Linear Assignment Problems and Transferable Development Rights

James E. Moore II

Guiding a public authority’s response to imperfections in the markets for urban land and transportation is at the core of urban and regional planning. Externalities in land use and transportation imply that purely competitive markets will very likely fail to produce Pareto optimal allocations of these resources. Many approaches to land use planning have been suggested (Hagman and Misczynski 1978), but zoning is the tool most widely used by local governments to regulate land use. Such exercises in police power are presumed to produce net social benefits by separating dissimilar land uses and reducing externalities. However, there is a substantial literature concerning the theoretical and empirical inefficiency of zoning decisions (Creine et al. 1967; Reuter 1973; Ohls et al. 1974; Peterson 1974). Even in the (hypothetical) case of an optimal zoning plan, rationing the right to develop land would still increase land rents for development properties by a quantity equal to the marginal social cost these developments would impose (Mills 1989).

Unfortunately, these rents accrue at the expense of the economic opportunities available to others. This problem of windfalls and wipeouts has been the source of substantial litigation, largely because it is the source of substantial inequity.

Transferable development rights (TDRs) have received considerable attention in urban economics and planning literatures. Urban economists have treated TDR markets in the context of spatial equilibrium models (Mills 1980; Carpenter and Heffley 1981). Ghosh (1986) has applied such models to policy studies. The consensus among interested economists is that TDR markets improve efficiency by mitigating the external effects of land use and that they improve equity because restricted-property owners voluntarily enter contracts to sell their development rights. The arbitrary rationing of gains associated with direct controls is avoided (Barrows and Prenguber 1975). Also, in a concentrated urban area where the profitability of developed land is high, it is reasonable to hope that the administrative costs of a TDR market would be lower than the costs of other control measures. TDR programs are premised on the contention that market-controlled growth is likely to be both more efficient and more equitable than the outcomes associated with conventional land use control measures.

The fundamentals of transferable development rights are simple. Within a development zone, local government suspends or sharply curtails the right of landowners to develop their land as they choose. Property owners are allocated a fixed quantity of development rights, i.e., permitted capital intensities to which they may develop their property. These rights are typically realized in terms of air rights, but may be extended to include subterranean rights as well. In more sophisticated TDR schemes development rights may vary by land use or by location within the development zone.

ABSTRACT

The Koopmans and Beckmann (1957) assignment model approach to problems of urban economics has led to considerable discussion. Recent papers (Gordon and Moore 1989; Moore and Gordon 1990; Gordon and Moore in press) review some of the issues raised and suggest extensions that resolve many of the conceptual dilemmas suggested by the original model. Most of the existing approaches surveyed in these previous papers make little use of the duality properties of the assignment model. In this paper duality information provided by the linear version of the Koopmans and Beckmann model is used to explain the utility of transferable development rights (TDR) as land use control policies. In particular, it is shown that aggregate rents accruing to capital are affected by land use plans in ways the current literature ignores. This view emphasizes the importance of TDR policies to city planners and sheds light on how planners can guide optimal urban development.

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Property owners may sell their development rights to other landowners interested in developing their own properties beyond the TDR limits they have been assigned. The landowner who forgoes development is compensated via prevailing market price for the restrictions he accepts. The externalities of land use are made partially (or completely) internal to the decision to develop. And a transaction has occurred that may be taxed or subsidized or to which planning authorities may attach additional restrictions. Taxation, however, can be problematic because it is unclear whether transferable development rights are real or personal property. Most applications of TDR have been fairly conservative exercises, even to the point of restricting development rights transfers to adjacent sites. However, the potential exists for much broader applications, including empowerment of the public sector to accumulate and disburse development rights in a given jurisdiction subject to the standards of a comprehensive land use plan.

It is conventionally assumed that the extent to which the various actors involved bear the cost of TDR is principally a function of the intensity and elasticity of development demand. We can examine this contention and other aspects of TDR in the context of a very simple mathematical programming model of urban form. The aggregate duality information provided by this formulation provides some insight into how an idealized TDR market should be organized and would be expected to behave. In this simplest case land use externalities are not modeled explicitly, though they are treated in an implicit way. In particular, it can be shown that aggregate rents accruing to capital are affected by zoning plans in ways the current literature ignores, and that, unlike the neoclassical economic approach to TDR discussions, the solutions to assignment models reveal that rental gains and losses may accrue to the owners of both land and capital.

