max}}$, for which both the Continuous Wavelet Transform and the Gabor Transform are meaningful is determined by the sampling rate of the signal, and is its the Nyquist frequency, $\omega_{\text{Nyquist}}$, (Oppenheim and Schafer, 1999)

\[
\omega_{\text{max}} < \omega_{\text{Nyquist}} = \frac{\pi}{\Delta t} \quad (2.43)
\]
where $\Delta t$ is the sampling time interval, which implies minimum meaningful scale for the Wavelet Transform

$$a_{\text{min}} > \frac{\omega_0 \Delta t}{\pi}$$  \hspace{1cm} (2.43)

The minimum meaningful frequency (maximum scale) is determined by the length of the window in a way that there should be at least one period $T = 2\pi / \omega$ contained within the window of the wavelet or of the Gabor function, and ultimately by the length of the signal in the sense that the window length should always be shorter than the length of the signal. For a Gaussian window (with infinite support and nonuniform amplitude), this condition is expressed in terms of $\sigma$, e.g. that there should be at least one period contained in $6\sigma$ (Foufula-Georgiou and Kumar, 1994). This implies for the Gabor Transform, as defined in this report,

$$\omega_{\text{min}} = \frac{2\pi}{T_{\text{max}}} > \frac{2\pi}{6\sigma} = \frac{2\pi}{6(0.71\sigma)} > \frac{2\pi}{N \Delta t}$$  \hspace{1cm} (2.44)

For the Wavelet Transform, the window length is flexible and increases proportionally with the scale, and once $\omega_0$ and $\sigma$ have been set for the mother wavelet so that the wavelet does not violate much the admissibility condition, the upper bound for maximum scale is determined by the length of the signal. For the Wavelet Transform with the Morlet wavelet as defined in this report

$$6\sigma_{1, (b, a)} = 6a_{\text{max}} \sigma_{1, 0} = 6a_{\text{max}} (0.71\sigma) < N \Delta t$$  \hspace{1cm} (2.45)

which implies

$$a_{\text{max}} < \frac{N \Delta t}{6\sigma} = \frac{N \Delta t}{6(0.71\sigma)}$$  \hspace{1cm} (2.46)
3. INSTANTENEOUS FREQUENCY OF SIGNALS

This chapter presents the theoretical background related to the definition of instantaneous frequency of signals (Hilbert Transform, analytic signals associated with real signals, asymptotic signals, and principle of stationary phase), and two methods for determination of instantaneous frequency of asymptotic signals, one from the phase (called “Marseille” method by Carmona et al., 1997) and the other one from the amplitude (referred to as “Carmona” method in this report) of the Continuous Wavelet Transform.

3.1 Definition of Instantaneous Frequency

Frequency is the derivative of phase with respect to time, and physical signals are real valued. Then, an intuitively reasoning would suggest that instantaneous frequency \( \omega(t) \) of a signal \( f(t) \) could be defined by first writing it as

\[
f(t) = A(t) \cos \Phi(t)
\]

where both \( A(t) \) and \( \Phi(t) \) are real, and then differentiating \( \Phi(t) \)

\[
\omega(t) = \frac{d\Phi}{dt}
\]

One problem with this approach is that there are infinitely many ways of writing \( f(t) \) as in eqn (3.1), and additional constraints are needed for such a representation to be unique. This can be done with the help of the Hilbert Transform, as follows.

3.1.1 Hilbert Transform

The Hilbert Transform of a real function \( f(x) \) is by definition

\[
H_f(x) = \frac{1}{\pi} P \int f(x + y) \frac{dy}{y}, \ x \in R
\]

where \( P \) indicates principal value of the integral (i.e. up to a scaling factor, the Hilbert Transform is the principal value of the convolution of \( f(x) \) with \( 1/x \)). Its Fourier Transform has a very simple and convenient form, which describes its usefulness

\[
\hat{H}_f(\omega) = -i \text{sgn}(\omega) \hat{f}(\omega), \ \omega \in R
\]
Equation (3.4) implies that \( \hat{f}(\omega) + i\hat{H}_f(\omega) \) is nonzero only for nonnegative frequencies. This motivates the definition of an analytic signal associated with a real signal.

### 3.1.2 Analytic Signal Associated with a Real Signal and Hilbert Transform Method for Determination of Instantaneous Frequency

The analytic signal \( Z_f(x) \) associated with a real signal \( f(x) \) is by definition the complex signal the real part of which is \( f(x) \) and the imaginary part of which is the Hilbert Transform of \( f(x) \)

\[
Z_f(x) = f(x) + iH_f(x) \quad (3.5)
\]

Its Fourier Transform is

\[
\hat{Z}_f(\omega) = \hat{f}(\omega) + i\hat{H}_f(\omega)
\]

\[
= \hat{f}(\omega) + \text{sgn}(\omega) \hat{f}(\omega)
\]

\[
= \begin{cases} 
2\hat{f}(\omega), & \omega \geq 0 \\
0, & \omega < 0 
\end{cases} \quad (3.6)
\]

where \( H(\omega) \) is the Heaviside step function. Equation (3.6) shows that \( Z_f(x) \) has energy only in the nonnegative part of the spectrum (i.e. it is a member of a Hardy space)

\[
Z_f \in H(R) = \left\{ f \in L_2(R): \hat{f} = 0, \quad \omega < 0 \right\} \quad (3.7)
\]

in which domain its Fourier Transform is twice the Fourier Transform of \( f(x) \). This result is very convenient for analysis of real and causal processes, as it offers the advantages of the complex function spaces without introducing additional information (for real and causal processes, all the information is contained in the domain \( \omega > 0 \), and negative frequencies have no physical meaning).

