Traffic Equilibrium with Shared Mobility Services in A Coupled Morning-evening Commute Framework

Wei Gu\textsuperscript{1}, Maged Dessouky\textsuperscript{1}, Jong-Shi Pang\textsuperscript{1}, H. Michael Zhang\textsuperscript{2,*}

\textsuperscript{1}Daniel J. Epstein Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, CA 90089  
weig@usc.edu  maged@usc.edu  jongship@usc.edu

\textsuperscript{2}Department of Civil and Environmental Engineering, University of California Davis, Davis, CA 95616  
hmzhang@ucdavis.edu

Abstract

In this research, we develop a general equilibrium model to capture the complex interactions between solo-driving, rideshare and ridesourcing services such as Uber and Lyft that allows travelers to switch between different transportation modes in a coupled morning-evening commute. The model is formulated as a mixed complementarity problem. Then the existence of an equilibrium solution and the properties of the solution are investigated, and the conditions on the model parameters under which the equilibrium will be unique are provided. The proposed model is then validated on the Sioux-Falls network. The results show that our model captures the mode switches between morning and evening that are missed by decoupled morning and evening commute models. In particular, our numerical examples show that modeling morning and evening commutes separately tends to overestimate the number of drivers and total vehicle miles traveled (VMT) in the network when accounting for travelers’ capabilities for mode switching. With a coupled model, transportation planners can better understand appropriate incentives to increase vehicle occupancy and reduce VMT.

\textit{Key words}: traffic equilibrium, mixed complementarity problem, rideshare, ridesourcing, coupled morning-evening commute
1 Introduction

App-based transportation services, such as ridesourcing services (also called e-hailing services in the literature, e.g., Ban et al., 2019) provided by Uber, Lyft, Didi, Grab and Ola or casual rideshare enabled by SCOOP, WAZE, Zipcar, and Turo are growing rapidly. For example, Uber has hit its milestone in 2018 to serve over 10 billion trips within more than 700 cities of 80 countries (Uber, 2018). There are over 75 million riders and 3.9 million drivers in total, producing more than 14.1 billion dollars of annual net revenue (Iqbal, 2020). These emerging transportation services are transforming the travel behavior of individuals and urban mobility patterns, and provide significant challenges to transportation planners and policy makers on how to assess the impact of these services on transportation systems, and how to facilitate or regulate these services because conventional planning tools are inadequate to model their more complex interactions between drivers, riders, and the private enterprises that link the drivers and riders together.

Due to heavy traffic, commuters suffer from long travel delays in both the morning and evening commutes in many urban areas. The emerging shared mobility transportation services, ridesourcing and ridesharing, provide more travel mode choices for commuters in both morning and evening commutes. Furthermore, these new modes of travel compete or cooperate in this space to reduce travel demand and hence traffic congestion. For example, a person can combine a rideshare service in the morning, but use a ridesourcing service for the evening return trip to reduce the pairing cost, and provide more flexibility in evening trips.

While these new transportation services significantly increase the options that travelers have for their commute and the travel choice in the morning commute directly impacts the feasible options for the evening commute, the net effect of these new services on the long term efficiency, sustainability, and equity of urban transportation systems remain to be better understood. There is a clear need to not only understand the nature and effect of these new mobility services better, but also to understand, model, and study the interactions between the various modes of transportation, and integrate them in a unified transportation planning model that includes morning and evening commute trips. However, prior research models typically treat these two commute trips separately (Xiao et al., 2016; Ma and Zhang, 2017; Liu and Li, 2017; Su and Wang, 2019; Lin et al., 2020). To the best of our knowledge, there is no research to provide a general equilibrium model to both capture the complex interactions between solo driving, ridesharing, and ridesourcing and allow travelers to switch between different transportation modes in a coupled morning-evening commute. In order to address the research gap, we develop a general modeling framework to simultaneously consider the morning and evening commute. The objective of this research is to understand the impact of the new shared mobility modes on the coupling of the morning and evening commute: traffic congestion, travelers’ behavior of mode choices, and efficiency of the overall urban transportation systems.

Without a coupled modeling approach, planners could solve a traffic assignment problem using estimated cost and value of time data to predict and understand traveler behavior during the morning commute. The same could be done for the evening commute but there would have to be the added constraint that if an individual chose to drive in the morning that they would also need to drive in the evening. A coupled modelling approach would be able to capture traveler behavior for the entire day and allow for travelers to switch one type of commute mode in the morning to another in the evening. For various reasons, travelers may switch from one type of commute mode in the morning to a different type in the evening: rideshare passengers and ridesourcing passengers may switch among these two types. This capturing of mode switches is especially important if the travel cost data is different in the morning and evening.
times. For example, a traveler with a high inconvenience cost for ridesharing in the afternoon, which may be due to the need to pick up their children from after school activities, will not use this mode in the afternoon, but may consider this option in the morning with the appropriate incentives since they would be able to take ridesourcing in the afternoon. With a coupled model, transportation planners can better understand appropriate incentives that captures the entire day to increase vehicle occupancy and reduce vehicles miles traveled. We note that even when the cost structure for the morning and evening commutes are the same, a coupled model could yield a different equilibrium solution (e.g., number of drivers, total VMT, etc) than the separate models if the traffic network is not symmetrical.

In this paper, we first develop an equilibrium model that considers multiple transportation modes (solo driving, ridesourcing and ridesharing) and integrates the morning and evening commutes. Then we show an equilibrium solution exists for our proposed model, and provide the conditions under which the solution is unique. Furthermore, we run some experiments to show the effectiveness of our coupled model in capturing possible mode switch behavior with the appropriate incentives compared with treating the commute trips separately. The remainder of this paper is organized as follows. In Section 2, we review the relevant literature. Section 3 provides the coupled morning-evening traffic equilibrium model, while Section 4 analyzes mathematical properties of the proposed model. In Section 5, experimental results are given to illustrate our model. Section 6 concludes this paper and points out some possible directions for future research.

2 Literature Review

There have been extensive efforts to model the emerging shared mobility transportation services, and many of these papers mainly focus on the transportation network companies’ (TNCs’) daily operations (Furuhata et al., 2013; Mourad et al., 2019; Wang and Yang, 2019; Yan et al., 2019; Tafreshian et al., 2020). In this section, we review the papers that are most relevant to our research, which includes two categories: (1) equilibrium of shared mobility transportation systems, and (2) shared mobility services in the morning commute.

There has been some research that formulates the new shared mobility modes as a Traffic Assignment Problem (Sheffi, 1985; Patriksson, 2015). Xu et al. (2015b) proposed a traffic equilibrium model with rideshare service, while Ban et al. (2019) considered the ridesourcing service in their general equilibrium model. Di and Ban (2019) provided a general traffic equilibrium modeling framework, which included travelers’ mode choices, rideshare equilibrium and ridesourcing operations. Considering an OD-based surge pricing strategy, Ma et al. (2020) proposed a rideshare user equilibrium model with ride-matching constraints. Li et al. (2020) studied a path-based rideshare equilibrium model to simultaneously produce route choices, mode choices, and matching decisions. Instead of using a mixed complementary formulation, Wang et al. (2020) established a convex programming formulation for the rideshare user equilibrium problem. To better understand vacant trips generated by ridesourcing service, Xu et al. (2020) put forward a network equilibrium model to capture both cruising and deadheading trips of ridesourcing vehicles.

