A Sequential Programming Model of Urban Land Development

JAMES E. MOORE II\(^1,2\) and PETER GORDON\(^1,3\)

\(^1\)School of Urban and Regional Planning, \(^2\)Department of Civil Engineering and \(^3\)Department of Economics, University of Southern California, Los Angeles, CA 90089-0042, U.S.A.

Abstract—The development of urban land is treated as a sequence of activity shifts resulting from locators' efforts to maximize net revenues by mitigating congestion costs and other externalities. This decentralized process is modelled by solving a series of linear assignment problems that track urban land use through time. Network costs are endogenous, and locators update site bids by combining recent experience with a ceteris paribus point of view. Vacancy is treated as a null activity that can be assigned to any number of sites. Locators that have been outbid by vacancy are displaced from the land use system, but may continue to formulate bids and compete for sites.

INTRODUCTION

The deficiencies of monocentric models of urban land use are widely recognized, but no satisfactory alternatives have been forthcoming. We describe a sequential model of urban land development that captures the dynamics of urban land use. This general equilibrium formulation accounts for interactions between the markets for land and transportation with a series of temporally linked assignment models. Site bids are updated from a ceteris paribus perspective based on the congestion and other external costs experienced by locators. With the exception of a conventional network equilibrium problem, all the optimization problems involved are unimodular linear programs subject to very efficient solution if treated as bipartite graphs. We suggest that the model's duality conditions might serve as a template for a number of the decisions made by urban planners.

Although numerical exercises abound (this paper included), the theoretical foundations of mathematical programming have not been routinely used to explore urban land use problems. More frequently, mathematical explorations of urban systems have tended to emphasize more general economic approaches. A few of the more important exceptions include those of Koopmans and Beckmann [16], Herbert and Stevens [13], Mills [20], Evans [5], Wilson and Senior [22], Hartwick and Hartwick [11], Los [19], Guldmann [10], Hopkins [14], Brill et al. [3], and Kim [15].

These citations ignore a large and important literature pertaining to facility location problems. Facility models tend to be mixed integer programs or quadratic assignment problems, and are usually classified as "location" rather than "land use" formulations. Because of the difficulties associated with treating mixed integer constraints, these location problems tend to be analyzed from a mathematical programming rather than an economic perspective. Like location models, this research focuses on discrete activities; but only in the context of unimodular (integer in/integer out) formulations. As a result, important duality conditions are considerably simplified. Because the constrained optimization problems addressed by this research are smooth rather than discrete, the theoretical foundation of the current effort subsumes more of the land use literature than it does the location literature. (A complete review of location theoretic models can be found in Brandeau and Chiu [2].)

Koopmans and Beckmann

The Koopmans and Beckmann [16] model is unusual in that it includes both indivisible activities and the (spatially dependent) costs of interaction between activities:

 maximize: 

 \[ \sum_j \sum_m a_{jm} \cdot X_{jm} - \sum_j \sum_m \sum_n f_{ij} \cdot c_{mn} \cdot X_{im} \cdot X_{jn}, \] 

 (1)
subject to:

\[ \Sigma_{\mu} X_{m} = 1 \quad \text{(for all } m \text{ in } M), \]  
\[ \Sigma_{\alpha} X_{m} = 1 \quad \text{(for all } i \text{ in } I), \]  
\[ X_{m} = 0, 1 \quad \text{(for all } i \text{ in } I, \text{ for all } m \text{ in } M), \]

where

\( a_{\mu} = \) the exogenous component of the profit ("semi-net revenue" in Koopmans and Beckmann's text) available to activity \( i \) at site \( m \),

\( f_{ij} = \) the exogenous transportation flow between activities \( i \) and \( j \),

\( c_{\mu\alpha} = \) the exogenous unit transportation cost between locations \( m \) and \( n \),

\( X_{m} = \) an endogenous, binary variable equal to one if activity \( i \) is assigned to location \( m \), and equal to zero otherwise, and

\( X_{m} = \) defined similarly to \( X_{m} \).

Koopmans and Beckmann identified a set of general conditions under which the optimal solution to the smooth version of this quadratic program could not be integer, although it has been shown by Heffley [12] and others that there also exist special conditions under which integer outputs are optimal. In general, however, the solution to this discrete quadratic programming problem is not price-sustainable. This result has made the Koopmans and Beckmann model a less popular tool for investigating the market for urban land. Gordon and Wingo [9] addressed the issue of price-sustainability in this context by noting that the static nature of the Koopmans–Beckmann model attaches too high a degree of simultaneity to the location decisions of competing agents. These simultaneous location decisions have no plausible empirical interpretation in a world in which the decisions of competitors are almost always sequential rather than simultaneous.

**Beckmann, McGuire and Winsen**

Beckmann et al.’s [1] path flow model of user-equilibrium transportation costs identified endogenous network link and path costs under conditions of elastic demand:

minimize:

\[ \Sigma_{k} \int_{0}^{f_{k}} c_{k}(w) \, dw - \Sigma_{n} \int_{0}^{f_{m}} g_{m}^{-1}(u) \, du, \]

subject to:

\[ \Sigma_{s} f_{ss} = f_{mm} \quad \text{(for all } m, n \text{ in } M), \]  
\[ \Sigma_{k} \Sigma_{n} \Sigma_{s} \partial_{kss} f_{kmm} = f_{k} \quad \text{(for all } k \text{ in } K), \]  
\[ f_{ss} \geq 0 \quad \text{(for all } s \text{ in } S; \text{ for all } m, n \text{ in } M), \]

where

\( c_{k}(\cdot) = \) an increasing congestion cost function of the traffic volume on link \( k \),

\( g_{m}^{-1}(\cdot) = \) an inverse demand for travel function of the equilibrium cost of travel,

\( f_{k} = \) the endogenous material flow on link \( k \),

\( f_{mm} = \) the endogenous material flow on path \( s \) from origin \( m \) to destination \( n \),

\( f_{m} = \) the endogenous material flow from origin \( m \) to destination \( n \), and

\( \partial_{kss} = \) an indicator variable equal to one if link \( k \) is on path \( s \) from origin \( m \) to destination \( n \), and equal to zero otherwise.