Complex function \( Z_f(x) \) is an analytic function in the upper complex plane (hence the attribute analytic), and has a unique polar coordinate representation

\[
Z_f(t) = |Z_f(t)|e^{i\arg Z_f(t)} \quad (3.8)
\]

where
\[ |Z_f(t)| = \sqrt{[\Re Z_f(t)]^2 + [\Im Z_f(t)]^2} \]  
(3.9)

\[ \arg Z_f(t) = \arctan \frac{\Im Z_f(t)}{\Re Z_f(t)} \]  
(3.10)

and where \( \Re \) and \( \Im \) indicate real and imaginary parts of a complex number.

Then a unique representation of the form in eqn (3.1) can be defined, referred to as the \textit{canonical} representation of a real signal

\[ f(t) = A(t)\cos \Phi(t) \]  
(3.11)

where

\[ f(t) = \Re Z_f(t) \]  
(3.12)

\[ A(t) = |Z_f(t)| \]  
(3.13)

\[ \Phi(t) = \arg Z_f(t) \]  
(3.14)

The instantaneous frequency then can be defined as in eqn (3.2), i.e.

\[ \omega(t) = \frac{d}{dt} \arg Z_f(t) \]  
(3.15)

This method for determination of instantaneous frequency is referred to as the Hilbert Transform Method. It is the most direct method for determination of instantaneous frequency, and is easy to implement. Also, besides the frequency modulation, \( \omega(t) \), it also gives the amplitude modulation of the signal, \( A(t) \). However, this method has the following disadvantages: (1) \( \omega(t) \) defined by this method does not always have the physical meaning of frequency, and (2) numerical evaluation of derivatives is often unstable, especially for real life signals.

### 3.2 Asymptotic Signals

For some real signals \( f(t) = A(t)\cos \Phi(t) \), the associated analytic signal is \( Z_f(t) = A(t)e^{j\Phi(t)} \), but this is not true in general. Signals for which this is true are called \textit{asymptotic} signals, formally defined as follows.
**Definition:** Given a real square integrable signal \( f(t) = A(t)\cos\Phi(t) \) where \( A(t) \geq 0 \) and \( \Phi(t) \in [0,2\pi] \) for all \( t \in \mathbb{R} \), \( f(t) \) is said to be **asymptotic** if it is oscillatory enough so that its associated analytic signal is \( Z_f(t) = A(t)e^{i\Phi(t)} \). Here **oscillatory enough** means that the variations of \( f(t) \) due to \( \cos\Phi(t) \) (i.e. change of phase) are much faster than the variations due to \( A(t) \) (i.e. amplitude modulation).

An example of a signal that is not asymptotic is \( f(t) = \cos\frac{1}{t} \), for which the associated analytic signal is \( Z_f(t) = f(t) + iH_f(t) = \sin\frac{1}{t} + i\left(\cos\frac{1}{t} - 1\right) = e^{\frac{i}{t}} - i \neq e^{\frac{i}{t}} \).

The above property of asymptotic signals follows from the following lemma.

**Lemma:** Let \( f(t) = A(t)\cos\Phi(t) \), where \( \lambda \gg 1 \) (i.e. large positive number), \( A(t) \in C_2(\mathbb{R}) \) (space of twice continuously differentiable functions) and \( \Phi(t) \in C_4(\mathbb{R}) \) (space of four times continuously differentiable functions). Then, as \( \lambda \to \infty \), the analytic signal associated with \( f(t) \) is \( Z_f(t) = A(t)e^{i\Phi(t)} + O(\lambda^{-3/2}) \).

### 3.3 Method of Stationary Phase for Approximation of Integrals

The **Method of Stationary Phase** is based on the following principle.

**The Principal of Stationary Phase** (proposed by Lord Kelvin): Let

\[
I(x) = \int_a^b A(x)e^{i\lambda\phi(x)} \, dx \tag{3.16}
\]

where \( A(z) \) and \( \phi(z) \) are analytic on the complex plane and real valued on the real line. Then, as \( \lambda \to \infty \) (i.e. the integrand is very oscillatory) the dominant terms in the asymptotic expansion of \( I(x) \) arise from the immediate neighborhood of the end points, and intermediate points at which \( \lambda\phi(x) \) is stationary, i.e. \( \lambda\phi'(x) = 0 \).

The physical interpretation of this principle is as follows. When \( \lambda \to \infty \) and \( \lambda\phi'(x) \neq 0 \), \( \omega = \lambda\phi'(x) \) is very large, which means that the signal is very oscillatory, so oscillatory that the amplitude \( A(x) \) changes very little during one cycle and the integral in eqn (3.16) evaluated over one cycle is zero. At points \( x = x_0 \) where \( \phi'(x) = 0 \) (i.e. \( \phi(x) \) is stationary), this cancellation does not occur. So only contributions from the stationary points and the boundary points add to the value of the integral.
Method of Stationary Phase: Let \( A(z) \) and \( \phi(z) \) be analytic complex functions that are real on the real line. If \( A(x)e^{i\phi(x)} \) has only one stationary point, \( x = x_0 \) (of first order), then

\[
\lim_{\lambda \to \infty} I(x) = \lim_{x \to a} \int_a^b A(x)e^{i\phi(x)} \, dx = \frac{\sqrt{2\pi}e^{i\pi/4}\text{sgn}[\phi''(x_0)]}{\sqrt{\lambda |\phi''(x_0)|}} A(x_0)e^{i\phi(x_0)} + O\left(\frac{1}{\lambda}\right) \quad (3.17)
\]

In case of more stationary points, the limit of the integral is the sum of the contributions from all the stationary points. Equation (3.17) shows that for large \( \lambda \)

\[
I(x) = \text{value of the integrand at } x = x_0 \times \text{(correction term)} \quad (3.18)
\]

The correction term depends only on \( \phi''(x) \). Equation (3.17) holds for \( \phi''(x) = 0 \) but \( \phi''(x_0) \neq 0 \), i.e. for stationary points of first order. For the general case of stationary points of order \( n \), i.e. when \( \phi^{(n)}(x) = 0 \) but \( \phi^{(n+1)}(x_0) \neq 0 \), the correction term depends on \( \phi^{(n+1)}(x_0) \neq 0 \) (see eqns (4.10) and (4.11) in Deplat et al., 1992).