With the traffic equilibrium including shared mobility services as a lower level problem, researchers extended the literature of the Transportation Network Design Problem (Yang and Bell, 1998; Chen et al., 2011; Farahani et al., 2013; Xu et al., 2016). For example, Di et al. (2018) extended the rideshare equilibrium framework of Xu et al. (2015b) to optimize the deployment of high-occupancy toll (HOT)
Since the shared mobility transportation market may influence traffic congestion, some research explored the equilibrium for a shared mobility transportation market, and its impacts on or interaction with the equilibrium in a traffic network. Xu et al. (2015a) combined an elastic demand traffic equilibrium model with an economic pricing model to determine the rideshare price. Under the scenario of mixed ridesourcing and taxi market, He and Shen (2015) established a spatial equilibrium model to balance supply and demand in the market, and at the same time evaluated travelers’ possible adoption to the emerging ridesourcing service; Qian and Ukkusuri (2017) investigated the equilibrium of the competitive market by modeling it as a multiple-leader-follower game: passengers are the leaders who aim to minimize the cost, while drivers are the followers seeking to maximize the profit. Li et al. (2019) studied the impact of regulation on TNCs based on a queuing theoretic market equilibrium model. Ke et al. (2020a) explored the effects of key decision variables of a ridesourcing platform (such as price and vehicle fleet size) on its revenue and social welfare. With a macroscopic fundamental diagram to characterize traffic congestion, Ke et al. (2020b) proposed a ridesourcing market equilibrium model with congestion effects. Zhang and Nie (2020) put forward a matching-based market equilibrium model to explore the influence of regulation on both ridesourcing and rideshare services.

Some papers extended the Morning Commute Problem (Vickrey, 1969; Newell, 1987, 1988; Daganzo and Garcia, 2000; Nie and Zhang, 2009; Shen and Zhang, 2009; Liu and Nie, 2011; Qian et al., 2012; Xiao and Zhang, 2014) by considering the emerging transportation services. Xiao et al. (2016) explored the morning commute problem with rideshare service and parking space limitation. Ma and Zhang (2017) studied the integration of rideshare in the morning commute from home to the central business district under a dynamic rideshare payment. Liu and Li (2017) investigated the dynamic rideshare user equilibrium during the morning commute under the fixed-ratio charging-compensation scheme (FCS), while Wang et al. (2019) examined an extended version under the variable-ratio charging-compensation scheme (VCS). Considering parking space constraints, Su and Wang (2019) addressed the problem of regulating the supply of rideshare services in the morning commute. Lin et al. (2020) studied the influence of HOV/HOT lines on dynamic rideshare during morning commute.

None of the literature reviewed above considers the joint travel decisions, our study extends the current literature by providing a general equilibrium model framework to capture both rideshare and ridesourcing services between morning and evening commutes, and the possible switches across rideshare and ridesourcing modes.

3 Mathematical Model

3.1 Problem description

We propose an extended path-induced cycle-based traffic equilibrium problem of morning and evening commutes, taking into account the emergent travel trends of ridesharing and ridesourcing that offer alternative modes of travel supplementing the traditional mode of commute: solo driving. The goal of the model is to study the morning and evening commute trip flows in the network caused by traffic congestion and the travelers’ choices of commute types to minimize their disutilities. Most importantly, our approach is holistic, combining morning travel from an origin to a destination and evening return from the same destination (which therefore is the origin of the evening trip) to the morning’s origin; this
round trip constitutes the commute cycle (cc). Specifically, each such cycle is composed of a morning trip taken on a path and an evening trip taken on a possibly different (reverse) path with possibly a different mode. The cycle flows encompass travelers’ commute behavior; the equilibrium will determine the travelers’ cycle selections by equilibrating the cycle flows with the travelers’ disutilities associated with the particular round-trip (i.e., cycle) choices based on an extension of Wardrop’s user equilibrium principle.

As illustrated in Fig. 1, there are 3 types of commuters: (a) drivers, labelled as 1; (b) rideshare passengers, labelled as 2; and (c) ridesourcing passengers, labelled as 3. Trip makers travel from home to work in the morning as one of these 3 types of commuters and from work to home in the evening. For various reasons, travelers may switch from one type of commute mode in the morning to a different type in the evening: rideshare passengers and ridesourcing passengers may switch among these two types. As a result, there are 5 types of mode combinations between morning commuters and evening commuters in total (see Fig. 1). Labeled by \(j \in \{1, \cdots, 5\}\), each of these 5 mode combination types is incident to a unique pair of commuters, which we label as \(j_{am}\) and \(j_{pm}\), both being indices in \{1, 2, 3\}, respectively. For instance, the combination \(j = 3\) means morning rideshare passenger (2) switched to evening ridesourcing passenger (3); thus for this \(j\), we have \(j_{am} = 2\) and \(j_{pm} = 3\). Based on a set of travel costs, the model aims to determine a user equilibrium of trips under a set of reasonable and realistic assumptions and subject to traffic congestion. In the process, the model also determines the switches of the commuter types in the morning and the evening trips. The correspondences between the mode combinations and commuter types are listed in the following table:

<table>
<thead>
<tr>
<th>mode combination</th>
<th>morning commuter type</th>
<th>evening commuter type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(j)</td>
<td>(j_{am})</td>
<td>(j_{pm})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

A morning OD pair \(k = (o;d)\) becomes the OD pair \(\bar{k} = (d;o)\) in the evening. That is to say, the origin and destination of morning OD-pair \(k \in K\) becomes the destination and origin of evening OD-pair \(\bar{k}\), respectively. Corresponding to each morning OD-pair \(k\) and evening reverse OD-pair \(\bar{k}\) is a set \(C_k\) of commute cycles \(c_k\) each consisting of a pair of morning and evening paths used by this cycle; conversely, associated with each commute cycle \(c \in C\) \(\triangleq \bigcup_{k \in K} C_k\) is a unique morning OD-pair and its evening reverse OD-pair. Therefore, the association \(k \in K \mapsto c_k \in C\) is multi-valued while its inverse is single-valued. It is possible that two commute cycles \(c_k\) and \(c_{k}'\) associated with the same OD-pair \(k\) can use the same morning path and yet differ in the evening paths; similarly, it is also possible for these two commute cycles to differ in the morning paths but use the same evening path. Flows on such commute cycles are the primary decision variables of the model.
Figure 1. Multiple modes and combinations of morning and evening commutes. The arrow $i \rightarrow i'$ means that traveler type $i \in \{1, 2, 3\}$ in morning becomes traveler type $i' \in \{1, 2, 3\}$ in evening.

### 3.2 Model notations and assumptions

Notations used in this study are summarized as follows, including input sets and parameters in Table 1 and decision variables in Table 2.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}$</td>
<td>Set of nodes, $n \in \mathcal{N}$</td>
</tr>
<tr>
<td>$\mathcal{A}$</td>
<td>Set of (directed) arcs, $a \in \mathcal{A}$</td>
</tr>
<tr>
<td>$\mathcal{K}$</td>
<td>Set of morning origin-destination (OD) pairs; subset of $\mathcal{N} \times \mathcal{N}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Evening return OD-pair corresponding to morning OD-pair $k \in \mathcal{K}$</td>
</tr>
<tr>
<td>$c$</td>
<td>Set of commute cycles associated with OD pair $k$ (and its evening reverse pair $\bar{k}$)</td>
</tr>
<tr>
<td>$p_{am}^c$</td>
<td>The morning path used by commute cycle $c \in C$</td>
</tr>
<tr>
<td>$p_{pm}^c$</td>
<td>The evening path used by commute cycle $c \in C$</td>
</tr>
<tr>
<td>$d_k$</td>
<td>Total person trip demand of OD pair $k$ (morning demand = evening return demand)</td>
</tr>
<tr>
<td>$O_{1;am}^k$</td>
<td>Operation cost of OD pair $k \in \mathcal{K}$ for morning drivers</td>
</tr>
<tr>
<td>$O_{1;pm}^{\bar{k}}$</td>
<td>Operation cost of reverse OD pair $\bar{k}$ for evening drivers</td>
</tr>
<tr>
<td>$I_{i;am}^k$</td>
<td>Unit inconvenience of OD pair $k \in \mathcal{K}$ for morning commuter type $i \in {1, 2, 3}$</td>
</tr>
<tr>
<td>$I_{i;pm}^{\bar{k}}$</td>
<td>Unit inconvenience of reverse OD pair $\bar{k}$ for evening commuter type $i \in {1, 2, 3}$</td>
</tr>
<tr>
<td>$E_{2;am}^k$</td>
<td>Unit payment of OD pair $k \in \mathcal{K}$ for morning rideshare commuter</td>
</tr>
<tr>
<td>$E_{2;pm}^{\bar{k}}$</td>
<td>Minimum unit payment of reverse OD pair $\bar{k}$ for evening rideshare commuter</td>
</tr>
<tr>
<td>$\gamma_{2;am}^k$</td>
<td>Conversion factor of rideshare under-capacity to surcharge over minimum unit payment of OD pair $k \in \mathcal{K}$ for morning rideshare commuter</td>
</tr>
<tr>
<td>$\gamma_{2;pm}^{\bar{k}}$</td>
<td>Conversion factor of rideshare under-capacity to surcharge over minimum unit payment of reverse OD pair $\bar{k}$ for evening rideshare commuter</td>
</tr>
<tr>
<td>$W_{i;am}^k$</td>
<td>Waiting time of OD pair $k \in \mathcal{K}$ for morning commuter type $i \in {2, 3}$</td>
</tr>
<tr>
<td>$W_{i;pm}^{\bar{k}}$</td>
<td>Waiting time of reverse OD pair $\bar{k}$ for evening commuter type $i \in {2, 3}$</td>
</tr>
<tr>
<td>$M$</td>
<td>Capacity in terms of number of rideshare passengers for each vehicle</td>
</tr>
<tr>
<td>$\delta_{a;p}$</td>
<td>Arc-path incidence indicator; $\delta_{a;p} = \begin{cases} 1 &amp; \text{if path } p \text{ uses arc } a \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$t_a(\bullet)$</td>
<td>The Bureau of Public Roads (BPR) travel time function for arc $a \in \mathcal{A}$ as a function of traffic flow</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Conversion factor of time (minutes) to money (dollars)</td>
</tr>
</tbody>
</table>
Table 2. Decision variables