### Assignment Models Of Urban Structure

#### Linear Assignment Models

The linear assignment problem matches plants (or, more generally, any land-using activities that might locate in cities) with an equal number of available sites. The profits available at each location are different for each activity, and the objective is to associate plants with sites such that the total value of profits is maximized. This weighted bipartite matching is possibly the simplest meaningful model of urban land use. It can be formulated as the simple linear program:

\[
\text{maximize} \quad \sum_i \sum_m a_{i,m} \cdot x_{i,m}
\]  

subject to

\[
\sum_m x_{i,m} = 1 \quad \text{(for all i in I)} \ q_i \quad (1.2)
\]

\[
\sum_i x_{i,m} = 1 \quad \text{(for all m in M)} \ r_m \quad (1.3)
\]

\[
x_{i,m} = 0,1 \quad \text{(for all i in I, for all m in M)}
\quad (1.4)
\]

where

\[
a_{i,m} = \text{the profitability of site m for activity i} \quad \text{and}
\]

\[
x_{i,m} = \text{an endogenous, binary variable equal to one if activity i is assigned to location m, and equal to zero other wise.}
\]

Constraints (1.2) ensure that plant i is located at exactly one site, constraints (1.3) ensure that site m receives exactly one plant, and q_i and r_m are the LaGrange multipliers associated with constraints (1.2) and (1.3), respectively. The fact that the right-hand side of the constraint set consists of a vector of ones identifies this formulation as the assignment version of the transportation problem. All transportation linear programs are unimodular. That is, integer inputs will always provide integer outputs. Consequently, constraints (1.4) are redundant. This linear program is actually a combinatorial programming problem subject to very efficient solution procedures.

The optimal values of the LaGrange multipliers indicate how quickly the optimal value of the objective function changes as the constraint associated with each multiplier is relaxed. If a constraint is not binding at the optimal solution, i.e. if the constraint has no influence on the optimal value of the objective function, then relaxing the constraint can have no impact on the solution and the associated LaGrange multiplier will have value zero. In a mathematical programming context constraints often describe the availability of various resources (land and capital in this case), and LaGrange multipliers are often interpreted as prices or rents. Because they measure the rate at which a system's performance can be improved as resource consumption is increased, these multipliers measure the maximum amount economic agents affected by the value of the objective function can afford to pay to purchase additional resources.

Though each bidding activity can locate at only one site, each activity formulates competing bids for all sites. The profit available from each activity at each site has been characterized as "semirevenues" (Koopmans and Beckmann 1957) because the value does not account for rents paid to the owners of land and capital. External costs might also be excluded, but this is a function of temporal
Linear Assignment Problems

assumptions (Gordon and Wingo 1981; Gordon and Moore 1989). Viewed from this perspective the values \( a_{i,m} \) are the maximum land and capital costs each plant operator \( i \) could afford to bear at site \( m \). If plant operators own the capital in their plants then they retain the plant rents. The standard assumption, however, is that plant operators rent land and capital inputs from others.

Every plant operator contributes one row to a matrix of seminet revenues, \( A \). The value of each element in the row matches the seminet revenues the operator could earn at the corresponding site. It follows that the matrix of seminet revenues must necessarily summarize information on the supply and demand for site characteristics. A procedure for determining the elements in the matrix \( A \) is described in Moore and Gordon (1990). This procedure relies on the theory of Lancaster (1966) to establish the microeconomic foundations of assignment problems. It is important to note that locator \( i \) has no interest in the seminet revenues computed by any other locator \( j \), and the values in locator \( i \)'s row are determined independently from the values in other rows.

Koopmans and Beckmann’s Quadratic Programming Model

The linear assignment model is a special case of Koopmans and Beckmann’s (1957) quadratic programming model of urban land use. The quadratic version of the assignment problem is important, but can be unusually difficult to solve because it includes both indivisible activities and the (spatially dependent) cost of interactions between activities. These economic interdependencies between activities contribute quadratic terms to the objective function. Koopmans and Beckmann defined coefficients for these interactions in terms of exogenous traffic intensities. Thus, the cost of interindustry shipments varied with the locations of the activities involved. More generally the cost of interactions might also include the value of spatial externalities.

Like the linear problem, the quadratic assignment problem matches plants or other urban facilities with an equal number of available sites. The profits available at each location are different for each activity, and the objective is to associate plants with sites such that the total value of profits net of interaction costs is maximized. The standard version of the quadratic assignment model is:

\[
\sum_i \sum_m a_{i,m} x_{i,m} - \sum_i \sum_m \sum_j \sum_n f_{ij} c_{m,n} x_{i,m} x_{j,n}
\]  

subject to

\[
\sum_m x_{i,m} = 1 \quad \text{(for all } i \text{ in } l) \quad q_i \quad (2.2)
\]

\[
\sum_i x_{i,m} = 1 \quad \text{(for all } m \text{ in } M) \quad r_m \quad (2.3)
\]

\[
\sum_i X_{i,m} = 0, 1 \quad \text{(for all } i \text{ in } 1, \text{ for all } m \text{ in } M) \quad (2.4)
\]

where

\( a_{i,m} = \) the exogenous component of the profit ("seminet revenue" in Koopmans and Beckmann’s text) available to activity \( i \) at site \( m \);

\( f_{ij} = \) the exogenous transportation flow between activities \( i \) and \( j \);

\( c_{m,n} = \) the exogenous unit transportation cost between locations \( m \) and \( n \);

\( X_{i,m} = \) an endogenous, binary variable equal to one if activity \( i \) is assigned to location \( m \), and equal to zero otherwise; and

\( X_{j,n} = \) defined similarly to \( X_{i,m} \).