Network equilibrium path costs are important because accessibility is assumed to have an important bearing on any site’s economic attractiveness. Unfortunately, this convex programming formulation is intractable because it implies the pre-computation of all network paths. However, the following link-flow formulation is also available:

minimize:

\[ \Sigma_{k} \int_{0}^{f_{k}} c_{k}(w) \, dw \]
subject to:

$$\sum_n f_{kn} = f_k \quad \text{for all } n \in M,$$

$$\sum_{\text{links } k \text{ outbound from node } m} f_{kn} - \sum_{\text{links } k \text{ inbound to node } m} f_{kn} = f_{mn} \quad \text{for all } k \in K,$$

$$f_{kk} \geq 0 \quad \text{for all } k \in K; \text{ for all } n \in M,$$

where:

$$\epsilon_k(\cdot) = \text{an increasing congestion cost function of the traffic volume on link } k,$$

$$f_k = \text{the endogenous material flow on link } k,$$

$$f_{kn} = \text{the endogenous material flow on link } k \text{ from any origin to destination } n,$$

$$f_{mn} = \text{the exogenous material flow from origin } m \text{ to destination } n.$$

The consumer surplus term that appeared in the objective function of the path-flow formulation has been suppressed. This does not necessarily imply that origin-destination flows are exogenous since the elastic demand model can be converted to an equivalent fixed demand model [6, 7]. This link-flow formulation can be stated in a polynomial number of steps, and is subject to iterative solution [18]; but it does not determine unique path flows. Optimal link-flows can, however, be post-processed by solving an all-shortest-paths problem of (in the worst case) order \(n^3\). Since user-equilibrium conditions [18, 21] imply that the travel time on all used paths between all origins and destinations are equal, the minimum cost path between any origin and destination must necessarily be an equilibrium cost path.

**Gordon and Moore**

Gordon and Moore [8] recently combined Beckmann et al.’s [1] network equilibrium concepts with the assignment component of the Koopmans and Beckmann model [16] to formulate a sequential model of urban land development. Time and space are both discrete, and location is defined by the configuration of an existing transportation network. The model bootstraps the system through time periods sequentially rather than treating all periods simultaneously. This tractability follows from the realistic assumptions the model incorporates concerning the information available to locators, and how this information is used over time.

The computational requirements of the algorithm are bounded either by the solution of an all-shortest-paths problem, or by obtaining the solution to the user-equilibrium network flow problem. As noted above, convergent algorithms are available for the user-equilibrium flow problem. If necessary, the computational burden associated with these algorithms can be mitigated by relying instead on an approximation technique.

An extension of Gordon and Moore [8] is now presented. This extension generalizes the previous model by incorporating vacancies, and recognizing the possibility that locators who cannot offer competitive bids during the current time period may be able to locate in a subsequent period. Locators and potential locators are assumed to update their site bids by anticipating experiences at alternative sites, rather than by relying on current costs. Further, we suggest that the values of the dual variables associated with the constraints accounting for bidding activities can be used to predict the technical characteristics of new locators.

**A MODEL OF SEQUENTIAL LAND DEVELOPMENT DECISIONS**

The assumption that economic agents engage in myopic location behaviors is central to our investigation of urban form. Locators are assumed to make decisions from a ceteris paribus perspective. This is not a restrictive assumption since perfect foresight is not an empirically useful construct. Further, accounting for the influence of perfect foresight is an analytically expensive proposition. Instead of invoking clairvoyance, we assume that locators rely on the simplest possible forecasting model to make decisions involving future costs and benefits [9].
Economic agents assume that:

- the recent past is the best indicator of the future, and that
- their individual decisions will not affect the prevailing equilibria in the markets for urban land and transportation.

That is, an agent's decision provides the agent with no new information concerning the future. The model thus incorporates imperfect information without relying on computationally problematic constraints pertaining to information or entropy. Other market assumptions apply except that, as in the case of the Koopmans–Beckmann problem, land uses are assumed to be indivisible.

**Specifying a matrix of semi-net revenues**

The model described below focuses on how each locator updates his associated elements in an augmented version of Koopmans and Beckmann's (exogenous) matrix of the semi-net revenues each bidding activity $i$ attaches to each physical site $m$. We assume that each of these bidding activities can locate at exactly one site, and that each physical site will be made available only to the activity offering the highest bid. It follows that the matrix of semi-net revenues must necessarily summarize information on the supply and demand for site characteristics.

We denote Koopmans and Beckmann's coefficient matrix as $A_3$, a submatrix of an augmented matrix $A$. If there are $I$ bidding (nonvacancy) activities, $M$ physical sites, and $H$ characteristics associated with each site, then the $(I \times M)$ matrix $A_3$ can be represented as the product of an $(I \times H)$ matrix $A_1$, listing the demands of each activity $i$ for each site characteristic $h$; and an $(H \times M)$ matrix $A_2$, describing the supply of each characteristic $h$ (including measures of network accessibility) at each site $m$. It is logical that the semi-net revenue any activity associates with any site is a function of the demand and supply of characteristics.