<table>
<thead>
<tr>
<th>Primary:</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^j_c$</td>
<td>Flow of travelers of commute cycle $c \in \mathcal{C}$ of mode combination type $j \in {1, ..., 5}$</td>
</tr>
<tr>
<td>$E^{2;am}_k$</td>
<td>Unit payment of OD pair $k \in \mathcal{K}$ for morning rideshare commuter type</td>
</tr>
<tr>
<td>$E^{2;pm}_k$</td>
<td>Unit payment of the reverse OD pair $\bar{k}$ for evening rideshare commuter type</td>
</tr>
<tr>
<td>$u_k$</td>
<td>(Least) disutility of OD pair $k \in \mathcal{K}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Derived:</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^{am}_a$</td>
<td>Vehicular flow of arc $a \in \mathcal{A}$ in the morning</td>
</tr>
<tr>
<td>$f^{pm}_a$</td>
<td>Vehicular flow of arc $a \in \mathcal{A}$ in the evening</td>
</tr>
<tr>
<td>$d^k_j$</td>
<td>Travel demand of OD pair $k \in \mathcal{K}$ of mode combination type $j \in {1, ..., 5}$ in number of travelers</td>
</tr>
<tr>
<td>$\pi^j_c$</td>
<td>Total cost on commute cycle $c \in \mathcal{C}$ of mode combination type $j \in {1, ..., 5}$</td>
</tr>
<tr>
<td>$\alpha^{am}_k$</td>
<td>Average number of morning rideshare passengers per vehicle of OD pair $k \in \mathcal{K}$</td>
</tr>
<tr>
<td>$\alpha^{pm}_k$</td>
<td>Average number of evening rideshare passengers per vehicle of the reverse OD pair $\bar{k}$</td>
</tr>
<tr>
<td>$c^{am}(f^{am})_a$</td>
<td>Travel cost on arc $a \in \mathcal{A}$ in morning commute as a function of arc flows $f^{am} \triangleq {f^{am}<em>a}</em>{a \in \mathcal{A}}$</td>
</tr>
<tr>
<td>$c^{pm}(f^{pm})_a$</td>
<td>Travel cost on arc $a \in \mathcal{A}$ in evening commute as a function of arc flows $f^{pm} \triangleq {f^{pm}<em>a}</em>{a \in \mathcal{A}}$</td>
</tr>
<tr>
<td>$C^{n;am}_c(h)$</td>
<td>Total cost on morning path $p^{am}<em>c$ used by commute cycle $c \in \mathcal{C}$ and commuter type $i \in {1, ..., 3}$ as a function of commute cycle flow $h \triangleq {(h^j_c)</em>{c \in \mathcal{C}}}^5_{j=1}$</td>
</tr>
<tr>
<td>$C^{n;pm}_c(h)$</td>
<td>Total cost on evening path $p^{pm}_c$ used by commute cycle $c \in \mathcal{C}$ and commuter type $i \in {1, ..., 3}$ as a function of commute cycle flow</td>
</tr>
</tbody>
</table>

Assumptions about the model are as follows:

- There is no distinction between rideshare and solo drivers; each is operating a vehicle.
- All drivers are willing to share vehicles and provide rideshare services.
- There is the same passenger capacity in each vehicle.
- The rideshare capacity constraints are enforced at the aggregate level, not at the vehicle level. That is, for each OD pair, the total number of rideshare passengers over the total number of drivers has to be less than or equal to the vehicle capacity.

Several other remarks about the model:

- All variables are expressed as real numbers; in particular, the travel demands $d^k_j$ are considered as traveler (i.e. people) flows, so are $d^j_c$ and the commute cycle flows $h^j_c$. The am and pm arc flows $f^{am}_a$ and $f^{pm}_a$ are vehicular flows, which are the source of traffic congestion.
- The commute cycle flows $h^j_c$ for $j = 1, 3, 4,$ and 5 contribute to the arc flows (either morning or evening or both), thus to congestion, while the other commute types do not.
- Each ridesourcing vehicle is assumed to pick up only one passenger.
- We postulate a minimum per-passenger rideshare fee which becomes the charged payment if the rideshare vehicle is at capacity. As a result of this postulate, the unit passenger rideshare payment is a decision variable to be determined from the model. The payment can be equal to the set minimum, or a higher value if the rideshare is below capacity. This addition to the minimum payment (if positive) is equal to a multiplicative factor of the under-capacity.
3.3 Connections among variables and functions

(1) Relationships between arc and cycle flows:

Arc flows have 3 components as shown in Fig. 2 and are calculated as the summation of the associated path flows as follows. Here $\delta_{a,p^{m}}$ and $\delta_{a,p^{m}}$ are the arc-path indicators in the morning and in the evening, respectively. Note that the flows of rideshare passengers are not considered here since they have no influence on traffic congestion.

\[ \forall a \in A \begin{cases} f_{a}^{am} = \sum_{c \in C} \left( \sum_{j=1,4,5} h_{c}^{j} \times \delta_{a,p^{m}} \right) \\ f_{a}^{pm} = \sum_{c \in C} \left( \sum_{j=1,3,5} h_{c}^{j} \times \delta_{a,p^{m}} \right) \end{cases} \]

Figure 2. Components of flow between arc $(x, y)$ in network
(the flows of rideshare passengers do not influence traffic congestion)

(2) Arc cost functions:

The notations $b_{a}^{am}$ and $b_{a}^{pm}$ offer the possibility that the arc cost on the same arc $a$ may be different in the morning and evening, respectively. Assuming separable arc costs, we have,

\[ b_{a}^{am}(f_{a}^{am}) = t_{a}(f_{a}^{am}) \quad \forall a \in A \]
\[ b_{a}^{pm}(f_{a}^{pm}) = t_{a}(f_{a}^{pm}) \quad \forall a \in A \]

where $t_{a}(\bullet)$ represents the Bureau of Public Roads (BPR) travel time function for arc $a \in A$ as a function of traffic flow. The cost on arc $a$ is a function of the flow on that arc only. Note that the arc costs in the morning are different from those in the evening due to the difference in flows, even on the same arc.