3.4 Estimation of Instantaneous Frequency From the Phase of the Continuous Wavelet Transform (Marseille Method)

This section describes the method for estimation of instantaneous frequency of signals from the phase of the continuous wavelet transform proposed by Deplart et. al. (1992). We will refer to it shortly as the Marseille Method (Carmona at al., 1998). The theoretical background for this method is first described and then the algorithm is presented.

3.4.1 Continuous Wavelet Transform of Asymptotic Signals

The objective is to determine the instantaneous frequency of the asymptotic signal

\[
s(t) = A_s(t) \cos \phi_s(t) \quad (3.19)
\]

from its Continuous Wavelet Transform

Let

\[
Z(t) = A_s(t)e^{i\phi_s(t)} \quad (3.20)
\]
be its associated analytic signal, and let \( \psi_{(b,a)}(t) \) be a family of progressive wavelets (i.e. a wavelet whose Fourier Transform is nonnegative). The wavelet transform of \( s(t) \) is then a Hardy function. We will use the Morlet wavelet but the derivation of the method will be presented for a general progressive and asymptotic wavelet

\[
\psi(t) = A_\psi(t) e^{i\phi_\psi(t)} \tag{3.21}
\]

The continuous wavelet transform of \( s(t) \) is

\[
T_s(b,a) = < s, \psi_{(b,a)} >_{L_2} = \int_{-\infty}^{\infty} s(t) \overline{\psi_{(b,a)}(t)} dt = \frac{1}{a} \int_{-\infty}^{\infty} s(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt = \frac{1}{2a} \int_{-\infty}^{\infty} Z(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \tag{3.22}
\]

Substitution of eqns (3.20) and (3.21) into the last line of (3.22) gives

\[
T_s(b,a) = \frac{1}{2a} \int_{-\infty}^{\infty} A_s(t) A_\psi\left(\frac{t-b}{a}\right) e^{i\phi_{(b,a)}(t)} dt \tag{3.23}
\]

where

\[
\Phi_{(b,a)}(t) = \phi_s(t) - \phi_\psi\left(\frac{t-b}{a}\right) \tag{3.24}
\]

is the phase of \( T_s(b,a) \).

If both the signal and wavelet are asymptotic, then the integrand in eqn (3.23) is also asymptotic. Let \( t = t_0 \) be a stationary point of first order of this integrand, i.e. such that

\[
\Phi_{(b,a)}(t_0) = \phi_s'(t_0) - \phi_\psi'\left(\frac{t_0-b}{a}\right) = 0 \tag{3.25}
\]

and

\[
\Phi_{(b,a)}''(t_0) = \phi_s''(t_0) - \frac{1}{a^2} \phi_\psi''\left(\frac{t_0-b}{a}\right) \neq 0 \tag{3.26}
\]
Then by the Method of Stationary Points, $T_s(b, a)$ can be approximated by

$$T_s(b, a) \approx \frac{1}{2a} \sqrt{2\pi} \frac{e^{i(\pi/4)\text{sgn}[\Phi'_{(b,a)}(t_0)]}}{\sqrt{\Phi''_{(b,a)}(t_0)}} A_s(t) A_{\psi} \left( \frac{t-b}{a} \right) e^{i\Phi_{(b,a)}(t)}$$

(3.27)

which can be written as the value of the integrand at $t = t_0$ corrected by some factor, i.e.

$$T_s(b, a) \approx \left\{ \frac{\pi}{2} Z(t_0) \bar{\psi} \left( \frac{t_0 - b}{a} \right) \right\}_{\text{corr}(b,a)}$$

(3.28a)

where

$$\text{corr}(b,a) = a \left[ \Phi''_{(b,a)}(t_0) \right]^{1/2} \exp \left\{ -i(\pi/4)\text{sgn}[\Phi''_{(b,a)}(t_0)] \right\}$$

(3.28b)

If $t = t_0$ is a stationary point of order $k$ (i.e. $\Phi'_{(b,a)}(t_0) = 0$ and $\Phi^{(k+1)}_{(b,a)}(t_0) \neq 0$), then

$$T_s(b, a) \approx \left[ \frac{1}{2} \Gamma \left( \frac{4}{3} \right) \right]^{1/3} Z(t_0) \bar{\psi} \left( \frac{t-b}{a} \right)_{\text{corr}(b,a)}$$

(3.29a)

where

$$\text{corr}(b,a) = a \left[ \Phi^{(k+1)}_{(b,a)}(t_0) \right]^{1/(k+1)} \exp \left\{ -i \left( \frac{\pi}{2(k+1)} \right) \text{sgn}[\Phi^{(k+1)}_{(b,a)}(t_0)] \right\}$$

(3.29b)

Without loss of generality, from now on we assume that $t = t_0$ is a unique stationary point and that it is of first order. For composite signals, this means that they have components which do not interact so the analysis can be restricted to a domain where the wavelet coefficients of all but one component are negligible.

### 3.4.2 The Ridge of the Continuous Wavelet Transform and Its Relation to Instantaneous Frequency

**Definition:** The ridge of the wavelet transform of $s(t)$, $T_s(b, a)$, is the set of points $(b,a)$ in the domain of the transform where the phase of $s(t)\bar{\psi}_{(b,a)}(t)$ is stationary, i.e. the points that satisfy
Let \( a_r(b) \) be the ridge. Substituting \( t_0 = b, a = a_r(b) \) and \( \phi_s'(t_0) = \omega_s(t_0) = \omega_s(b) \) in the stationary phase condition (3.25) gives

\[
\omega_r(b) = \frac{\phi_s'(0)}{a_r(b)}
\]  

Further by substitution for \( \phi_s'(0) = \omega_0 \) for the Morlet wavelet gives

\[
\omega_r(b) = \frac{\omega_0}{a_r(b)}
\]  

Equation (3.32) shows that the instantaneous frequency of the signal at time \( b \) is the frequency of the wavelet at scale \( a_r(b) \). So, once the ridge of the wavelet transform is determined, the instantaneous frequency can be determined easily from eqn (3.32). The ridge can be determined from the amplitude of the wavelet transform or from the phase, as it will be seen later.