If the elements of $A_3$ are denominated in dollars, it follows that the elements of $A_1$ are measured in dollars or willingness-to-pay for each unit of attribute, while the elements of $A_2$ are measurable in physical terms (buildable square feet, days per year of clean air, miles to the coast line, degree of slope, etc.). $A_1$ thus summarizes the demand information obtained from conventional (site) consumer utility maximization. In this case, there are $I$ consumer groups and $H$ characteristics. Utility maximization would be over characteristics rather than commodities, as per the theory of Lancaster [17]. For consumer group (plant operator) $i$ facing attribute set $Q$ and income constraint $Y_i$, we have:

\[
\text{maximize: } U_i[q_1, \ldots, q_H],
\]

subject to:

\[
p_1 \cdot q_1 + \cdots + p_H \cdot q_H \leq Y_i, \quad (14)
\]

\[
q_h \geq 0 \quad \text{(for all } h \text{ in } H), \quad (15)
\]

where $p_h$ is the exogenous price of attribute $q_h$. All first-order conditions solved simultaneously yield demands for all characteristics in terms of prices and incomes. Alternatively, the results of hedonic investigations could be inserted. Each such optimization generates one row of $A_3$. Interestingly, this procedure establishes the microeconomic foundations of assignment problems.

In the model of (13–15), the arrival, departure, and ongoing bidding of activities constitute the principal mechanisms for spatial rearrangement. Consequently, an important distinction exists between the temporal entry of new locators and the arrival of new bidders. New bidders are not synonymous with new locators because not all bidders are able to obtain locations. Unsuccessful bidders are consigned to a null site, or queue. Activities bid nothing for access to the queue, and there is no constraint on the number of activities that can locate there simultaneously. New bidders contrast with existing bidders in that the latter have been previously located either at physical sites or in the queue.

The situation is further simplified by the introduction of a nonbidding or null activity called "vacancy" that bid nothing for physical sites and can be simultaneously assigned to any number
of sites. When nonvacancy activities offer (sufficiently) positive bids for sites, existing vacancies are displaced.

This perspective implies that the matrix \( A \) should be augmented by appending an \( I + 1 \)st row accounting for vacancies, and an \( M + 1 \)st column corresponding to the null location, or queue. The augmented matrix that results is \( A \), a matrix of objective function coefficients used to initialize the model's solution algorithm (see Table 1).

**Overview of the model**

The model is structured for iterative solution; and, as noted above, is premised on the assumption that each production activity can gain access to only one site, including the null site; and each physical site is able to accommodate exactly one activity, including vacancy. Also as has been noted, there is no logical restriction on the number of sites to which the vacancy activity may be assigned, nor on the number of activities that may be assigned to the queue. Treated in this fashion, vacancy can outbid activities that become wholly unprofitable; and unprofitable activities can be retired from the system without terminating their capacity to formulate bids.

The contents of the queue can be further updated at each iteration of the procedure by inspecting the dual variables associated with Koopmans and Beckmann's activity constraints. The most profitable of these activities have technical characteristics that are most likely to be representative of new bidders. Alternatively, new bidders might be treated as the exogenous outcome of regional market operations, in which case their arrival would be divorced from duality considerations.

With vacancies and unsuccessful bidders both treated explicitly, our proposed model of urban growth and structure can be summarized as follows:

- The procedure is initialized either by accepting an exogenous match between sites and activities, or by solving an assignment linear program of the form defined by Koopmans and Beckmann. Quadratic interaction terms are ignored because it is assumed that locators have no a priori knowledge of one another's location decisions. Without knowledge of where other activities will locate, locators cannot account for the costs of inter-activity shipments and spatial externalities. Locators \( i \) base their bids for discrete sites \( m \) entirely on \( A \), the augmented version of Koopmans and Beckmann's matrix of semi-net revenues.
- Given this initial match between activities and sites, and an exogenous matrix of inter-activity shipments, spatial externalities are realized, and flows on the transportation network are organized into a user-equilibrium configuration. Congestion costs are endogenous, and vacant sites may serve as transshipment points.
- Locators use this new information concerning transportation and spatial externality costs to update their vector of location bids. That is, the semi-net revenues each activity attaches to each site are modified relative to current experience, but from a ceteris paribus perspective. Each locator assumes that no other locator will move, and that network path costs are insensitive to his own location. Given an existing configuration of activities from which to draw inferences, locators can estimate the spatial externality and (efficient) transportation cost associated with each site.

—In the case of vacant sites, the bid each locator should make is clearly defined: no other activities will be displaced if a locator moves to a vacant site.

<table>
<thead>
<tr>
<th>Table 1. The augmented matrix ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i/m )</td>
</tr>
<tr>
<td>( i )</td>
</tr>
<tr>
<td>( j )</td>
</tr>
<tr>
<td>( i-1 )</td>
</tr>
<tr>
<td>( i )</td>
</tr>
<tr>
<td>( 1 )</td>
</tr>
<tr>
<td>( 2 )</td>
</tr>
<tr>
<td>( \vdots )</td>
</tr>
<tr>
<td>( I )</td>
</tr>
<tr>
<td>( I + 1 )</td>
</tr>
<tr>
<td>( M + 1 )</td>
</tr>
</tbody>
</table>
In the case of an occupied site, the bidding locator must make an additional assumption concerning the costs of his interactions with the activity currently occupying the site. We suggest that the bidding locator would use current spatial externality and transportation costs as the logical estimate of future interaction costs with the displaced activity.