(3) Cost functions for commuter types:

We assume throughout that the commute cycle costs are additive; that is, each such cycle cost is equal to the sum of the costs on the arcs used by the two morning and evening paths, respectively, in the cycle. Besides the travel cost due to congestion, there are many other components that contribute to the total cost, which differ between the commuter types. For each trip, the total travel cost equals to the summation of travel cost due to congestion and specific costs for the commuter type. The cost structures of different commuter types in our model are as follows. Unlike Xu et al. (2015) which builds the arc costs in a rideshare model, here we directly formulate the commute cycle cost functions.
a) Drivers:

**total cost** = **travel cost** (BPR function) + **operation cost** (different constants between ODs) + **inconvenience of rideshare** (function of number of passengers) - **income from rideshare** (different constants between ODs, times number of passengers)

\[
C_{c}^{1,am}(h) = \psi \times \sum_{a \in A} (\delta_{a,pm} \times b_{a}^{am}(f_{am})) + O_{k}^{1,am} + \alpha_{k}^{2,am} \times \left( I_{k}^{1,am} - E_{k}^{2,am} \right)
\]

\[
C_{c}^{1,pm}(h) = \psi \times \sum_{a \in A} (\delta_{a,pm} \times b_{a}^{pm}(f_{pm})) + O_{k}^{1,pm} + \alpha_{k}^{2,pm} \times \left( I_{k}^{1,pm} - E_{k}^{2,pm} \right)
\]

∀ \( c \in C_k \) ∀ \( k \in K \)

Note that the right-hand sides are functions of the arc flows; yet we write the left-hand sides as functions of the commute cycle flows; this is done with the understanding that once the arc flows in the right-hand sides are substituted by their connections to the commute cycle flows through the expressions (1), the substituted left-hand sides are indeed function of the latter flows. The operation cost includes parking, depreciation, insurance, maintenance and so on, which may differ between OD pairs and in the morning and evening; e.g., there is no parking cost in the evening commute.

The inconvenience of drivers with extra passengers includes anxiety for riding with strangers, detours, etc, which is increased with the number of passengers. Income to a driver with extra passengers equals the total payment by the rideshare passengers in the vehicle. The inconvenience and price of rideshare may differ between OD pairs and could be different between morning and evening. Although the detour is not considered in the model for simplification, it could be viewed as considered here in the inconvenience function: the inconvenience is increased with the number of passengers, which is a relationship between inconvenience, detour distance and number of passengers - the more passengers, the more detour distance, at the same time the more inconvenience.

b) Rideshare passengers:

**total cost** = **travel cost** (BPR function) + **waiting cost for rideshare** (different constants between ODs) + **inconvenience of rideshare** (function of number of passengers) + **payment for rideshare** (different constants between ODs)

\[
C_{c}^{2,am}(h) = \psi \times \sum_{a \in A} (\delta_{a,pm} \times b_{a}^{am}(f_{am})) + W_{k}^{2,am} + \alpha_{k}^{2,am} \times I_{k}^{2,am} + E_{k}^{2,am}
\]

\[
C_{c}^{2,pm}(h) = \psi \times \sum_{a \in A} (\delta_{a,pm} \times b_{a}^{pm}(f_{pm})) + W_{k}^{2,pm} + \alpha_{k}^{2,pm} \times I_{k}^{2,pm} + E_{k}^{2,pm}
\]

∀ \( c \in C_k \) and associated \( \bar{k} \)

Here we treat the waiting cost as different constants between ODs. The inconvenience of rideshare passengers, similar to the inconvenience of drivers, increases with the number of passengers. The total
payment of rideshare passengers is the same as the income received by the drivers with extra passengers.

c) Ridesourcing passengers:

total cost = travel cost \text{(BPR function)} + \text{waiting cost for ridesourcing} \text{ (different constants between ODs)} + \text{inconvenience of ridesourcing} \text{ (different constants between ODs)} + \text{payment for ridesourcing} \text{ (different constants between ODs)}

\[
C^{3;am}_c(h) = \psi \times \sum_{a \in A} (\delta_{a;i;p^{am}} \times b^{am}_a(f^{am}) + \underbrace{W^{3;am}_k + \delta^{3;am}_k + E^{3;am}_k}_{\text{a constant}}) \quad \forall c \in C_k \quad \forall k \in \mathcal{K}
\]

\[
C^{3;pm}_c(h) = \psi \times \sum_{a \in A} (\delta_{a;i;p^{pm}} \times b^{pm}_a(f^{pm}) + \underbrace{W^{3;pm}_k + \delta^{3;pm}_k + E^{3;pm}_k}_{\text{a constant}}) \quad \forall c \in C_k \quad \text{and associated } \bar{k} \quad \text{(5)}
\]

The waiting cost is treated as a constant which differs between ODs. Inconvenience of ridesourcing is also a constant that differs between ODs since we assume no pooling for ridesourcing. Unit payment for ridesourcing passengers should be higher than that of rideshare passengers.

(4) Cost functions for mode combination types:

The total cost of each mode combination and each commute cycle is the summation of the costs in the morning and evening. We have

\[
\pi^j_c = C^{u;am}_c(h) + C^{v;pm}_c(h) \quad \forall c \in C, \quad \forall j \in \{1, \ldots, 5\} \quad \text{with } u = j_{am} \text{ and } v = j_{pm} \quad \text{(6)}
\]

3.4 Flow conservation equations

Demand distribution equations are used to balance total trip demands with commute cycle flows and ensuring morning trip demands equal evening trip demands

- per mode combination type

\[
d^j_k = \sum_{c \in C_k} h^j_c = d^j_k \quad \forall j \in \{1, 2, \ldots, 5\}, \quad \forall k \in \mathcal{K} \quad \text{(7)}
\]

- morning and evening trip demands, aggregated to total trip demands

\[
d_k = \sum_{j=1}^5 d^j_k = \sum_{j=1}^5 \sum_{c \in C_k} h^j_c, \quad \forall k \in \mathcal{K} \quad \text{(8)}
\]

3.5 Rideshare capacity and addition to minimum fare

In this section, we first compute the average number of rideshare passengers in each vehicle for each OD pair and then describe the rideshare payment scheme.

rideshare passenger flow = average number of rideshare passengers per vehicle \times number of vehicles, with the average number slightly adjusted to avoid division by zero

Written as a fraction, the formulas for the (approximate) average number of rideshare passengers per
vehicle are as follows: for a small scalar $\varepsilon > 0$,

$$
\alpha^2_{k} = \frac{d^2_k + d^3_k}{d^2_k + \varepsilon} \quad \text{and} \quad \alpha^2_{k} = \frac{d^2_k + d^4_k}{d^2_k + \varepsilon} \quad k \in \mathcal{K}, \quad \forall \bar{k}
$$

(9)

For the overall equilibrium model, existence of a solution can be established as long as $\alpha^2_{k}$ and $\alpha^2_{k}$ are continuous functions of the commute combination demands $d^2_k$, thus of the commute cycle flows $h^2_k$. The above are examples of such functions.

Based on the pair of averages $(\alpha^2_{k}, \alpha^2_{k})$ we propose a corresponding pair additions to the minimum rideshare fares. The averages are decision variables subject to the upper bound $M$. The under-capacity will translate into added payment to the minimum rideshare fee. We model this preliminary consideration by the following constraints:

$$
0 \leq \gamma^2_{k}(M - \alpha^2_{k}) - (E^2_{k} - E^2_{k}) \quad \perp \quad E^2_{k} - E^2_{k} \geq 0 \quad \forall k \in \mathcal{K} \\
0 \leq \gamma^2_{k}(M - \alpha^2_{k}) - (E^2_{k} - E^2_{k}) \quad \perp \quad E^2_{k} - E^2_{k} \geq 0 \quad \forall \bar{k},
$$

(10)

where $\perp$ is the perpendicularity notation, which in this context has several consequences for the morning and evening ridesharing; we describe only the morning ones:

- if $\alpha^2_{k} = M$ (i.e., if rideshare is at capacity), then $E^2_{k} = E^2_{k}$ (i.e., the payment is at its minimum);
- if $E^2_{k} > E^2_{k}$ (i.e., if the payment exceeds its minimum), then $\alpha^2_{k} < M$ (i.e., rideshare must be below capacity) and, more importantly, $E^2_{k} - E^2_{k} = \gamma^2_{k}(M - \alpha^2_{k})$ (i.e., the addition to the minimum payment is a constant factor of the under-capacity);
- in the other two cases, that is, if $E^2_{k} = E^2_{k}$, then rideshare may or may not be at capacity; similarly, if $\alpha^2_{k} < M$, then rideshare payment may be equal to or exceed the minimum.