The constraint set is unchanged except that equations (2.4) are no longer redundant.

Koopmans and Beckmann identified a set of general conditions under which the optimal solution to the smooth version of this quadratic program (constraints 2.4 replaced by nonnegativity requirements) could not be integer. Consequently, they concluded that the solution to the discrete version of the problem is unstable from a market perspective. At the optimal integer solution, at least two locators are left with an incentive to exchange positions. If this exchange occurs it changes the structure of interaction costs experienced by all activities and creates incentives for additional site exchanges. In the absence of relocation costs this process could be perpetual.

It has subsequently been shown by Heffley (1976, 1982) and others that there exist conditions under which integer outputs are optimal solutions to the smooth problem. However, a price-sustainable optimum is unlikely if interaction costs are large relative to seminet revenues. This result has made the Koopmans and Beckmann model a less popular tool for investigating the market for urban land, though its importance as conceptual framework is undisputed.

If the quadratic terms in Koopmans and Beckmann’s objective function are assumed to represent externalities, then excluding these terms purges the model of the market imperfections used to justify zoning interventions. However, Gordon and Wingo (1981), Gordon and Moore (1989), and Moore and Gordon (1990) have noted that the static character of the quadratic assignment model attaches an unrealistic degree of simultaneity to the location decisions made by competing agents. These simultaneous location
decisions have no plausible empirical interpretation in a world in which the decisions of competitors are almost always sequential rather than simultaneous. Locators do not confront an empty plane when they evaluate sites. They face an existing configuration of land uses and are able to anticipate associated externalities. In an intertemporal context these external costs can be included in the calculation of seminet revenues. The linear assignment model provides a much more meaningful behavioral context than the quadratic version.

Duality and Rents

Because the assignment model is a linear program the LaGrange multipliers associated with its constraints can be identified by solving a dual linear program. This dual program is unique to each assignment problem and vice-versa. Solving one problem also identifies the solution to the other. This pair relationship is not restricted to linear assignment problems. All linear programs exist as primal/dual pairs. The dual formulation of the simple assignment model is

\[
\text{minimize} \quad \sum_i q_i + \sum_m r_m \tag{3.1}
\]

subject to

\[
\sum_i q_i + \sum_m r_m \geq a_{i,m} \quad \text{(for all } i \text{ in I; for all } m \text{ in M)} \quad X_{i,m} \tag{3.2}
\]

\[
q_i, r_m \text{ unrestricted in sign} \quad \text{(for all } i \text{ in I; for all } m \text{ in M).} \tag{3.3}
\]

The LaGrange multipliers for the assignment problem are the decision variables in the dual formulation, and the decision variables in the assignment problem are the LaGrange multipliers for the dual formulation. Consequently, the dual formulation has the same information content as the original model. Further, a number of useful relationships are known to exist between the optimal solutions to any given linear program and its dual. Often, these relationships are most meaningfully interpreted in terms of the incentives they imply for market transactions. If the owners of land and capital make these inputs available to the plant operator offering the highest bid, then the optimal values of \( q_i \) and \( r_m \) identify equilibrium plant rents and site rents, respectively.

Expression (3.1) reveals that solving the dual formulation minimizes the sum of rents paid to the owners of land and capital. This is a plausible interpretation, because the optimal values of \( q_i \) and \( r_m \) are related to the optimal values of \( X_{i,m} \) in the following way.

\[
q_i + r_m = a_{i,m} \quad \text{for } X_{i,m} = 1 \tag{4.1}
\]

\[
q_i + r_m \geq a_{i,m} \quad \text{for } X_{i,m} = 0 \tag{4.2}
\]

This implies that at any location other than the optimal site (or an equivalent site), the combined land and capital rents determined by the market exceed the seminet revenues available, and the plant operator would experience a loss at suboptimal sites. Thus, the configuration that maximizes the system’s seminet revenues also implies that there is no incentive for any locator \( i \) at optimal site \( m \) to consider exchanging locations with anyone else and that all locators are in spatial equilibrium.