- Given a matrix of updated site bids and a vector of activity specific relocation costs, the period-specific operation of this land market can be simulated by solving another assignment linear program.
- In any time period, a new activity or activities might join the queue and begin bidding on locations. The arrival of a new bidder, if it occurs, increases the value of \( I \) to \( I + 1 \). The dimensionality of \( A \) is correspondingly increased.
  - New bidders are able to unambiguously estimate the transportation and spatial externality costs associated with each vacant site. No activities are relocated if a new locator occupies a previously vacant site.
  - When bidding on an occupied site, we suggest the new bidder ignores interaction costs with the activity it would displace from the site. With no prior experience, the new bidder has no way to determine where he expects the displaced activity to relocate. Consequently, the new bidder has no way to evaluate the interaction costs this relocation implies.
  - Similarly, existing activities ignore the impact of new bidders when updating previous bids.
- In any time period, the most likely activities to be attracted to the system will be those with the most intensive demand for the site characteristics available there. The economic attractiveness of the system is defined by the duality conditions associated with the assignment linear program solved during the previous time period, i.e. during the previous iteration of the procedure. Thus, the primal variables are used to track the decisions of existing and newly arrived activities, while the dual variables (which describe the value of additional activities) are used to track the decisions of agents outside the system who are contemplating entry.

Algorithmic specification of the model

The principal advantage of our solution procedure is that complex information about congestion and other externalities is assumed to flow from recent experience, allowing the sequential use of linear programs to emulate the decisions of locators. A flowchart describing this approach appears in Fig. 1. The steps identified in this chart are discussed below.

- **Step 0: Data on semi-net revenues**
  - **Step 0a:** Perform a profit and/or utility maximization calculation to compute activity-specific bids for various site attributes. Define an \((I \cdot H)\) matrix \( A_1(0) = [a_{i,n}(0)] \) that attaches an \( H \)-vector of bids for site characteristics to each activity \( i \) in \( I \).
  - **Step 0b:** Assess the characteristics of discrete local sites, possibly including network connectivity indices relating to site accessibility. Define an \((H \cdot M)\) matrix \( A_2(0) = [a_{2,m}(0)] \) that attaches an \( H \)-vector of characteristic values to each physical site \( m \) in \( M \).
  - **Step 0c:** Premultiply \( A_1(0) \) with \( A_2(0) \) to create an \((I \cdot M)\) matrix, \( A_3(0) \), consisting of the semi-net revenues accruing to each bidding activity \( i \) in \( I \) at each physical site \( m \) in \( M \).

\[
A_3(0) = A_1(0) \cdot A_2(0) = [\Sigma_{i} a_{i,n}(0) \cdot a_{2,m}(0)] = [a_{3,im}(0)].
\]  \hspace{1cm} (16)

- **Step 0d:** Extend the row and column dimensions of matrix, \( A_3(0) \), to create an \([I + 1] \cdot (M + 1)]\) matrix, \( A(0) \), by defining a null activity (vacancy) that bids zero for each site; and a null site (an unbounded queue containing all bidding activities not currently located in the urban system) for which all activities bid zero.

- **Step 1: Initialization**
  - Given a matrix \( A(0) = [a_{im}(0)] \) from **Step 0d**, solve the following (assignment) linear program for the \([I + 1] \cdot (M + 1)]\) vector \( X(0) \):
    - maximize:
      \[
      \Sigma_{i=1}^{I+1} \Sigma_{m=1}^{M+1} a_{im}(0) \cdot X_{im}(0),
      \]  \hspace{1cm} (17)
Fig. 1. Algorithmic representation of the sequential land development model.
subject to:
\[
\sum_{i=1}^{I+1} X_{in}(0) = 1 \quad \text{(for sites } m = 1 \rightarrow M; \text{)}
\]
\[
\sum_{m=1}^{M+1} X_{im}(0) = 1 \quad \text{(for all activities } i = 1 \rightarrow I; \text{)}
\]
i.e. for all activities \(i\), except vacancy. \(19\)
\[
X_{im}(0) \geq 0 \quad \text{(for all activities } i = 1 \rightarrow I + 1, \text{)}
\]
for all sites \(m = 1 \rightarrow M + 1). \(20\)

Or, recognize some exogenous vector \(X(0)\).

- **Step 2: Accounting**
  Given the column vector \(X(t)\) determined in Step 1 or the previous iteration, form the symmetric \([[I + 1] \cdot (M + 1)] \cdot [[I + 1] \cdot (M + 1)])\) matrix \(Y(t) = X(t) \cdot X(t)^T\), i.e. form the outer (matrix) product of the vectors \(X(t)\) and \(X(t)^T\). Compute:

\[
Y(t) = Y_{mm}(t) = X_{m}(t) \cdot X_{n}(t) \quad \text{(for all } i, j = 1 \rightarrow I + 1; \text{ for all } m, n = 1 \rightarrow M + 1). \quad 21
\]

- **Step 3: (User) equilibrium transportation flows**
  Given the matrix \(Y(t) = [Y_{mm}(t)]\) determined in Step 2; an exogenous matrix \(F = [f_{ij}]\) consisting of traffic intensities between all activities \(i\) and \(j\); and an exogenous vector \(c(f) = [c_k(f_k(t))]\) consisting of flow-dependent congestion cost functions specific to each network link \(k\); solve the link-flow version of the user-equilibrium network assignment problem. Note that there are no network links to the null site.