### 3.4.3 The Skeleton of the Continuous Wavelet Transform

**Definition**: The skeleton of the wavelet transform of \( s(t), T_s(b, a) \), is the wavelet transform evaluated on the ridge, i.e. \( T_r(b, a_r(b)) \).

Equation (3.28a) implies that on the ridge

\[
T_r(b, a_r(b)) \approx \left( \frac{\pi}{2} \right) \frac{Z(b)\psi(0)}{\text{corr}(b, a_r(b))}
\]  

The correction term requires \( \Phi''_{(b,a)} \) evaluated on the ridge. Equation (3.26) implies

\[
\Phi''_{(b,a_r(b))}(b) = \phi_s''(b) - \frac{1}{[a_r(b)]^2} \phi_s''(0)
\]

For the Morlet wavelet, the second term on the right hand side of eqn (3.34) is zero, and \( \phi_s''(b) \) evaluated using eqn (3.32) is

\[
\phi_s''(b) = \omega_s'(b) = \frac{a_r'(b)\omega_0}{[a_r(b)]^2}
\]
Then

\[ \Phi_{(b,a,(b))}''(b) = -\frac{a'(b)\omega_0}{\overline{a_r(b)}^2} \]  \hspace{1cm} (3.36)

which implies that the correction term in eqn (3.33) depends only on the ridge and on the analyzing wavelet.

Equation (3.33) can be used to reconstruct \( Z(b) \) from the skeleton of the wavelet transform as

\[ Z(b) = T_s(b,a_r(b)) \sqrt{\frac{2 \text{ corr}(b,a_r(b))}{\psi(0)}} \]  \hspace{1cm} (3.37)

The instantaneous amplitude of the signal is then

\[ A(b) = |Z(b)| = \left| T_s(b,a_r(b)) \sqrt{\frac{2 \text{ corr}(b,a_r(b))}{\psi(0)}} \right| \]  \hspace{1cm} (3.38)

i.e. the amplitude of the skeleton times a correction.

### 3.4.4 Ridge Extraction from the Phase of the Continuous Wavelet Transform

The ridge can be extracted from the amplitude or from the phase of \( T_s(b,a) \). Theoretically at least, extraction from the phase is more accurate; in practice it may not be so because it involves differentiation of phase.

Let \( \Psi(b,a) \) be the phase of \( T_s(b,a) \). Then eqn. (3.27) implies that

\[ \Psi(b,a) = \arg T_s(b,a) = \frac{i\pi}{4} \text{ sgn} \Phi_{(b,a)}(t_0) + \Phi_{(b,a)}(t_0) \] \hspace{1cm} (3.39)

and the derivatives of \( \Psi(b,a) \) are essentially those of \( \Phi_{(b,a)}(t_0) \), defined by eqn (3.24).

Differentiation of \( \Psi(b,a) \) with respect to scale gives

\[ \frac{\partial \Psi(b,a)}{\partial a} = -\frac{t_0 - b}{a^2} \Phi_{(b,a)}' \left( \frac{t_0 - b}{a} \right) \] \hspace{1cm} (3.40)

On the ridge (where \( t_0 = b \))
\[ \frac{\partial \Psi(b,a)}{\partial a} \bigg|_{a=a_r(b)} = 0 \]  \hspace{1cm} (3.41)

Then, for given time \(b\), the ridge \(a_r(b)\) can be found by iteration, as the fixed point of eqn (3.41).

Similarly, the derivative of \(\Psi(b,a)\) with respect to \(b\) evaluated on the ridge is

\[ \frac{\partial \Psi(b,a)}{\partial b} \bigg|_{a=a_r(b)} = \frac{1}{a} \psi'(0) \]  \hspace{1cm} (3.42)

and for the Morlet wavelet is

\[ \frac{\partial \Psi(b,a)}{\partial b} \bigg|_{a=a_r(b)} = \frac{\omega_0}{a_r(b)} \]  \hspace{1cm} (3.43)

Again, the ridge \(a_r(b)\) can be found by iteration, as the fixed point of eqn (3.43).

### 3.4.5 Algorithm for the Marseille Method for Instantaneous Frequency Extraction and Signal Reconstruction

The algorithm consists of the following steps

1. Set initial value of \(a\) (an arbitrary value within the search domain).
2. Do for time steps \(ib=1, nb\)
   
   a. Find the ridge \(a_r(b)\) as the fixed point of eqn (3.43)

   \[ \frac{\partial \Psi(b,a)}{\partial b} \bigg|_{a=a_r(b)} = \frac{\omega_0}{a_r(b)} \]

   where

   \[ \Psi(b,a) = \arg T_s(b,a) \]

   as follows:

   - Compute \(\frac{\partial \Psi(b,a)}{\partial b}\) using a finite difference scheme and the current values of \(a\) and \(b\);
• Compute $a_{\text{new}}$ as

$$a_{\text{new}} = \omega_0 l \frac{\partial \Psi(b, a)}{\partial b}$$

• Repeat the iteration until $a$ and $a_{\text{new}}$ are sufficiently close.
• Set $a = a_{\text{new}}$

b. Compute the instantaneous frequency $\omega_s(b)$ from eqn (3.32)

$$\omega_s(b) = \frac{\omega_0}{a_s(b)}$$

c. Compute the instantaneous amplitude $A(b)$ from eqn (3.38)

$$A(b) = |Z(b)| = \left| T_s \omega_s(b) \sqrt{\frac{2}{\pi}} \frac{\text{corr}(b, a_s(b))}{\Psi(0)} \right|$$

where from eqns (3.28b) and (3.36)

$$\text{corr}(b, a) = a \Phi''(b, a, (b))^{1/2} \exp \left\{ -i(\pi / 4) \text{sgn} \left[ \Phi''(b, a, (b)) \right] \right\}$$

$$\Phi''(b, a_s(b), (b)) = -\frac{a_s'(b) \omega_0}{[a_s(b)]^3}$$

d. Compute the reconstructed signal at time $b$

$$s_{\text{rec}}(b) = A(b) \cos \left[ \omega_s(b)b \right]$$

End of do loop over time steps $ib$.