Plant rents accrue to the owners of the mobile factor (capital) while site rents accrue to the owners of the immobile factor (land). Lind (1973) showed that the standard (Wingo, Alonso, Muth, E. Mills) economic model of urban land use is really a special case of the assignment model in which there are large numbers of bidders and competition of such intensity that all plant rents must be bid for sites. In the case of few bidders and discrete sites it is sufficient for owners of capital (or developers) to outbid a next-highest bidder for a given site and retain any remaining profitability.

A modified version of Koopmans and Beckmann’s linear example is instructive. The objective is to define an optimal set of assignments for the four activities and sites in Figure 1. Table 1 shows the (4x4) matrix (A) that lists the seminet revenues available to each activity at each site. An optimal vector of activity assignments is indicated by asterisks. The maximum value for the seminet revenues that can be generated by this system is 57. In this case, the vector of equilibrium plant rents is

\[
q^* = (16, 5, 7, 3), \tag{5.1}
\]

and the vector of equilibrium land rents is

\[
r^* = (15, 4, 0, 7). \tag{5.2}
\]

![Figure 1. A location assignment problem: four activities and four sites.](image-url)
Linear Assignment Problems

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Z_{base} = 57.

Table 1. An optimal (base case) solution to a linear assignment problem.

Expressions (4.1) and (4.2) are evaluated for the optimal solution in Figure 2. If a planning authority were to assign perfectly informed, rational locators to their optimal locations, the dual optimal vectors q* and r* are representative of the equilibrium site and plant prices that would emerge from small auction competition in the markets for land and capital. If locators were not assigned to sites, but were instead confronted by the equilibrium capital and land prices q* and r*, their individual decisions would reproduce the land use configuration that is optimal for the group. Thus, the determination of optimal land uses and the identification of equilibrium prices in the markets for land and capital are the same problem. Solving one problem also solves the other.

The dual constraints evaluated in Figure 2 reveal that optimal plant and site rents demanded from each plant operator exactly absorb the semirent revenues earned at each activity's optimal site. Note also that the optimal site rent paid by each locater meets or exceeds the rent any other locater is willing to pay at the site, implying that the landowner has no incentive to lease the site to anyone else.

Further, the combined land and capital cost of undertaking any activity at any site other than the optimal site always meets or exceeds the semirent revenues available. Thus, this endogenous land use configuration is price-sustained.

The fact that equation (4.2) holds for each optimally located activity does not imply that there are no normal profits to be made. Any appropriate definition of an activity's semirent revenues must include a minimum acceptable payment to the plant operator's retained earnings account. Thus, normal profits are assumed to have already been netted out of the revenues locaters expect to earn at any site. After accounting for the additional market rents demanded by owners of land and capital, the net revenues available at each locater's optimal site are identically zero. This is understandable; the owners of land and capital will demand the highest market rents locaters can afford to pay. To demand less would be to forego opportunity. If the

-owners of these inputs are perfectly informed, they will demand rents high enough to exhaust semirent revenues. Locators are competing for access to sites, and will offer as much as they can afford given the semirent revenues available at each site. A consequence of this competitive situation and the spatial equilibrium it implies is that the optimal value of the assignment problem's objective function equals the optimal value of the dual formulation's objective function. This is a necessary condition associated with the optimal solution to any linear program.

Because linear programs have linear objective functions subject to convex feasible regions, these problems do not exhibit local optima (Dorfman et al. 1958). However, it does not follow that the vectors X*, q*, and r* are unique. If the optima's objective plane intersects the feasible region along a face or an edge instead of at a vertex, then alternative optima of equivalent value will exist. Further, even if X* is unique, q* and r* need not be. For example,

\[ q^* = (14, 4, 7, 2) \]  \[ r^* = (15, 6, 1, 8) \]

(6.1)  \hspace{2cm}  (6.2)

\[ X_{im} = 0 \text{ implies } q_i \cdot r_m \leq 0 \]
\[ X_{im} = 1 \text{ implies } q_i \cdot r_m = 0 \]

[Diagram of spatial equilibrium]

Figure 2. An optimal assignment: spatial equilibrium.
also satisfy equations (4.1) and (4.2) and would sustain the optimal land use configuration shown in Figure 2.

The possibility of alternative primal optima implies that the equilibrium land use configuration identified by the market will be drawn from a set of economically equivalent alternatives. Once any alternative is achieved, the incentive for further change disappears. However, the separate possibility of alternative dual solutions raises more interesting conceptual questions. Consider the small example summarized in Table 2. At the optimal solution, the dual vectors \( r \) and \( q \) must satisfy

\[
\begin{align*}
q_1 + r_1 &= 7 \text{ since } X_{1,1} = 1 \\
q_1 + r_2 &\geq 1 \text{ since } X_{1,2} = 0 \\
q_2 + r_1 &\geq 3 \text{ since } X_{2,1} = 0 \\
q_2 + r_2 &= 8 \text{ since } X_{2,2} = 1.
\end{align*}
\]

(7.1) 
(7.2) 
(7.3) 
(7.4)

Any combination of \( r \) and \( q \) satisfying expressions (7.1) through (7.4) defines a price system that will sustain the optimal land use configuration.