An additional consequence of the above complementarity conditions is that the addition to the minimum payment is bounded above by a constant multiplicative factor of the rideshare under-capacity so that the total payment will not be unreasonably high. Unfortunately, given $\alpha^2_{k}$ and $\alpha^2_{k}$, the complementarity conditions (10) do not determine the additional payments over the minimum payments uniquely. To achieve this uniqueness, we postulate the following models to determine the additional payments, given the average $\alpha^2_{k}$ of rideshare passengers,

$$
\text{minimize } E^2_{k} \quad \text{subject to } 0 \leq E^2_{k} - E^2_{k} \leq \gamma^2_{k}(M - \alpha^2_{k}).
$$

(11)

In essence, this yields

$$
E^2_{k} - E^2_{k} = \gamma^2_{k}(M - \alpha^2_{k}), \quad \text{if } \alpha^2_{k} \leq M,
$$

(12)

which essentially fixes the excess payment to be equal to the multiplicative factor $\gamma^2_{k}$ times the under-capacity when the ratio $\alpha^2_{k}$, which is a decision variable of the model, satisfies the upper bound $M$. Since such a bound is guaranteed to be satisfied only through a solution of the model, we need to impose the bound explicitly in defining the model. Thus instead of using (12) directly, we employ the Karush-Kuhn-Tucker optimality conditions of the above simple bounded quadratic program (11) which we write
in the form of the following complementarity conditions, where \( \lambda_{k}^{2;am} \) is a multiplier for the upper bound constraint of \( E_{k}^{2;am} - \bar{E}_{k}^{2;am} \) in this program:

\[
\begin{align*}
0 \leq \gamma_{k}^{2;am} (M - \alpha_{k}^{am}) - (E_{k}^{2;am} - \bar{E}_{k}^{2;am}) & \perp \lambda_{k}^{2;am} \geq 0 \\
0 \leq (E_{k}^{2;am} - \bar{E}_{k}^{2;am}) - \gamma_{k}^{2;am} (M - \alpha_{k}^{am}) + \lambda_{k}^{2;am} \perp E_{k}^{2;am} - \bar{E}_{k}^{2;am} \geq 0,
\end{align*}
\]

where \( \perp \) is the perpendicularity notation, which in this context asserts the complementary slackness condition of the quadratic program (13). From quadratic programming and linear complementarity theory (Cottle et al., 2009), we know that the unique \( E_{k}^{2;am} \) satisfying the above complementarity conditions is a piecewise affine, thus a Lipschitz continuous function of \( \alpha_{k}^{am} \), and thus of the commute cycle flows \( h_{j}^c \). A similar set of pm conditions is as follows:

\[
\begin{align*}
0 \leq \gamma_{k}^{2;pm} (M - \alpha_{k}^{pm}) - (E_{k}^{2;pm} - \bar{E}_{k}^{2;pm}) & \perp \lambda_{k}^{2;pm} \geq 0 \\
0 \leq (E_{k}^{2;pm} - \bar{E}_{k}^{2;pm}) - \gamma_{k}^{2;pm} (M - \alpha_{k}^{pm}) + \lambda_{k}^{2;pm} \perp E_{k}^{2;pm} - \bar{E}_{k}^{2;pm} \geq 0,
\end{align*}
\]

3.6 User equilibrium

We apply the user equilibrium principle that describes a complementary relation between the daily commute flows and the travelers’ minimum disutilities; it is based on the combined morning-evening round trips, allowing switches of commute types. This type of equilibrium distinguishes itself from the separate morning equilibrium and evening equilibrium. Yet the disutilities pertain to each OD pair \( k \) and the flows \( h_{j}^c \) of all the cycles \( c \in C_{k} \) joining that OD pair and across the 3 commute types. That is to say, for each OD pair \( k \), the chosen mode combinations \( c \in C_{k} \) joining this OD pair among the 3 types in Figure 1 will all have travel costs equal to the least disutility of the OD pair in question, and this common cost is the smaller than the travel costs of the unchosen mode combinations joining the same OD pair. This is exactly Wardrop’s user equilibrium principle for the commute cycle flows instead of the path flows in a traditional traffic equilibrium problem. This equilibrating process incorporates the switches of commuter types between morning and evening trips.

Thus the user equilibrium conditions for the combined morning and evening commutes among the 5 mode combination types are:

\[
\begin{align*}
0 \leq h_{j}^c \perp \pi_{j}^c - u_{k} & \geq 0, \quad \forall j \in \{1, \cdots, 5\}; \quad \forall k \in K \text{ and } \forall c \in C_{k},
\end{align*}
\]

where \( \perp \) is the perpendicularity notation, which in this context asserts the complementarity between the commute cycle flows and the travelers’ deviations from the minimum disutilities. In words, if a traveler chooses the combination \( c \in C_{k} \), then the cycle cost/disutility of this combination must be the minimum of all costs for this OD pair \( k \).

3.7 The overall equilibrium model

In this section, we summarize the aforementioned sections and develop a general equilibrium model to capture the complicated interactions between drivers, rideshare passengers and ridesourcing passengers that allows travelers to switch between different transportation modes in a coupled morning-evening commute. The model is formulated as a mixed complementarity problem as follows.

\[
\begin{align*}
0 \leq h_{j}^c \perp \pi_{j}^c - u_{k} & \geq 0, \quad \forall j \in \{1, \cdots, 5\}; \quad \forall k \in K \text{ and } \forall c \in C_{k},
\end{align*}
\]
\[ d_k = \sum_{j=1}^{5} d^j_k = \sum_{j=1}^{5} \sum_{e \in C_k} h^e_j, \quad \forall k \in K \]  

(16)

\[ 0 \leq \gamma^2_{am}(M - \alpha^am_k) - \left( E^2_{am,k} - E^2_{am,k} \right) \quad \perp \quad \lambda^am_k \geq 0 \quad \forall k \in K \]

\[ 0 \leq \left( E^2_{am,k} - E^2_{am,k} \right) - \gamma^2_{am}(M - \alpha^am_k) + \lambda^am_k \quad \perp \quad E^2_{am,k} - E^2_{am,k} \geq 0 \quad \forall k \in K \]

\[ 0 \leq \gamma^pm_{k} (M - \alpha^pm_{k}) - \left( E^pm_{k} - E^pm_{k} \right) \quad \perp \quad \lambda^pm_k \geq 0 \quad \text{associated} \ \forall k \]

\[ 0 \leq \left( E^pm_{k} - E^pm_{k} \right) - \gamma^pm_{k} (M - \alpha^pm_{k}) + \lambda^pm_k \quad \perp \quad E^pm_{k} - E^pm_{k} \geq 0 \quad \text{associated} \ \forall k \]

where constraints (15) are the user equilibrium conditions for the combined morning and evening commutes among the 5 mode combination types; constraints (16) are the flow conservation equations to balance total trip demands with commute cycle flows and ensure morning trip demands equal to evening trip demands; constraints (17) ensure solution uniqueness for \( E^2_{am,k} \) and \( E^2_{pm,k} \) if \( \alpha^am_k \) and \( \alpha^pm_k \) are determined, and satisfy the rideshare capacity, i.e., number of rideshare passengers must be no larger than capacity for each vehicle times the number of drivers. Here \( \lambda^am_k \) and \( \lambda^pm_k \) are multipliers. Actually, constraints (17) are equivalent to constraints (18) as follows:

\[ E^2_{am,k} - E^2_{am,k} = \gamma^2_{am}(M - \alpha^am_k) \quad \text{if} \quad \alpha^am_k \leq M \quad \forall k \in K \]

\[ E^2_{pm,k} - E^2_{pm,k} = \gamma^pm_{k} (M - \alpha^pm_{k}) \quad \text{if} \quad \alpha^pm_k \leq M \quad \text{associated} \ \forall k \]

(18)

4 Model Analysis

In this section, properties of our model are analysed. First, we show that the proposed (mixed) complementarity formulation is equivalent to a variational inequality, and there exists an equilibrium. Then we discuss the properties of the coupled morning-evening commute model when it reaches an equilibrium. Finally, we present the conditions under which the proposed model will have a unique solution.