This algorithm was implemented in the FORTRAN program mars_rdg.for.
3.5 Ridge Extraction from the Modulus of the Continuous Wavelet Transform—
“Carmona” Method

This method was proposed by Carmona et al. (1997) and is also described in Carmona et al. (1998). It was used by Staszewski (1998) in a parametric identification of nonlinear systems. This method is less accurate for extraction of the instantaneous frequency but is more robust for signals that have noise (hence for real life signals). A nice property of the method is that it does not require knowledge of the noise (e.g. like the Kalman Filter), which is the case in most real life problems. It does however allow incorporation of prior information about the ridge, e.g. its smoothness. Therefore, it can be interpreted as a Bayesian method.

This method is based on the wavelet Plancherel formula

\[
\|R_f\|^2 = c_\psi \|f\|^2, \quad f \in L^2(R)
\]  

(3.44)

according to which the modulus squared of the wavelet transform can be interpreted as a time-scale energy density, and on the following lemma (Carmona et al., 1998).

Lemma: Let \( s(t) = A(t)\cos(\phi(t)) \in L^2(R) \) be an asymptotic signal with wavelet transform \( T_s(b,a) \). Then

\[
T_s(b,a) = \frac{1}{2} A(b)e^{i\phi(b)}\overline{\psi}(a\phi'(b)) + O\left(\frac{A}{|A|}, \frac{\phi'}{|\phi'|^2}\right)
\]  

(3.45)

Proof: A rigorous proof explaining the remainder can be obtained by Taylor expansion. The principal part can be explained as follows. Let \( Z_s(t) = A(t)e^{i\phi(t)} \) be the analytic signal associated with \( s(t) \). Then
\[ T_s(b,a) = \int_{-\infty}^{\infty} s(t) \overline{\Psi}_{(b,a)}(t) dt \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) \overline{\Psi}_{(b,a)}(\omega) d\omega \]
\[ = \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Z}_s(\omega) \overline{\psi}(a\omega) e^{iab} d\omega \]
\[ \approx \frac{1}{2} \overline{\psi}(a\phi'(b)) \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Z}_s(\omega) e^{iab} d\omega \]
\[ = \frac{1}{2} \overline{\psi}(a\phi'(b))Z_s(t) \]
\[ = \frac{1}{2} A(b)e^{i\phi'(b)} \overline{\psi}(a\phi'(b)) \]

The approximation leading from line 3 to line 4 is based on the fact that, because the signal is narrow-band, with energy concentrated near \( \omega = \omega_s(b) = \phi'(b) \), the product \( \hat{Z}_s(\omega) \overline{\psi}(a\omega) \) will be significant only near this frequency. Because \( \overline{\psi}(\omega) \) is concentrated near \( \omega = \omega_0 \), the central frequency of the wavelet, the energy density \( A(b)e^{i\phi'(b)} \overline{\psi}(a\phi'(b)) \) will be concentrated along the \( a \)-axis near the curve \( a\phi'(b) = \omega_0 \), which is exactly the ridge of the wavelet transform.

\[ a_s(b) = \frac{\omega_0}{\phi'(b)} = \frac{\omega_0}{\omega_s(b)} \]  

(3.47)

defined earlier in this chapter. Using this last argument, then, the ridge can be determined, for each time \( b \), by finding the scale \( a \) where the modulus of the wavelet transform is the maximum. This is certainly the simplest and most direct approach. For noisy signals, however, there is a better approach described in what follows.

### 3.5.1 Ridge Estimation in Presence of Noise

If the signal is contaminated with noise, its wavelet transform (modulus and phase) will also be contaminated with noise. The signal-to-noise ratio however will be the largest near the ridge of the transform, because most of the energy of the signal is concentrated near the ridge (unless the noise is also narrow-band and with frequency near that of the signal). This is why using the transform to estimate the instantaneous frequency is better than using the signal directly. Further, extracting the ridge from the modulus of the transform is more robust than that from the phase, because the extraction from the modulus does not involve differentiation of phase. Another advantage of using the
modulus is that this procedure can easily incorporate a-priori information on the ridge as a constraint, e.g. that the ridge is smooth. The problem then reduces to solving a constrained optimization problem, of finding among all candidate curves $a, (b)$ the one that minimizes the following penalty function

$$ F[a, (b)] = -\int \mid \mathcal{F}_x (b, a, (b)) \mid^2 db + \int \left\{ \lambda a'' (b) + \mu a'' (b) \right\}^2 db $$

(3.48)

Here minimization of the penalty function is equivalent to maximization of the modulus of the transform along the ridge. Constants $\lambda$ and $\mu$ can be chosen by the analyst depending on the problem and on how much weight is to be placed on the smoothness constraint (e.g. the larger the values of these constants, the larger the weight of the constraint).

This optimization problem can be solved by formulating and solving the Euler equations to find the local extrema, but a more robust method for noisy signals is one based on simulated annealing. If the record is noisy, then the modulus of the transform will have many spurious local extrema. The simulated annealing algorithm allows jumping over these spurious extrema and searching further for the global extremum. The following section describes the principles on which the simulated annealing method is based and an algorithm for its implementation.