The shaded area in Figure 3 identifies all such price systems. Which coordinate identifies the prices associated with the state toward which the system tends? All of the shaded price combinations are equally viable from an economic perspective and, in the absence of additional information, are equally likely. It is tempting to treat selection of the appropriate rent coordinate as a planning exercise, but this is a difficult and controversial objective. Identifying the ideal rent coordinate is not an exercise in efficiency. It is an exercise in fairness. For example, the vectors

\[
\begin{align*}
q^* &= (3.5, 4.0) \text{ and} \\
r^* &= (3.5, 4.0)
\end{align*}
\]

(8.1) 
(8.2)

allocate rents equally across the owners of land and capital. While the symmetry of this coordinate seems to have a naïve appeal, further consideration raises hard questions. Why should planners allocate a higher rent to the owner of site 2 than the owner of site 1? These rents could be equalized, but it is obvious from the parallel diagonals in Figure 3 that if land (capital) rents are constrained to be equal then the capital (land) rents cannot be. This need not generally be the case, but even if it is possible to equalize rents it is unclear that doing so constitutes a defendable exercise in fairness. Should rents be equalized, or should the marginal value of the rents be equalized? Equalizing the marginal value of rents would maximize the social benefits the rents provide (Musgrave and Musgrave 1980), but even this objective might lead to unexpected outcomes. For example, should total rent benefits be maximized even if this implies that all rents will be allocated to one individual?

These are important questions, but they are not important because of their answers. There are no universally accepted answers. Rather, the questions are important because treating them makes explicit planning assumptions that would otherwise remain implicit. In practice, selecting an optimal rent coordinate is an intractable exercise. Prices are not visible control variables because the information requirements associated with using them as such are impossibly large. The administrative burden of creating prices must usually be distributed across market participants. The transactions of many competing buyers and sellers propel the system along a trajectory of multidimensional rent coordinates. Though the planner cannot expect to know the details of this rent space, the planner should understand how different land use policies that might be represented in the primal problem are likely to affect the size

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\[ Z_{\text{base}}^* = 15. \]

Table 2. A problem with one primal solution and alternative dual solutions.

![Figure 3. Market feasible land and capital rents.](image-url)
and dimensionality of the box in Figure 3. While additional decisions by planners might well enhance equity, the planner should not forsake the opportunity to treat questions of efficiency.

**Vacant Sites and Unlocated Plants**

Consider extending the linear assignment problem to include vacancies and unlocated activities. As noted above, constraints (1.2) and (2.2) require that every activity be located somewhere. In the case of a very unproductive site this requirement might dictate that the site owner offer a subsidy to prevent the optimal locator's net revenues from becoming negative. Obviously, landowners would not voluntarily accept negative site rents. In a more realistic system, not all locations would be able to afford a site, and not all locations would be productive enough to attract a plant. There might be more activities than there are sites, or it might be that, though there are surplus sites available, a marginal bidder can only offer nonnegative site rents for locations that have been leased to higher bidders.

These possibilities can be accommodated by introducing two modifications to the current formulation. First, unsuccessful bidders are consigned to a null site, which implies a queue. Activities bid nothing for access to the queue because no revenues can be earned there, and (unlike other sites) there is no constraint on the number of activities that can occupy the queue simultaneously. Second, vacancies are treated explicitly. Vacancy is a null activity that bids nothing for physical sites and can be simultaneously assigned to any number of sites. When nonvacancy activities offer sufficiently positive bids for sites existing vacancies are displaced. Under this approach vacancy can outbid activities that become wholly unprofitable, and unprofitable activities can be retired from the system without terminating their capacity to formulate future bids. This perspective implies that the matrix A in Table 1 should be augmented by appending an I+1st row accounting for vacancies and an M+1st column corresponding to the null location or queue. The augmented matrix, A', that results appears in Table 3.

### Modeling the Market Implications of Land Use Controls

#### Zoning

In neoclassical economic analysis all rents are assumed to accrue to landowners. In the context of an assignment problem this result occurs only as a special case. It is entirely plausible for some rents to accrue to the owners of capital. For this reason the dual variables corresponding to plant rents merit attention. It is simple to demonstrate that plant rent values can be substantially suppressed by the imposition of zoning restrictions such as open space requirements. Under reasonably general conditions the pattern of windfalls and wipeouts is therefore much more complicated than the literature suggests.