Maximize:
\[
\Sigma_k \int_0^{f_k(t)} c_k(w) \, dw,
\]
subject to:
\[
\Sigma_{m=1}^{M} \Sigma_{i=1}^{I+1} \Sigma_{j=1}^{I+1} Y_{mji}(t) \cdot f_{ij}(t) = f_k(t) \quad \text{(for all links } k \text{ in } K), \quad 23
\]
\[
\Sigma_{i=1}^{I+1} \Sigma_{j=1}^{I+1} Y_{mji}(t) \cdot f_{ij} = f_{in}(t)
\]
\[
\text{(for site pairs } m, n = 1 \rightarrow M; \text{ i.e. for all site pairs that exclude the null site),} \quad 24
\]
\[
\Sigma_{\text{outbound links } k \text{ from node } m} \Sigma_{m'=1}^{M} \Sigma_{i=1}^{I+1} Y_{m'ji}(t) \cdot f_{m'j}(t)
\]
\[
- \Sigma_{\text{inbound links } k \text{ to node } m} \Sigma_{m'=1}^{M} \Sigma_{i=1}^{I+1} Y_{m'i}(t) \cdot f_{m'i}(t) = f_{mn}(t)
\]
\[
\text{(for site pairs } m, n = 1 \rightarrow M; \text{ i.e. for all site pairs that exclude the null site),} \quad 25
\]
\[
f_{ij}(t) \geq 0 \quad \text{(for all links } k \text{ in } K; \text{ for all activity pairs } i, j = 1 \rightarrow I + 1), \quad 26
\]
where \(f_{ij}(t)\) is the endogenous component of the flow between activities \(i\) and \(j\) that uses link \(k\).
Constraints (24) define the right-hand side of constraints (25), and could clearly be substituted.
Most of the exogenous variables \(Y_{mji}(t)\) have value 0, and, consequently, most of the endogenous variables \(f_{ij}(t)\) do not appear in this formulation. (The number of endogenous variables is equal to \((K \cdot M^2) + K.\))

This constraint set does not accommodate input substitution. However, this can be accomplished with additional assumptions concerning the availability of alternative technologies and imports. In such circumstances, \(f_{ij}(t)\) might logically be replaced by \(f_{k_{mji}}(t)\). In the most general case, the exogenous traffic intensities in the matrix \(F\) could be endogenized completely in the context of an elastic demand model [6, 7].

- **Step 4: (User) equilibrium transportation costs**
  Given the complete inventory of activity and vacant sites, the equilibrium link costs \(c_k(f_k(t))\) determined in Step 3, and the free-flow costs \(c_k(0)\) for network links \(k\) not currently subject to endogenous flow; determine the shortest path between each activity site pair \(m, n = 1 \rightarrow M\). The costs of traversing these minimum paths are the equilibrium transportation costs...
associated with flows between activity \( i \) at site \( m \) and activity \( j \) at site \( n \). Define these equilibrium costs to be the \( (M \cdot M) \) matrix \( C^*(t) = [c_{mn}^*(t)] \).

Compute:

\[ C^*(t) = [c_{mn}^*(t)] \] (for all site pairs \( m, n = 1 \rightarrow M \); i.e. for all site pairs that exclude the null site). \( (27) \)

This requires solution of a conventional all-shortest-paths problem for a network with exogenous arc weights, all of which are nonnegative.

- **Step 5: Spatial externalities**

  Given the vector \( X(t) = [X_{mk}(t)] \) identified in **Step 2**, an exogenous matrix \( E = [e_y(d_y)] \) consisting of distance-dependent externality cost functions specific to each activity pair \( i, j = 1 \rightarrow I + 1 \), and an exogenous matrix \( D = [d_{mn}] \) consisting of spatial (non-network) distances between all discrete location pairs \( m, n = 1 \rightarrow M \); determine the matrix of potential spatial externalities imposed by each activity \( j \) at (fixed) location \( n \) on each activity \( i \) at (variable) location \( m \). Define these potential externalities to be the \( (I \cdot |I \cdot M|) \) matrix \( E^*(t) = [e_{mn}^*(t)] \).

Compute:

\[ E^*(t) = [e_{mn}^*(t)] = [e_y(\Sigma_{n=1-M}d_{mn} \cdot X_{mk}(t))] \] (for all \( i, j = 1 \rightarrow I + 1 \); for all \( m = 1 \rightarrow M \)), \( (28) \)
\[ e_y(\cdot) = 0 \text{ if } i \text{ and/or } j = (I + 1), \text{ i.e. if } i \text{ and/or } j \text{ are vacancies.} \]

- **Step 6: Update location bids for existing site and activity inventories**

  Given the matrix \( A(0) = [a_{im}(0)] \) identified in **Step 0**; the matrix \( X(t) = [X_{mk}(t)] \) identified in **Step 2**; the matrix \( F = [f_y] \) identified in **Step 3**; the matrix \( C^*(t) = [c_{mn}^*(t)] \) identified in **Step 4**; and the matrix \( E^*(t) = [e_{mn}^*(t)] \) identified in **Step 5**; update the bids for each locator \( i = 1 \rightarrow I + 1 \) for each site \( m = 1 \rightarrow M + 1 \) based on each locator's semi-net revenues and anticipated experiences at all locations. Define these location bids to be the \( (I + 1) \cdot (M + 1) \) matrix \( A(t + 1) \).

Compute:

\[ A(t + 1) = [a_{im}(t + 1)] = [a_{im}(0) - \Sigma_{j=1-I+1}((\Sigma_{n=1-M}c_{mn}^*(t) \cdot f_{yj} \cdot X_{mk}(t)) + e_{mn}^*(t))] \]

(for all \( i = 1 \rightarrow I + 1 \), for all \( m = 1 \rightarrow M + 1 \)), \( (29) \)
\[ a_{im}(0) = 0 \text{ if } i = (I + 1), \text{ i.e. if } i \text{ is vacancy; and/or } m = M + 1, \text{ i.e. if } m \text{ is the null location.} \]

- **Step 7: Update activity inventory**

  Given the matrix \( A(t + 1) = [a_{im}(t + 1)] \) identified in **Step 6** of the current iteration, and the optimal dual values identified in **Step 8** of the previous iteration, extend the dimension of \( A(t + 1) \) to \( (I + 2) \cdot (M + 1) \) by appending a row of semi-net revenues for one new bidding activity, or by accepting the arrival of an exogenously determined locator. Set \( I = I + 1 \). If no new locator has arrived, set \( I = I \).