### 3.5.2 Simulated Annealing Algorithm

This method is a combinatorial optimization method and is appropriate to use when the space of possible functions that minimize the objective function is discrete and very large so that exhaustive search is prohibitively expensive (Press et al., 1994). In our problem, the region $(a_{\min}, a_{\max}) \times (b_{\min}, b_{\max})$ is discretized and the space of candidate ridge functions consists of all possible chains of elements $(a, b)$ such that $b$ increases monotonically but $a$ can be any one of the prescribed discrete values in the domain. The simulated annealing method consists of random steps but in a “cooling” environment, and is based on analogy with the way thermodynamic systems cool, i.e. liquids crystallize and metals anneal. If the cooling is slow, the crystals/atoms reach naturally a minimum energy state, in which billions of molecules/atoms are perfectly lined up. If the rate of cooling is fast, the system reaches a higher energy, polycrystalline or amorphous state. At high temperatures, the molecules of a liquid move freely with respect to each other, and their mobility decreases with cooling. If the cooling is slow, the molecules/atoms have time to redistribute themselves to a minimum energy state. The mobility of these natural systems is described by the Boltzmann probability distribution of energy as function of the
absolute temperature of the system, $P(E) \sim \exp[-E/(kT)]$ (higher energy states are less likely, more so at low temperatures, but are still possible even at low temperature).

The principle of slow cooling and annealing was first implemented in numerical algorithms by Metropolis (1953). According to his algorithm, the probability of a thermodynamic system to change energy from state $E_1$ to state $E_2$ is described by $P(E_1,E_2) \sim \max\{1, \exp[-(E_2 - E_1)/(kT)]\}$. This distribution is such that whenever $E_1 > E_2$, $P(E_1,E_2) = 1$ while when $E_1 < E_2$, $P(E_1,E_2) < 1$, i.e. the system will always go from a higher energy to a lower energy state, and sometimes go from a lower energy state to a higher energy state. For a general system, the algorithm must have the following elements: (1) description of possible system configurations, (2) a generator of random changes in the system, (3) an objective function, and (4) a control parameter like temperature and an annealing schedule.

The steps of this algorithm are as follows

1. Divide the domain $(a_{min}, a_{max}) \times (b_{min}, b_{max})$ where candidate ridges are defined in a discrete grid of equally spaced points, indexed $(ia, ib)$, $ia=1,\ldots, na$ and $ib=1,\ldots, nb$ and compute the wavelet transform on this grid.

2. Define values for parameters $\lambda$ and $\mu$, $isame_{max}$ for the stopping criterion (step e), $C_{temp}$ and seeds for the random number generators (explained below).

3. Define an initial trial ridge. The ridge is defined by the array $ia_r(ib)$, $ib=1,\ldots, nb$ where $ia_r(ib)$ is an index of the scale grid corresponding to index $ib$ of the time grid. Compute initial value of the temperature as

$$T = C_{temp} / \ln 2$$

where $C_{temp}$ is a constant, and compute the initial value of the objective function $F_{obj}$.

4. Start a do loop over random steps for $k=1$ to $niter$

   a. Compute an updated value of the temperature as per the following schedule:

   $$T = C_{temp} / \ln (2+k)$$

   b. Choose a nearest neighbor ridge, i.e. a new ridge that differs from the current ridge only at time point $ib=l$. First select $l$ as a random number between 2 and $nb-1$. Then select random shift $ie = \pm1$. Define the new ridge and check
whether it is on the grid; if it is not, continue choosing a new nearest neighbor ridge until it is in the search domain. This is the candidate ridge for step \( k \).

c. Calculate the objective function for the candidate ridge, \( F_{\text{obj}_\text{cand}} \).

d. Decide whether to update the ridge as follows. If the objective function for the candidate ridge is less than or equal to the one for the current ridge, update the ridge. If it is not, then select a random number \( \sigma \) uniformly distributed between 0 and 1. Update the ridge only if

\[
\sigma \leq \exp\left[-\frac{(F_{\text{obj}_\text{cand}} - F_{\text{obj}})}{T}\right].
\]

Otherwise continue.

e. Check whether the stopping criterion is satisfied. Define a-priori number of steps \( \text{isame}_\text{max} \). If the ridge has not changed for the last \( \text{isame}_\text{max} \) steps, then exit the do-loop. If not continue searching (i.e. go to step a).

f. End do-loop over random steps \( k \).

5. Once the ridge has been determined, find the instantaneous frequency using

\[
\omega_s(b) = \frac{\omega_0}{a_s(b)}
\]

Smooth \( \omega_s(b) \) if necessary.

The above algorithm was implemented in the Fortran program \texttt{carm_rdg.for} for ridge detection and instantaneous frequency estimation by Carmona’s method.

### 3.5.3 Signal Reconstruction

Although the signal can be reconstructed from the skeleton of the wavelet transform, e.g. using eqn (3.45), for noisy signals the skeleton is not smooth. Carmona et al. (1998) proposed to solve an optimization problem of finding a signal \( s_r(t) \) such that its wavelet transform along the ridge is close enough to the one for the original signal, and it is also a smooth function. This method is not reviewed in this report.

### 3.6 The “Simple” Method

The most straight forward method for determining the ridge of the Continuous Wavelet Transform (and hence the instantaneous frequency) of a signal from its amplitude is by finding for each time \( b \) the scale \( a \) for which the amplitude of the transform is maximum. In this report, this method is referred to as the “simple” method. This method does not allow for using any a priori information about the ridge such as its smoothness in the
“Carmona” method, but is much faster than the “Carmona” method, which may require
thousands (or hundreds thousands) of iterations, and is more stable than the “Marseille”
method because it deals with the amplitude of the transform instead of the phase. This
method is based on the heuristic that in a narrow time interval the energy of the signal is
concentrated near its instantaneous frequency for that time interval.