Recall the optimal assignment identified in Figure 2. Consider a new solution subject to the added constraint that there be a vacant zone. This open space requirement can be enforced by appending a constraint of the form

\[ X_{5,m} = 1 \]  \hspace{0.5cm} (9.1)

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Table 3. The augmented matrix A'.
where \( i = 5 \) denotes vacancy and \( m \) is the location to which vacancy will be assigned. The solution to this modified problem will identify the activity that can be displaced from the system at least cost. The remaining activities will be redistributed away from the vacant zone in an optimal manner. More generally, both the activity to be displaced and the most efficient location for the resulting vacancy can be found simultaneously by appending a constraint of the form

\[
\sum m X_{i,m} = 1
\]

(9.2)

instead of the form (9.1). Under constraint (9.2) the location of the vacancy need not be identified beforehand by the planning authority. Since the optimal value of the assignment problem’s objective function is equal to the optimal value of the dual objective function, this solution also minimizes reductions in the total rents paid to the owners of land and capital.

Neither of these new constraints could possibly induce an increase in the optimal value of the objective function, \( Z^* \). Constraints describe restrictive relationships that must hold regardless of what value the objective function takes on. Appending additional constraints to any problem can only cause the value of the objective function to decay. At best the additional constraints might be dominated by existing constraints, in which case the relationships the new constraints imply would already be represented in the model. If this is the case the value of the objective function would be unaffected. Improvements in \( Z^* \) cannot come about unless existing constraints have been deleted or replaced by less restrictive relationships.

In the example detailed in Figure 2 vacancy failed to outbid any competing activity at any site. Consequently, all locations were occupied. Imposing a vacancy on the system will almost certainly cause the value of the assignment problem’s objective function to decrease by some quantity \( \Delta Z^* \). Consider modifying the original problem to include constraint (9.1) where \( m = 2 \). The optimal solution appears in Table 4 and is illustrated in Figure 4a. Site 2 is vacant, as required by (9.1). Activity 1 has been shifted from site 2 to site 4. Activity 2 has been shifted from site 4 to site 3. Activity 4 has been displaced to the queue because the plant operator cannot outbid operators 1 through 3 at any of the available sites.

If externalities are present in the system, restricting site 2 to use as open space may reduce these costs, but doing so is expensive, even if the system remains maximally efficient in

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<td>2</td>
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<td>3</td>
<td>22.*</td>
<td>4.</td>
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<tr>
<td>4</td>
<td>16.</td>
<td>7.</td>
</tr>
<tr>
<td>Vacancies</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.</td>
<td>0.*</td>
<td>0.</td>
</tr>
<tr>
<td>( r_m ) &amp; 18. &amp; 0. &amp; 3. &amp; 10. &amp; 0. &amp; 0. &amp; 0.</td>
<td>0.</td>
<td></td>
</tr>
<tr>
<td>( \Delta r_m ) &amp; +3. &amp; (-4.) &amp; +3. &amp; +3.</td>
<td>0. &amp; 0. &amp; 0.</td>
<td>0.</td>
</tr>
<tr>
<td>( Z_{\text{exox}}^* ) &amp; 46. &amp; 46. &amp; 46. &amp; 46.</td>
<td>46. &amp; 46. &amp; 46.</td>
<td>46.</td>
</tr>
</tbody>
</table>

Table 4. Result of zoning for an exogenous vacancy at site 2: change have been computed relative to the base case.

<table>
<thead>
<tr>
<th>Sites</th>
<th>( q_i )</th>
<th>( \Delta q_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i/m</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Activities</td>
<td></td>
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<tr>
<td>1</td>
<td>25.</td>
<td>20.*</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>22.*</td>
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<td>4</td>
<td>16.</td>
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<tr>
<td>Vacancies</td>
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</tr>
<tr>
<td>0.</td>
<td>0.*</td>
<td>0.</td>
</tr>
<tr>
<td>( r_m ) &amp; 20.</td>
<td>11.</td>
<td>10.</td>
</tr>
<tr>
<td>( \Delta r_m ) &amp; +5. &amp; +7.</td>
<td>+10.</td>
<td>(-7.)</td>
</tr>
<tr>
<td>( Z_{\text{endox}}^* ) &amp; 54. &amp; 54.</td>
<td>54.</td>
<td>54.</td>
</tr>
</tbody>
</table>

Table 5. Result of zoning for an endogenous vacancy at site 3: changes have been computed relative to the base case.
Linear Assignment Problems

every other respect. This expense is accounted for by the resulting change in the optimal value of the objective function.

\[ \Delta Z_{\text{extra}} = Z_{\text{extra}} - Z_{\text{base}} = 46 - 57 = (-11). \]  

If the cost of mitigated externalities exceeds 11, restricting site 2 to use as open space has improved social welfare. If the cost of mitigated externalities does not exceed 11, the zoning exercise has resulted in a net welfare loss.