- **Step 8: Update location assignments**

  Given the matrix \( A(t + 1) = [a_{im}(t + 1)] \) identified in **Step 7**, and an exogenous \( (I + 1) \cdot |I \cdot I| \) vector \( R = [R_i] \) consisting of activity-specific relocation costs; solve the following linear program:

  maximize:
  \[
  \Sigma_{i=1-I+1} \Sigma_{m=1-M+1} (a_{im}(t + 1) - R_i \cdot (1 - X_{im}(t))) \cdot X_{im}(t + 1),
  \]

subject to:

\[
\Sigma_{i=1-I+1} X_{im}(t + 1) = 1 \quad \text{for sites } m = 1 \rightarrow M,
\]

i.e. for all sites \( m, \text{ except the null site} \), \( (31) \)

\[
\Sigma_{m=1-M+1} X_{im}(t + 1) = 1 \quad \text{for activities } i = 1 \rightarrow I + 1,
\]

i.e. for all activities \( i \) in \( I, \text{ except vacancy} \), \( (32) \)
\[ X_{im}(t + 1) \geq 0 \quad (\text{for all activities } i = 1 \rightarrow i + 1, \text{for all sites } m = 1 \rightarrow M + 1), \quad (33) \]

where \( X_{im}(t) \) is exogenous to time period \( t + 1 \).

- Set \( t = t + 1 \). Go to Step 2 and continue.

The computation bottleneck in this procedure is Step 3—the calculation of equilibrium transportation flows. At worst, this is a convex programming problem subject to linear constraints. All other optimization requirements are met by linear programming formulations.

**A NUMERICAL EXAMPLE**

The algorithm of Fig. 1 has been implemented for a densely connected trial network consisting of four nodes and 12 congestable arcs defining 20 directed paths. This network and the vector of bidding activities defined for this trial formulation appear in Fig. 2. The network equilibrium component of this problem is a verbose, path-flow formulation that is solved exactly via an application of the Frank–Wolfe procedure [23]. For a larger scale application, this combinatorially large path-flow formulation would necessarily be replaced by the corresponding link-flow problem [18]. The model is solved for six time periods and five activities. In the zeroth and first time periods, only activities 1–3 are assumed to be present and bidding for sites. In the second and third time periods, activities 1–4 are assumed to be bidding; and in the fourth and fifth time periods, all five activities are bidding for locations.

The updated \( A \) matrices corresponding to this trajectory appear in Appendix B. The matrices describing the transportation link costs, external effects, relocation costs, and inter-activity flows appear in Appendix A. Congestion cost functions are assumed to conform to the Bureau of Public Roads (BPR) fourth-degree polynomial [4].

Inter-activity shipments have been handled in a completely ad hoc fashion for the purpose of this small example. In the event that a located activity is outbid by vacancy and retires to the queue, inter-activity shipments involving this activity are set to zero. In the more general case, the absence of a key production activity would logically be expected to present an infeasibility. The obvious extension is to define one or more sites to be ports [14, 20] through which imports and exports enter and exit the city. This would ensure that all commodities are always available. If the cost of imported goods exceeds the local cost of production, it follows that this differential will be a factor in the technology choice decisions made by each bidder.

The results for periods 0 and 1 are summarized in Figs 3a and b, respectively. The value associated with each network link is the endogenous user equilibrium transportation cost for the link. The algorithm is initialized in period 0 by solving a linear assignment problem in which only semi-net revenues appear in the objective function (Steps 1 and 2). Activities 1 through 3 are initialized at sites 2, 4, and 1, respectively. Since only three bidding activities are present, at least one site must necessarily be vacant. In this case it is site 3, but no activity has been outbid by vacancy at all sites, and, consequently, there are no other vacancies and no activities located in

![Fig. 2. A simple four-node network that enumerates 20 directed paths.](image-url)
the queue. In period 1 no new bidders arrive, but locators have modified their bids (Steps 3–6) in light of the external and congestion costs they have experienced in their current locations as well as those they would expect at other sites, other things being equal, and relocation costs (Step 7). As a result, activity 2 relocates from site 4 to site 1, and activity 3 is displaced to the queue because it cannot generate a positive bid for any site.

Period 2 is distinguished by the exogenous arrival of activity 4 (Step 8). In this case, the site bids placed by activity 4 are semi-net revenues. However, in the more general case they would be based on semi-net revenues and observed external and congestion costs. Before offering site rents, new locators would logically solve a (ce teris paribus) all-shortest path problem to determine the inter-activity transportation costs expected at each site.

The results for periods 2 and 3 are summarized in Figs 4a and b, respectively. Activity 1 can only offer a positive bid for site 1, but is outbid at this site by activity 2, and thus joins activity 3 in the queue. New activity 4 locates at site 3. No new bidders arrive in period 4, but updates in observed external and congestion costs permit activity 1 to offer a successful bid for site 2, and lead to the relocation of activity 2 from site 1 to site 4 and of activity 4 from site 3 to site 1.

In period 4, a fifth bidder arrives and, like activity 4 before it, offers semi-net revenues. The results for periods 4 and 5 are summarized in Figs 5a and b, respectively. Given four sites, at least one bidder must necessarily be located in the queue. In period 5, there are two—activities 1 and 3. When activity 1 is again displaced to the queue, activity 5 locates at site 2. Activity 3 relocates to previously vacant site 3, permitting activity 2 to relocate to site 1. Period 6 introduces no new
bidders, but updated bids lead activity 2 to relocate back to site 4, permitting activity 5 to relocate to site 1 and activity 1 to leave the queue and locate at site 2. The results for these six time periods suggest an unrealistically intense set of relocations, but this is because relocation costs were chosen sufficiently low to ensure that relocation is induced when modest changes in external and congestion costs are realized. Sufficiently large relocation costs would prohibit such adjustments.