The steps of this algorithm are as follows

1. Divide the domain \((a_{\min}, a_{\max}) \times (b_{\min}, b_{\max})\) where candidate ridges are defined in a
discrete grid of equally spaced points, indexed \((ia, ib)\), \(ia=1, \ldots, na\) and \(ib=1, \ldots, nb\)
and compute the wavelet transform on this grid.

2. For each \(ib=1, \ldots, nb\)
   
   Find the index \(ia_r(ib)\) such that \(T_s(b(ib), a(ia))\) is maximum. This defines the
ridge.

3. End of do-loop over random steps \(k\).

5. Once the ridge has been determined, find the instantaneous frequency using

\[
\omega_s(b) = \frac{\omega_0}{a_s(b)}
\]

This algorithm was implemented in Fortran program dir_rdg.for.

3.7 General Remarks

The “Marseille”, “Carmona” and “Simple” methods can also be generalized to use the
Gabor Transform in the place of the Wavelet Transform. The theory for the “Marseille”
method can be found in Detrart et al. (1992). For the “Carmona” and “Simple” methods,
the procedure remains virtually unchanged except that the Gabor Transform is pre-
computed on a grid instead of the Wavelet Transform.
4. NUMERICAL EXAMPLES

This chapter presents results for the methods described in Chapter 3 applied to three well-defined chirp signals, one with constant frequency, another one with piecewise constant frequency and with a sharp jump in the frequency, and a third one with linearly increasing frequency. The amplitude of the signals is modulated so that they resemble 6 s pulses in structural response. The purpose of these examples is to test the codes, and to find out how close the estimates of instantaneous frequency are to the exact values, and how well these methods can localize a jump in frequency.

4.1 The Signals

All the three test signal have \( N = 600 \) points, equally sampled at \( \Delta t = 0.01 \) s, amplitude tapered by a Hanning window

\[
w(t) = 0.5(1 - \cos 2 \pi t)/(N \Delta t)
\]  

and unit maximum amplitude. Their analytical expressions are as follows.

Test signal 1:

\[
s(t) = w(t) \cos 2 \pi f_0 t, \quad f_0 = 1 \text{ Hz}
\]  

Test signal 2:

\[
s(t) = \begin{cases} 
w(t) \cos 2 \pi f_0 t, & t \in [0, 2] \cup (4, 6] \\
w(t) \cos 2 \pi (2 f_0) t, & t \in (2, 4]
\end{cases}, \quad f_0 = 1 \text{ Hz}
\]  

Test signal 3:

\[
s(t) = w(t) \cos 2 \pi \left( f_0 t + 0.5 \frac{f_1 - f_0}{N \Delta t} t^2 \right), \quad f_0 = 1 \text{ Hz}, \quad f_1 = 3 \text{ Hz/s}
\]  

Figure 4.1 shows the signals (top) and the amplitude of their Continuous Wavelet Transform as an image (center) and as a surface (bottom). The signals were generated by program `chirp.for` which can also add Gaussian noise to the signals.
Fig. 4.1a  Test signal 1, with uniform frequency and amplitude tapered by a Hanning window. It is defined by $s(t) = w(t) \cos 2\pi f_0 t$ with $f_0 = 1$ Hz and $w(t) = 0.5(1 - \cos 2\pi t / N\Delta t)$. The signal has $N=600$ points, sampled at $\Delta t = 0.01$ s. Top: the signal as a function of time. Center: amplitude of its wavelet transform as an image with lighter color representing larger amplitude. Bottom: amplitude of its wavelet transform as a surface.
Fig. 4.1b  Same as Fig. 4.1a but for test signal 2. It has piecewise uniform frequency and amplitude tapered by a Hanning window. It is defined by $s(t) = w(t) \cos 2\pi f_0 t, \quad t \in [0, 2] \cup (4, 6]$ and $s(t) = w(t) \cos 2\pi (2f_0) t, \quad t \in (2, 4]$, with $f_0 = 1$ Hz.
Fig. 4.1c  Same as Fig. 4.1a but for test signal 3. It has uniformly increasing frequency and amplitude tapered by a Hanning window. It is defined by $s(t) = w(t) \cos(2\pi(f_0 t + 0.5 f_1 t))$ with $f_0 = 1$ Hz, $f_1 = 0.5$ Hz/s.
and 3. The top parts of all these figures except Fig. 4.2 show the signals (solid line) and the skeletons of the Continuous Wavelet Transform (dashed line) as an approximation of the signal. The central parts show the phases of the signals $\phi_s(t)$ (Fig. 4.2) or the ridges $a_s(t)$ (Figs 4.3 through 4.5), and the bottom parts show the instantaneous frequency $f_s(t)$ estimated by the method (solid line or symbols) and the exact value (dashed lines).

Wherever the wavelet transform was used, the mother wavelet was such that $\omega_0 = 2\pi$ and $\sigma = 1$, and the shaded boxes in the plots of $f_s(t)$ show $2\sigma_\times 2\sigma_\times$ cells in the time frequency plane equivalent to those shown in Fig. 2.1 (for these examples, $\sigma = 0.63/ f$) and $\sigma_\times = 0.63 f / (2\pi)$. The results in Figs 4.2 through 4.5 were produced respectively by programs hilb_rdg.for, mars_rdg.for, carm_rdg.for and dir_rdg.for.

Figure 4.2 shows that the Hilbert Transform method is very accurate except near the beginning and the end of the signal and near the jumps in frequency for Signal 2 (the transition is gradual rather than a jump and $f_s(t)$ exhibits weak oscillations).