Consider replacing constraint (9.1) with constraint (9.2). The optimal solution to the new problem appears in Table 5 and is illustrated in Figure 4b. Site 3 is vacant, and activity 4 has been displaced directly to the queue. This solution accommodates the required vacancy but reduces total rents to the minimum extent possible. In this case the reduction in the optimal value of the objective function is only

\[ \Delta Z_{\text{extra}} = Z_{\text{extra}} - Z_{\text{base}} = 54 - 57 = (-3) \]  

which is clearly preferable to a value of (-11).

Unfortunately, the changes in equilibrium rents associated with either of these new solutions are not evenly distributed. Imposing even an optimal vacancy changes the relative availability of land and capital in the system. The relative scarcity of capital has been reduced, and that input has become less valuable. Land has been made scarcer, and the value of nonvacant sites has substantially increased.

Some of the rents previously paid to the owners of capital are paid to the owners of land. In either case the value of the vacant site is zero because it cannot be rented. The landowner has been wiped out. While the windfalls experienced by other landowners are sufficient to compensate the owner of the vacant site for losses, they are insufficient to also cover the incremental losses experienced by the owners of capital.

In fact, there is no way that the losers ever could be compensated in this situation because

\[ \Delta Z = \sum \text{windfalls} + \sum \text{windfalls} - \sum \text{wipesouts}. \]  

This is true in general, not just for the example summarized in Table 4 and Figure 4a. By strong duality, if there is a finite optimum for a primal linear program and a finite optimum for the corresponding dual linear program, then the optimal values of the primal and dual objective functions are necessarily equal. Since the dual variables \( q_i \) and \( r_m \) are the rents on land and capital, it follows that when \( \Delta Z \) is a negative number, the value of total wipesouts must exceed the value of total windfalls. This is true regardless of the value that prevails for each separate rent variable, so the problem of multiple optima in the dual does not affect this result. Further, \( \Delta Z \) will always be nonpositive for land use policies corresponding to new constraints in the primal linear program, because it is impossible to improve the optimal value of any function by further constraining it.

No externalities are represented in the structure of the model, and the economic incentive to decrease land use densities is similarly unaccounted for. However, the fact that there are no external effects does not imply that there are no externalities operating in the system being modeled, and this simple analysis still provides a useful benchmark. If land use externalities are operating, the welfare gains associated with reducing these externalities should be at least as large as \( \Delta Z_{\text{extra}} \). If this is not the case there is no economic justification for restricting any of the sites to open space use. Even if the zoning exercise is being undertaken for noneconomic reasons, \( \Delta Z_{\text{extra}} \) still bounds the net cost of the decision to impose a vacancy.

Transferable Development Rights

The logic of the linear assignment problem is conventionally assumed to imply a technological limit on the number of activities assigned to any site. In a TDR context this limit is institutional. Development is restricted by planning authorities, not technology. Consider the following modification to the linear assignment problem maximize

\[ \sum_i \sum_m a_{im} \cdot X_{im} \]  

subject to

\[ \sum_m X_{im} = 1 \text{ (for all} i \text{ in} I) \]  

\[ \sum_i X_{im} \leq 2 \text{ (for all} m \text{ in} M) \]  

<table>
<thead>
<tr>
<th>Sites</th>
<th>( qi )</th>
<th>( \Delta q_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.</td>
<td>0.</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>4</td>
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<td>0.</td>
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<tr>
<td>vacancies</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>( r_m )</td>
<td>6.</td>
<td>0.</td>
</tr>
<tr>
<td>( \Delta r_m )</td>
<td>(-9)</td>
<td>(-4)</td>
</tr>
</tbody>
</table>

Table 6. Implementing a transferable development rights approach: changes have been computed relative to the base case.
Figure 5. Optimal land use in a transferable development rights zone.

\[ X_{L,m} = 0, 1 \text{ (for all } i \text{ in } I, \text{ for all } m \text{ in } M). \]  (12.4)

This implies that, while no activity can be assigned to more than one site, up to two locators can be assigned to any given site. The new constraints are less restrictive than the previous versions and permit land to be used more intensively. However, the total number of plants that may be located is still constrained to four, an average of one per site.

The optimal solution to the new problem appears in Table 6 and is illustrated in Figure 5. The opportunity to intensify land use has diminished the relative scarcity of land, and land rents have been reduced. The scarcity and value of capital have been increased. Because the new constraints are less restrictive than the relationships they replaced, the optimal value of the objective function has increased. Further, the windfalls experienced by the owners of capital are more than sufficient to compensate landowners for their losses. This follows from equation (11.) Since \( \Delta Z_{TDR} \) is a positive number, the value of total windfalls must exceed the value of total wipeouts.