**IMPLICATIONS FOR URBAN PLANNING**

The algorithm of Fig. 1 summarizes a bidding process in which self-interested agents make sequential decisions about future locations based on current experience. Decisions are rational within the information constraints this perspective implies. At a minimum, the model forecasts the operation of the markets for urban land and transportation. Existing land use controls can easily be incorporated into this framework for purposes of policy investigation.

A forecasting capacity that tractably accounts for constraints on the information states of individual agents offers considerable utility to urban planners, regardless of their role in the urban economy. Even if urban planning activities are restricted to very limited market interventions, planners might:

- act as information brokers by formulating and solving the optimization problems represented in the algorithm, and by performing sensitivity tests;
- increase the rate at which information-constrained price adjustments occur in the market for land; and/or
- provide improved information concerning the anticipated impact of an individual firm’s location or relocation decision on equilibrium transportation costs and flows.

The algorithm’s market orientation aside, the model implies a much more important role for a public authority than the mere facilitation of market operations. Our perspective vests much more sophisticated use of foresight in the activities of urban planners than is attributed to individual locators. For example, the model is premised on an assumption of discrete space, and defines location in terms of the transportation network. Infrastructure investment decisions are one of the most important determinants of urban land use, and use of the model in the context of network design necessarily becomes an exercise in foresight.

As noted in the Introduction, the fundamental complexities of urban planning arise from the general equilibrium nature of the land use and transportation markets. Network costs determine accessibility, which contributes to the market incentives that drive land use decisions. Land use, however, is the substrate from which the demand for transportation is derived, and the spatial intensity of this demand determines endogenous transportation flows and costs. The proposed algorithm is particularly important in that it provides the logical computational link between conventional urban transportation planning procedures and the prediction of activity shifts.
Conventional transportation planning models recognize the endogeneity of network costs, but are not structured to exploit these prices in the context of mode choice, trip distribution between origins and destinations, the frequency of travel, or the organization of urban space. The proposed research focuses on a tractable, substantive market model that treats endogenous network costs as one of the fundamental determinants of urban form. With access to an urban systems model of this type and the information gathering agenda that it implies, planners could:

- develop multi-period forecasts of the information-constrained, general equilibrium in the markets for land and transportation;
- establish optimal budgets for transportation system improvements; and/or
- address the question of optimal transportation network design, i.e. the solution of a difficult embedded optimization problem focusing on how to identify discrete, system optimal investments in the transportation network given that user equilibrium flows will occur.

All of these planning exercises imply the anticipation of urban change. The model bootstraps the system forward by emulating the market exchange of vacancies and locators (as per the dynamic adjustments to the matrix A).

CONCLUSION

Recent work in urban economics has focused attention on how land, transportation, labor, and housing markets interact. Yet the dynamics of these interactions have proved elusive. Our model of sequential land development decisions provides a tractable, meaningful representation of the urban location decisions made by economic agents competing under conditions of growth and change. The computational requirements of the model are modest by any conventional standard of complexity.

Although most of the parameter values required by this algorithm are not immediately available, many can be derived or approximated. For example, inter-activity flows (the elements of the F matrix) are available from input–output data. The transportation engineering literature defines various calibrated congestion functions. Many of the elements of A can be derived from hedonic studies. Information concerning externalities is less complete, however, and will have to remain judgmental for the time being.

Acknowledgements—The authors gratefully acknowledge the assistance of Professor R. Kalaba, Departments of Economics, Electrical Engineering, and Biomedical Engineering, University of Southern California; conference contributions by Professor M. Wegener, Department of Civil Engineering, University of Tokyo, and Professor F. Sue, Kyoto Institute of Economic Research, Kyoto University; the efforts of two graduate research assistants, Moon Kim and Keith Hwang, School of Urban and Regional Planning, University of Southern California; and nontrivial improvements by the Editor-in-Chief and referees. The usual caveat applies.

REFERENCES

APPENDIX A

Input Data

\( A(0) = [a_{m}(0)] \): matrix of semi-net revenues

\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
1 & 200,000 & 160,000 & 40,000 & 24,000 & 0 \\
2 & 144,000 & 24,000 & 0 & 24,000 & 0 \\
(5) Bidding activities & 3 & 176,000 & 32,000 & 16,000 & 32,000 & 0 \\
4 & 160,000 & 64,000 & 88,000 & 8000 & 0 \\
5 & 16,000 & 24,000 & 40,000 & -8000 & 0 \\
(1) Null activity (vacancies) & 6 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( F = [f_{ij}] \): matrix of traffic intensities (inter-activity flows)\(^\dagger\)

\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
1 & 1000 & 5000 & 3000 & 2000 & 3000 & 0 \\
2 & 4000 & 1500 & 3000 & 7000 & 4000 & 0 \\
(5) Bidding activities & 3 & 4000 & 5000 & 1500 & 3000 & 1000 & 0 \\
4 & 2000 & 4000 & 3000 & 2000 & 1500 & 0 \\
5 & 1500 & 3000 & 2000 & 1000 & 2300 & 0 \\
(1) Null activity (vacancies) & 6 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( c(\cdot) \): vector of link specific (BPR) congestion functions

\[ c_{k}(t) = c_{k}(0) \cdot [1 + 0.15 \cdot (f_{k}(t)/\Omega_{k})^{4}] \]

\(^\dagger\)Traffic intensities involving activities not located at or bidding for physical sites are assumed to be zero.
**Sequential programming model of urban land development**

Ω = [Ωₖ]: symmetric matrix of link capacity constants

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>3000</td>
<td>9000</td>
<td>3600</td>
</tr>
<tr>
<td>2</td>
<td>3000</td>
<td>0</td>
<td>4500</td>
<td>6000</td>
</tr>
<tr>
<td>3</td>
<td>9000</td>
<td>4500</td>
<td>0</td>
<td>7500</td>
</tr>
<tr>
<td>4</td>
<td>3600</td>
<td>6000</td>
<td>7500</td>
<td>0</td>
</tr>
</tbody>
</table>