4.1 Existence of an equilibrium

The primary decision variables of the proposed model are:

- commute cycle flows: \( \{ h^e_j \mid j = 1, \ldots, 5 \} \in E \);
- average number of rideshare passengers: \( \{ \alpha^am_k; \alpha^pm_k \} \in K \);
- unit rideshare payments: \( \{ E^2_{am,k}; E^2_{pm,k} \} \in K \);
- least travel disutilities of OD pairs: \( \{ u_k \} \in K \).

Among the above variables, the basic ones are the commute cycle flows \( h^e_j \) and travel disutilities of the OD pairs. After substituting the flow variables into the expressions of \( \{ \alpha^am_k; \alpha^pm_k \} \in K \), we can in turn substitute the latter variables into the rideshare cost functions \( C^{am}_k(h), C^{pm}_k(h), C^{am}_k(h), \) and \( C^{pm}_k(h) \). The end result is that all the morning and evening commute cycle cost functions \( C^{am}_k \) and \( C^{pm}_k \) for \( i = 1, \ldots, 3 \) can be expressed as continuous functions of the flow variables \( h^e_j \). In summarizing the complementarity conditions below, it is understood that all these
substitutions are made. This results in the following two sets of conditions for our proposed mathematical model for the combined morning and evening commute user equilibrium problem:

- for all $k \in K$, all $c \in C_k$, and all $j = 1, \ldots, 5$ with $u = j_{am}$ and $v = j_{pm}$,

\[
0 \leq h_{c}^{j} \perp C_{c}^{u,am}(h) + C_{c}^{v,pm}(h) - u_k \geq 0, \quad \text{for all } c \in C_k
\]

- denoted $\pi_{j}^{c}(h)$

- for all $k \in K$,

\[
d_k = \sum_{j=1}^{5} \sum_{c \in C_k} h_{c}^{j}.
\]

Considering the variable $u_k$ as the multiplier of the OD-demand balancing constraints, this (mixed) complementarity formulation is equivalent to a variational inequality (VI) (Facchinei and Pang, 2003) defined by the pair of mapping $\Phi$ and polyhedral set $H$ as follows:

\[
\Phi(h) \triangleq (\pi_{j}^{c}(h))_{(c,j) \in C \times \{1, \cdots, 5\}}
\]

\[
H \triangleq \left\{ (h_{c}^{j})_{(c,j) \in C \times \{1, \cdots, 5\}} \geq 0 \mid \sum_{j=1}^{5} \sum_{c \in C_k} h_{c}^{j} = d_k, \quad \forall k \in K \right\}.
\]

Since the mapping $\Phi$ is continuous and the set $H$ is compact and convex, it follows VI($\Phi$, $H$) has a solution, thus so does our combined morning-evening commute model with mode switches.

### 4.2 Model properties under an equilibrium state

Main highlights of the proposed model include: (1) different from the morning commute problem (Xiao et al., 2016; Ma and Zhang, 2017; Liu and Li, 2017; Su and Wang, 2019; Lin et al., 2020), our model handles coupled morning-evening commute; (2) instead of using a discrete choice model (Ben-Akiva and Lerman, 1985; Train, 2009), the number of travelers for each commuter type is derived directly from our model, as the result of the user equilibrium conditions. In this section, we show that our proposed coupled morning-evening commute model could produce morning (evening) route choice equilibrium. Besides, the equilibrium of our model also leads to rational traveler behavior, which means that none of the travel mode combinations with higher cost will be selected.

To prove the aforementioned statements, we first derive an extended network for our problem. For each traveler, the total cost for the coupled morning-evening commute consists of three parts: (1) congestion cost on a selected path (consists of arcs) from home to work place in the morning; (2) specified cost for a selected travel mode combination, including the cost of travel mode from home to work place in the morning and the cost of travel mode from work place to home in the evening; (3) congestion cost on a selected path (consists of arcs) from work place to home in the evening. As shown in Fig. 3, we construct five virtual arcs representing the five travel mode combinations (am driver + pm driver, am rideshare passenger + pm rideshare passenger, am rideshare passenger + pm ridesourcing passenger, am ridesourcing passenger + pm rideshare passenger, am ridesourcing passenger + pm ridesourcing passenger), with the specified cost (travel cost due to congestion excluded) for relevant mode combination type as the cost function. To reach or leave the work place, each traveler must choose one of the five virtual arcs, which means that each traveler needs to choose one mode combination for traveling. With the virtual arcs, now we obtain the extended network for our problem, as shown in Fig. 3.
Figure 3. The extended network for our problem

Next, we redefine the cost functions for mode combination types, \(\pi_j\). Here we define the problem of traveling from \(x_1\) to \(y_1\) in Fig. 3 as the **morning route choice problem**, and the problem of traveling from \(y_2\) to \(x_2\) in Fig. 3 as the **evening route choice problem**. Note that here the morning route choice problem and the evening route choice problem are two sub-problems under our coupled morning-evening model framework, they are not new models with specific formulations.

In the coupled morning-evening commute model, each cycle \(c \in C\) includes two paths: the morning path \(P_{\text{am}}^c\) with path flow \(P_{\text{am}}^c\) and the evening path \(P_{\text{pm}}^c\) with path flow \(P_{\text{pm}}^c\). By definition, the set of commute cycles can be written as the Cartesian product of the set of morning paths and the set of evening paths, namely \(\{c\} = \{P_{\text{am}}^c\} \times \{P_{\text{pm}}^c\}\).

Denote the total cost for the morning route choice problem as \(\theta_{\text{am}}^c\), and the total cost for the evening route choice problem as \(\theta_{\text{pm}}^c\), then we have,

\[
\theta_{\text{am}}^c = \psi \times \sum_{a \in A} (\delta_{a,P_{\text{am}}^c} \times b_{a}^{\text{am}}(f_{\text{am}})) \quad \forall c \in C
\]

(19)

\[
\theta_{\text{pm}}^c = \psi \times \sum_{a \in A} (\delta_{a,P_{\text{pm}}^c} \times b_{a}^{\text{pm}}(f_{\text{pm}})) \quad \forall c \in C
\]

(20)

Denote the specified cost for travel mode combination \(j\) and OD pair \(k\) as \(\eta_j^k\), which is the cost from \(y_1\) to \(y_2\) in Fig. 3. Then the total cost of each mode combination and each commute cycle can be represented as

\[
\pi_j^k = \theta_{\text{am}}^c + \eta_j^k + \theta_{\text{pm}}^c \quad \forall k \in K, \quad \forall c \in C_k, \quad \forall j \in \{1, ..., 5\}
\]

(21)

Section 4.1 guarantees the existence of an equilibrium for the proposed coupled morning-evening commute model. In Theorem 1 below, we show the properties of the proposed model when it reaches an equilibrium.

**Theorem 1.** When the coupled morning-evening commute problem reaches an equilibrium, we have the following properties:

(a) The morning route choice problem reaches an equilibrium;

(b) Given travelers’ mode choice is fixed, the morning route choice problem is equivalent to a traditional traffic equilibrium problem;

(c) The evening route choice problem also has properties (a) and (b);

(d) Travelers are rational to mode choice, which means that no traveler will choose a more expensive travel mode combination. It coincides with the basic assumption of economic consumer theory (Mas-Colell et al., 1995).