Figure 4.3 shows that the “Marseille” method is very accurate for the signal with constant frequency (signal 1), less accurate for the signal with graduate change in frequency (Signal 3) and the least accurate for the signal with jumps in frequency (Signal 2). In fact, for Signal 2, the algorithm does not converge to the “true” ridge for all starting values of scale $a$, but converges to larger scales (smaller frequencies), corresponding to the smaller local maximum in the amplitude of the wavelet transform (see Fig. 4.1b, bottom).

Figure 4.4 shows that the “Carmona” method is also very accurate for the constant frequency signal (Signal 1). For the signal with jumps (Signal 2), this method is more stable than the Marseille method. However, the depicted change in frequency is gradual (with span of at least one second) rather than abrupt.

Figure 4.5 shows that the results by the “Simple” method are practically the same as those by the “Marseille” method. For such types of signals, the Marseille method does not offer any advantages.

Common observation in all four figures is that, for Signal 3, the accuracy is the poorest near the beginnings and the ends of the signals, when the amplitudes are small. Near $t = 0$, when $f = 1$ Hz, the error is larger than $\sigma_\times$, while near $t = 6$ s, when $f = 3$ it is within $\sigma_\times$. 
Fig. 4.2a Test signal 1 (top), and its phase (center) and instantaneous frequency (bottom) determined by the Hilbert Transform method.
Fig. 4.2b Test signal 2 (top), and its phase (center) and instantaneous frequency (bottom) determined by the Hilbert Transform method.
Fig. 4.2c Test signal 3 (top), and its phase (center) and instantaneous frequency (bottom) determined by the Hilbert Transform method.
"Marseille" Method

Fig. 4.3a Test signal 1 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the "Marseille" method.
Fig. 4.3b Test signal 2 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Marseille” method.
"Marseille" Method

Signal 3

Fig. 4.3c Test signal 3 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Marseille” method.
Fig. 4.4a Test signal 1 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Carmona” method.
Fig. 4.4b Test signal 2 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Carmona” method.
Fig. 4.4c Test signal 3 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Carmona” method.
Fig. 4.5a Test signal 1 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Simple” method.
Fig. 4.5b Test signal 2 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Simple” method.
"Simple" Method

Fig. 4.5c Test signal 3 (top), and its ridge (center) and instantaneous frequency (bottom) determined by the “Simple” method.
Fig. 4.6  Test signal 3 with uniformly increasing frequency and unit amplitude tapered by a Hanning window (see Fig.; 4.1c), and with additive Gaussian noise (with zero mean and standard deviation 0.2).  Top: the signal as a function of time.  Center: amplitude of its wavelet transform as an image with lighter color representing larger amplitude.  Bottom: its instantaneous frequency determined by the “Carmona” method (symbols) and by the “simple” method (solid line).
4.3 Instantaneous Frequency for Noisy Signals

Figure 4.6 shows results for Signal 3 with additive Gaussian noise, with mean zero and standard deviation 0.2 (this implies ratio between the amplitude of the signal and sigma of the noise equal to 5). The top part of the figure shows the noisy signal, as well as the signal without noise, for reference. The plot in the center shows the modulus of the wavelet transform of the noisy signal, and the plot in the bottom shows the instantaneous frequency estimated by the “Carmona” method (symbols) and by the “Simple” method (solid line). The parameters \( \lambda \) and \( \mu \) for the simulated annealing scheme were set to \( \lambda = 0.01 \) and \( \mu = 0.01 \), the maximum number of iterations was set to 1 million, the stopping criterion was set so that the objective function should not change in 500 consecutive steps before stopping. Also, the wavelet transform was subsampled so that every 20\(^{th}\) point in time was considered in the search, and it was evaluated for 100 values of scale between 0.2 and 1.1. Finer time-scale grid requires much more iteration steps. The result for this noisy signal shows that the instantaneous frequency is estimated with similar accuracy as for the signal without noise. Similar conclusion can be drawn even for signal to noise ratio much smaller than 1, e.g. 1/30 (not included in this report).
5. DISCUSSION AND CONCLUSIONS

This report presented a review of several methods for determination of instantaneous frequency. Three of these methods (“Marseille”, “Carmona” and “Simple”) use the Continuous Wavelet Transform or the Gabor Transform. These methods were applied to three test signals with known frequency changes, consisting of a 6 second pulse, resembling a pulse in a structural response strong motion record. The results show the following.

The methods based on the phase of the signal, i.e. its derivative (the Hilbert Transform method and the Marseille method), are most accurate theoretically but are not very stable in practical applications. The most stable are the methods based on the amplitude of a time-scale or time-frequency distributions (wavelet or Gabor transforms), even for signals with significant amount of noise (provided it is random, i.e. with approximately flat spectrum). The “Carmona” and “Simple” methods give very similar results for the signals without noise. For such signals, the “Carmona” method, which is much more computationally intensive, does not offer any advantages, and the “Simple” method can be used instead. The “Carmona” method, however, has advantages for noisy signals, because it allows introduction of a-priori information about the signal, e.g. its smoothness. The “Carmona” and “Simple” methods using the Gabor transform give very similar results as those for the wavelet transform illustrated in this report, and are therefore not presented.

There is a trade-off between accuracy in time and in frequency localization. These accuracies can be controlled by choosing the length of the window for the wavelet and Gabor transforms. Longer time-windows give better localization in frequency but poorer in time. This is important to bare in mind for signals with jumps in their frequency (Signal 2). The time of the jump and frequency of the jump cannot be determined with arbitrarily fine accuracy. If same accuracy for all frequencies is desired, the Gabor transform with carefully chosen time window should be used. The Hilbert transform method proved to be the most accurate for the detection of jump in frequency for the test signals, but is not stable to apply to real signals.

For pulse-like signals with continuous variation of their frequency (Signal 3), the accuracy of localization is the poorest where the amplitudes of the pulse are small. This is important to consider in interpretation of records of structural response to earthquakes, which are transient in nature and consist of many pulses.
6. REFERENCES