α(0) = [cₖ(0)]: symmetric matrix of free-flow transportation costs

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>15</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>0</td>
<td>21</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>21</td>
<td>0</td>
<td>31</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>27</td>
<td>31</td>
<td>0</td>
</tr>
</tbody>
</table>

E = [eₜ(dₜ)]: matrix of distance-dependent external costs†

<table>
<thead>
<tr>
<th></th>
<th>i \ j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>40,000</td>
<td>50,000</td>
<td>30,000</td>
<td>20,000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>40,000</td>
<td>0</td>
<td>20,000</td>
<td>50,000</td>
<td>30,000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>50,000</td>
<td>20,000</td>
<td>0</td>
<td>20,000</td>
<td>10,000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30,000</td>
<td>50,000</td>
<td>20,000</td>
<td>0</td>
<td>15,000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15,000</td>
<td>30,000</td>
<td>25,000</td>
<td>17,000</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

D = [dₜ]: matrix of inter-site distances (externality threshold = 10)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>9</td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>0</td>
<td>8</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>0</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

R = [Rᵢ]: vector of activity-specific relocation costs

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150,000</td>
<td>20,000</td>
<td>300,000</td>
<td>50,000</td>
<td>35,000</td>
</tr>
</tbody>
</table>

†External benefits would appear as negative numbers.
### APPENDIX B

**Summary Output Data**

$X(0) = [x_m(0)]$: matrix of initial location assignments (time period 0)

<table>
<thead>
<tr>
<th>$i \setminus m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Null site</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Physical sites (queue)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) Bidding activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) Null activity (vacancies)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>$i \setminus m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Null site</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Physical sites (queue)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) Bidding activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) Null activity (vacancies)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$A(1) = [a_m(1)]$: matrix of updated location bids formulated in time period 1

<table>
<thead>
<tr>
<th>$i \setminus m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Null site</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Physical sites (queue)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) Bidding activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) Null activity (vacancies)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$X(1) = [x_m(1)]$: matrix of location assignments for time period 1

<table>
<thead>
<tr>
<th>$i \setminus m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Null site</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Physical sites (queue)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) Bidding activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) Null activity (vacancies)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$A(2) = [a_m(2)]$: matrix of updated location bids formulated in time period 2

<table>
<thead>
<tr>
<th>$i \setminus m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Null site</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Physical sites (queue)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) Bidding activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) Null activity (vacancies)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*New arrival bidding semi-net revenues.*
Sequential programming model of urban land development

\[ X(2) = [x_{im}(2)] \]: matrix of location assignments for time period 2

<table>
<thead>
<tr>
<th>i \ m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

(4) Physical sites (queue)

(1) Null activity (vacancies)

\[ A(3) = [a_{im}(3)] \]: matrix of updated location bids formulated in time period 3

<table>
<thead>
<tr>
<th>i \ m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50,000</td>
<td>-46,091</td>
<td>-110,000</td>
<td>-126,000</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>46,315</td>
<td>4000</td>
<td>-20,000</td>
<td>4000</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>124,000</td>
<td>-268,000</td>
<td>-284,000</td>
<td>-268,000</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>110,000</td>
<td>14,000</td>
<td>57,532</td>
<td>-42,000</td>
<td>0</td>
</tr>
</tbody>
</table>

(4) Physical sites (queue)

(1) Null activity (vacancies)

\[ X(3) = [x_{im}(3)] \]: matrix of location assignments for time period 3

<table>
<thead>
<tr>
<th>i \ m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(4) Physical sites (queue)

(1) Null activity (vacancies)

\[ A(4) = [a_{im}(4)] \]: matrix of updated location bids formulated in time period 4

<table>
<thead>
<tr>
<th>i \ m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50,000</td>
<td>2978</td>
<td>-110,000</td>
<td>-126,000</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>124,000</td>
<td>4000</td>
<td>-20,000</td>
<td>-362,407</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-124,000</td>
<td>-268,000</td>
<td>-284,000</td>
<td>-268,000</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>61,676</td>
<td>14,000</td>
<td>38,900</td>
<td>-42,000</td>
<td>0</td>
</tr>
<tr>
<td>5†</td>
<td>16,000</td>
<td>24,000</td>
<td>40,000</td>
<td>-8000</td>
<td>0</td>
</tr>
</tbody>
</table>

(5) Bidding activities

(1) Null activity (vacancies)

†New arrival bidding semi-net revenues.
$X(4) = [x_{im}(4)]:$ matrix of location assignments for time period 4

<table>
<thead>
<tr>
<th></th>
<th>(1) Null site</th>
<th>(4) Physical sites (queue)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>i\m</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

$A(5) = [a_{im}(5)]:$ matrix of updated location bids formulated in time period 5

<table>
<thead>
<tr>
<th></th>
<th>(4) Physical sites</th>
<th>(1) Null site (queue)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>i\m</td>
<td>1</td>
<td>50,000</td>
</tr>
<tr>
<td>2</td>
<td>-12,129</td>
<td>4000</td>
</tr>
<tr>
<td>3</td>
<td>-124,000</td>
<td>-268,000</td>
</tr>
<tr>
<td>4</td>
<td>110,000</td>
<td>14,000</td>
</tr>
<tr>
<td>5</td>
<td>-19,000</td>
<td>-1758</td>
</tr>
</tbody>
</table>

$X(5) = [x_{im}(5)]:$ matrix of location assignments for time period 5

<table>
<thead>
<tr>
<th></th>
<th>(4) Physical sites</th>
<th>(1) Null site (queue)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>i\m</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>