Proof.
(a) As described in Section 3.6, the user equilibrium conditions for the coupled morning-evening commute problem are:

\[ 0 \leq h_c^j \perp \pi_c^j - u_k \geq 0, \quad \forall j \in \{1, \ldots, 5\}; \quad \forall k \in \mathcal{K} \text{ and } \forall c \in \mathcal{C}_k, \]

which, by definition, is equivalent to

\[
\begin{cases}
  \text{if } \pi_c^j - u_k = 0, \text{ then } h_c^j \geq 0 \\
  \text{if } \pi_c^j - u_k > 0, \text{ then } h_c^j = 0
\end{cases}
\]

By definition of (least) disutility, we have

\[ u_k = \min_{c \in \mathcal{C}_k, j \in \{1, \ldots, 5\}} \pi_c^j = \min_{c \in \mathcal{C}_k, j \in \{1, \ldots, 5\}} \left( \theta_{p_c^{am}}^m + \eta_k^j + \theta_{p_c^{pm}}^m \right). \]

Since for a fixed OD pair \( k \), \( \eta_k^j \) is independent of cycle \( c \), we have

\[ u_k = \min_{c \in \mathcal{C}_k, j \in \{1, \ldots, 5\}} \left( \theta_{p_c^{am}}^m + \eta_k^j + \theta_{p_c^{pm}}^m \right) = \min_{c \in \mathcal{C}_k} \left( \theta_{p_c^{am}}^m + \theta_{p_c^{pm}}^m \right) + \min_{j \in \{1, \ldots, 5\}} \eta_k^j. \]

By definition of a commuter cycle, it includes all possible combinations of morning paths and evening paths connecting OD pair \( k \), namely \( \{c\} = \{p_c^{am}\} \times \{p_c^{pm}\} \). Thus, we have

\[ u_k = \min_{c \in \mathcal{C}_k} \left( \theta_{p_c^{am}}^m + \theta_{p_c^{pm}}^m \right) + \min_{j \in \{1, \ldots, 5\}} \eta_k^j = \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m + \min_{j \in \{1, \ldots, 5\}} \eta_k^j + \min_{c \in \mathcal{C}_k} \theta_{p_c^{pm}}^m. \]

Then user equilibrium conditions for the coupled morning-evening commute problem can be written as:

\[ 0 \leq h_c^j \perp (\theta_{p_c^{am}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m) + (u_k^j - \min_{j \in \{1, \ldots, 5\}} \eta_k^j) + (\theta_{p_c^{pm}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{pm}}^m) \geq 0, \]

\[ \forall j \in \{1, \ldots, 5\}; \quad \forall k \in \mathcal{K} \text{ and } \forall c \in \mathcal{C}_k, \]

To show the morning route choice problem reaches an equilibrium, we still need to have that \( \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m \) is the (least) disutility of the morning route choice problem, notated as \( u_k^{am} \), namely,

\[
\begin{cases}
  \text{if } \theta_{p_c^{am}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m = 0, \text{ then } h_c^j \geq 0 \\
  \text{if } \theta_{p_c^{am}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m > 0, \text{ then } h_c^j = 0
\end{cases}
\]

Assume that \( \exists \theta_{p_c^{am}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m > 0 \) such that the path flow \( P_{p_c^{am}}^{am} > 0 \), i.e., the relevant cycle flow \( h_c^j > 0 \). Since we have \( \eta_k^j - \min_{j \in \{1, \ldots, 5\}} \eta_k^j \geq 0 \) and \( \theta_{p_c^{pm}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{pm}}^m \geq 0 \), there must exist \( (\theta_{p_c^{am}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m) + (\eta_k^j - \min_{j \in \{1, \ldots, 5\}} \eta_k^j) + (\theta_{p_c^{pm}}^m - \min_{c \in \mathcal{C}_k} \theta_{p_c^{pm}}^m) > 0 \) such that \( h_c^j > 0 \), i.e.,

\[ \pi_c^j - u_k > 0, \text{ then } h_c^j > 0 \]

which means that the coupled morning-evening commute problem also doesn’t reach an equilibrium. Contradiction happens. As a result, we can conclude that \( u_k^{am} = \min_{c \in \mathcal{C}_k} \theta_{p_c^{am}}^m \), namely the morning route choice problem must reach an equilibrium.

(b) From (a) we know that when the coupled morning-evening commute problem reaches an equilibrium, the morning route choice problem also reaches an equilibrium. Denote \( u_k^{am} \) as the (least) disutility of OD pair \( k \in \mathcal{K} \) for the morning route choice problem, the equilibrium of morning route choice problem
can be written as:

\[ 0 \leq P_{pam}^{am} \perp \theta_{pam}^{am} - u_{k}^{am} \geq 0, \quad \forall k \in K \text{ and } \forall c \in C_k, \]

where \( p_{c}^{am} \) is the morning path of cycle \( c \), with relevant path flow \( P_{c}^{am} \) and path cost \( \theta_{pam}^{am} \).

Since travelers’ mode choice is fixed, we have

\[ \sum_{c \in C_k} P_{pam}^{am} = d_{k}^{1} + d_{k}^{4} + d_{k}^{5}, \quad \forall k \in K \]

Since \( \delta_{a,pam} \) is the arc-path incidence matrix in the morning, the morning flow on arc \( a \), notate as \( f_{a}^{am} \), can be calculated as

\[ f_{a}^{am} = \sum_{c \in C} \left( P_{pam}^{am} \times \delta_{a,pam} \right), \quad \forall a \in A \]

Then the path cost \( \theta_{pam}^{am} \) can be written as

\[ \theta_{pam}^{am} = \psi \times \sum_{a \in A} \left( \delta_{a,pam} \times b_{a}^{am}(f_{a}^{am}) \right) \quad \forall c \in C \]

Actually, the formulation above is the same as the standard traffic equilibrium, which has unique arc flows.

(c) Similar to the proof in (a) and (b), we can prove the same properties for the evening route choice problem.

(d) Similar to the proof in (a), we can prove that the sub-problem from \( y_1 \) to \( y_2 \) in Fig. 3 also reaches an equilibrium. Since \( d_{k}^{j} \) is the travel demand of OD pair \( k \in K \) of mode combination type \( j \in \{1, \ldots, 5\} \), and \( \bar{u}_{k} \) represents the smallest cost of OD pair \( k \in K \) and travel mode combination \( j \in \{1, \ldots, 5\} \), by definition of equilibrium, we have,

\[
\begin{array}{ll}
\text{if} \quad \eta_{k}^{j} - \bar{u}_{k} = 0, \quad \text{then} \quad d_{k}^{j} \geq 0 \\
\text{if} \quad \eta_{k}^{j} - \bar{u}_{k} > 0, \quad \text{then} \quad d_{k}^{j} = 0 \quad \text{(also, the relevant } \quad h_{k}^{j} = 0) \\
\end{array}
\]

which means that travelers will not choose a travel mode combination that is more expensive. Thus, they are rational to mode choice.

4.3 Uniqueness of the equilibrium

In Section 4.2, we proved the properties of the coupled morning-evening commute problem when it reaches an equilibrium. In this section, we provide the conditions under which there will be a unique solution.

Let’s define the specified cost functions for commuter types (driver, rideshare passengers, and ridesourcing passengers). Since the congestion cost will not influence travelers’ mode choice, here we define the specified cost functions for commuter types as the cost for choosing a travel mode, which means that the congestion cost is excluded. Denote the specified cost function for commuter type \( i \) and OD pair \( k \) as \( \bar{C}_{i,k}^{am} \) in the morning and \( \bar{C}_{i,k}^{pm} \) in the evening, from Section 3.3 and Section 3.5, we have the specified cost functions for the three commuter types as follows,

a) Drivers:
In Theorem 3, we derive conditions under which the model will have unique α parameters, and all five mode combinations described in Fig. 1 could exist in the system. Although we cannot show the uniqueness of f_a, f_p, and d_k under these conditions.

Next we derive the conditions under which the solution will be unique. In Theorem 2, we generate the conditions that α parameters are uniquely determined by closed-form representations of the given parameters, and all five mode combinations described in Fig. 1 could exist in the system. Although we can prove that α and α are unique, we cannot show the uniqueness of f_a, f_p, and d_k under these conditions. In Theorem 3, we derive conditions under which the model will have unique α, α, f_a, f_p, and d_k, and four mode combinations described in Fig. 1 could exist in the system. No travelers will choose to use morning ridesourcing since it is too expensive.