Relative to the base case, average land use densities are unchanged (there are still four activities allocated to four sites), but open space and seminet revenues have both been increased. If the benefits of open space (including the mitigation of externalities) combined with the value of \( \Delta Z_{TDR} \) exceed the cost of any new externalities generated by the capital intensive land use at site 1, then the land use pattern resulting from transferable development rights must be preferred to the base case. Further, if the value of

\[ \Delta Z_{TDR} - \Delta Z_{exe} = 13 - (-3) = 16 \]  (13.)

exceeds the cost of any new externalities generated at site 1, then the land use pattern resulting from the transferable development rights solution must also be preferred to the optimal zoning pattern. While further relaxations of constraint (12.3) would produce ever diminishing improvements in \( Z \), external costs related to capital intensive land use would also be increased.

Both zoning and TDR approaches involve administrative burdens. The availability of an optimal zoning plan implies that the planner is in a state of perfect information. While it can be reasonably argued that an optimally structured market for TDR also implies a considerable information burden, it is considerably simpler to refine an inferior set of TDR allocations than to make an a priori specification of optimal land uses. Further, even an optimal zoning plan must be enforced. In a TDR market a land use plan is implemented and enforced via competition between market participants. Rather than enforcing land uses, the planner enforces voluntary contracts. In the example below, the TDR results can be generated at very low enforcement cost simply by granting each landowner in the development zone a transferable right to develop his property with a single activity and permitting site owners to purchase up to one additional development right.

While this arrangement might reduce equilibrium land rents, it would also allow the owners of capital (or developers acting as brokers and agents) with an incentive to pay the landowners to effect a transfer of the development rights. In this case the owners of capital would try to induce a transfer to site 1 of the development rights originally attached to site 3. The other landowners would be willing to undertake such transfers for smaller incentives, but any such suboptimal transactions could only be intermediate. Each transaction would lead to a new combination of (suboptimal) land and plant rents and a new pattern of incentives for further transfers. This process would continue until the net effect was a transfer of development rights from site 3 to site 1. During the course of these competitive transactions, a share of \( \Delta Z_{TDR} \) would be captured by the site owners who value it most highly.

- CONCLUSIONS

Evermore of the world’s cities are being planned in ways that allow (and rely on) markets for development rights. A linear program has been used to specify a computable general equilibrium model of urban structure that explains the role of TDRs in land use planning. The examples discussed here are simplistic but lead to the following important conclusions.

Under a conventional zoning plan, the value of wipeouts must exceed the value of windfalls. The opposite is true in the case of TDRs. Transferable development rights provide efficiency gains relative to ad hoc zoning plans because development rights can be more productively combined with complementary inputs if not locked to a site. Further, since TDR prices reflect their value, TDRs are combined with nonlocational inputs in more efficient proportions.
Rents accruing to capital are affected by both zoning and TDR plans. Discussions of windfalls and wipeouts should not be restricted to land values. Local capital values are affected as well.

Market-based strategies such as transferable development rights require much less information to implement than traditional zoning plans, because such strategies reflect the fact that markets fully exploit the information most readily available to participants. The combination of signals provided by prices has far more information content than planners could ever hope to re-create without permitting market exchanges.

Information questions aside, the enforcement costs of market-based strategies tend to be lower than the enforcement costs of command and control regulations. Optimal land use follows from the decisions of rational agents pursuing their interests in an optimally structured market. The cost of defining the market for TDR exchanges is likely to be much less than the cost of enforcing detailed land use plans. Markets make very expensive opponents, but very productive allies.

Reliably on transferable development rights does not imply an absence of urban planning. Indeed, the planning potential of TDRs is enormous, much greater than that of zoning because the information requirements are so much less burdensome. Still, this planning potential has gone largely unrealized. Consider Galperin's (1989) observation from the field. "Getting these (development) rights has long been a tedious and expensive process, and new rules governing such exchanges have developers worried about how to pull off such deals in the future. Besides money and patience, builders downtown need the approval of the city's Community Redevelopment Agency, Planning Commission and City Council. Unless all three sign off on an exchange of air rights, the deal dies." This need not be the case. Obtaining development rights has been an expensive, difficult activity only because planners have not participated in the organization of TDR markets. Those markets that do exist are of very limited scope, are implemented without clear economic objectives, and tend to be organized in ways that hamper TDR exchanges. This maximizes rather than minimizes administrative costs.

Though TDR schemes are conceptually simple, planning authorities are needed to set, clarify, and refine the rules of the TDR game. This includes defining the spatial and institutional boundaries of the market place; allocating and revising initial quotas of development rights; and, because a public plan would usually be at stake, monitoring the exchanges that take place. Too few of these important brokerage functions are currently being performed. Consequently the existing markets for transferable development rights are poorly organized. The viability and importance of transferable development rights are best demonstrated by the fact that such transfers continue to take place despite poor institutional support and an absence of attention from urban planners.

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