**Theorem 2.** Given α > and α > , if every OD pair k ∈ K and associated k satisfy the following conditions:
(a) $\bar{\alpha}^{2;am}_k \in \left[ \bar{C}^{2;am}_k(0), \bar{C}^{2;am}_k(M) \right]$

(b) $\bar{\alpha}^{2;pm}_k \in \left[ \bar{C}^{2;pm}_k(0), \bar{C}^{2;pm}_k(M) \right]$

(c) $\bar{\alpha}^{2;am}_k(\alpha^{2;am}_k) + \bar{\alpha}^{2;pm}_k(\alpha^{2;pm}_k) = \bar{C}^{2;am}_k + \bar{C}^{2;pm}_k

Then the coupled morning-evening commute problem will derive unique $\alpha^{2;am}_k$ and $\alpha^{2;pm}_k$ as follows

\[
\alpha^{2;am}_k = \frac{W^{3;am}_k + I^{3;am}_k + E^{3;am}_k - W^{2;am}_k - E^{2;am}_k - M \times \gamma^{2;am}_k}{I^{2;am}_k - \gamma^{2;am}_k}
\]

\[
\alpha^{2;pm}_k = \frac{W^{3;pm}_k + I^{3;pm}_k + E^{3;pm}_k - W^{2;pm}_k - E^{2;pm}_k - M \times \gamma^{2;pm}_k}{I^{2;pm}_k - \gamma^{2;pm}_k}
\]

Proof.

From Theorem 1(d) we know that when the coupled morning-evening commute problem reaches an equilibrium, the sub-region from $y_1$ to $y_2$ also reaches an equilibrium, namely,

\[
\begin{cases}
  \text{if } \eta^1_k - \bar{u}_k = 0, \text{ then } d^1_k \geq 0 \\
  \text{if } \eta^2_k - \bar{u}_k > 0, \text{ then } d^2_k = 0 \text{ (also, the relevant } b^j_k = 0)
\end{cases}
\]

Let’s consider $\bar{\eta}^2_k - \eta^2_k = (\bar{C}^{3;am}_k + \bar{C}^{3;pm}_k(\alpha^{2;pm}_k)) - (\bar{C}^{3;am}_k(\alpha^{2;am}_k) + \bar{C}^{3;pm}_k(\alpha^{2;pm}_k)) = \bar{C}^{3;am}_k - \bar{C}^{2;am}_k(\alpha^{2;am}_k)$. Since $\bar{I}^{2;am}_k \gamma^{2;am}_k$, $\bar{\eta}^2_k - \eta^2_k$ is a linear decreasing function in terms of $\alpha^{2;am}_k \in [0, M]$.

Thus, from (a) we have

\[
\alpha^{2;am}_k = M \Rightarrow (\eta^1_k - \eta^2_k)_{\min} = \bar{C}^{3;am}_k - \bar{C}^{2;am}_k(M) \leq 0
\]

\[
\alpha^{2;am}_k = 0 \Rightarrow (\eta^1_k - \eta^2_k)_{\max} = \bar{C}^{3;am}_k - \bar{C}^{2;am}_k(0) \geq 0
\]

We can conclude that there must exist a unique $\alpha^{2;am}_k \in [0, M]$ such that $\bar{\eta}^2_k = \eta^2_k$ and at the same time $\bar{C}^{2;am}_k(\alpha^{2;am}_k) = \bar{C}^{3;am}_k$. Furthermore, from $\bar{C}^{2;am}_k(\alpha^{2;am}_k) = \bar{C}^{3;am}_k$ we have

\[
W^{3;am}_k + I^{3;am}_k + E^{3;am}_k = W^{2;am}_k + E^{2;am}_k + M \times \gamma^{2;am}_k + (\bar{I}^{2;am}_k - \bar{\gamma}^{2;am}_k) \times \alpha^{2;am}_k
\]

namely

\[
\alpha^{2;am}_k = \frac{W^{3;am}_k + I^{3;am}_k + E^{3;am}_k - W^{2;am}_k - E^{2;am}_k - M \times \gamma^{2;am}_k}{I^{2;am}_k - \gamma^{2;am}_k} \in [0, M]
\]

Similarly, based on $\bar{I}^{2;pm}_k \gamma^{2;pm}_k$ and (b), we analyse $\bar{\eta}^3_k - \eta^3_k$ then we can conclude that there exists a unique $\alpha^{2;pm}_k \in [0, M]$ such that $\bar{\eta}^3_k = \eta^3_k$, and at the same time $\bar{C}^{2;pm}_k(\alpha^{2;pm}_k) = \bar{C}^{3;pm}_k$. Furthermore, from $\bar{C}^{2;pm}_k(\alpha^{2;pm}_k) = \bar{C}^{3;pm}_k$ we have

\[
\alpha^{2;pm}_k = \frac{W^{3;pm}_k + I^{3;pm}_k + E^{3;pm}_k - W^{2;pm}_k - E^{2;pm}_k - M \times \gamma^{2;pm}_k}{I^{2;pm}_k - \gamma^{2;pm}_k} \in [0, M]
\]

From the analysis above we have $\bar{C}^{2;am}_k(\alpha^{2;am}_k) = \bar{C}^{3;am}_k$ and $\bar{C}^{2;pm}_k(\alpha^{2;pm}_k) = \bar{C}^{3;pm}_k$. Add them up we have $\bar{C}^{2;am}_k(\alpha^{2;am}_k) + \bar{C}^{2;pm}_k(\alpha^{2;pm}_k) = \bar{C}^{3;am}_k + \bar{C}^{3;pm}_k$, which means that $\eta^5_k = \eta^5_k$.

Now we have $\eta^2_k = \eta^3_k = \eta^4_k = \eta^5_k$. Together with (c), we can conclude that $\eta^6_k = \eta^7_k = \eta^5_k$, which means that all five mode combinations exist in the system. □
Remark:

(1) Condition (a) means that the basic morning rideshare passengers’ cost is no larger than the morning ridesharing passengers’ cost, and the morning ridesharing passengers’ cost is no larger than the morning rideshare passengers’ cost when the vehicle is at capacity. Condition (b) can be explained similarly;

(2) Condition (c) means that the cost of mode combination 1 (am driver + pm driver) is equal to the cost of mode combination 6 (am ridesourcing + pm ridesourcing). It is used to guarantee that some travelers will choose to be drivers. If there is no drivers, there will also be no rideshare passengers. Thus, mode combinations 2 (am rideshare + pm rideshare), 3 (am rideshare + pm ridesourcing), 4 (am ridesourcing + pm rideshare) also cannot exist in the system.

(3) The assumption \( I^2_{k,am} > \gamma^2_{k,am} \) and \( I^2_{k,pm} > \gamma^2_{k,pm} \) means that the rideshare passenger inconvenience is larger than the conversion factor of rideshare under-capacity to surcharge both in the morning and in the evening. This situation happens when the inconvenience of rideshare is higher for the passengers than the payment for rideshare. Similar conditions under which the coupled morning-evening commute problem will have unique \( \alpha^2_{k,am} \) and \( \alpha^2_{k,pm} \) can be derived when \( I^2_{k,am} \leq \gamma^2_{k,am} \) or/and \( I^2_{k,pm} \leq \gamma^2_{k,pm} \).

Theorem 3. Given \( I^2_{k,pm} > \gamma^2_{k,pm} \), if every OD pair \( k \in K \) and associated \( \bar{k} \) satisfy condition (a) and one of conditions (b) and (c):

\[
\begin{align*}
(a) & \quad C^3_{k,pm} \in \left[ C^2_{k,pm}(0), C^2_{k,pm}(M) \right] \\
(b) & \quad (M - 1) \times \eta^2_{k,am} + E^2_{k,am} + I^2_{k,pm} \geq I^1_{k,am} \\
(c) & \quad (M - 1) \times \eta^2_{k,am} + E^2_{k,am} + I^2_{k,pm} \leq I^1_{k,am} \\
& \quad C^1_{k,am}(0) + C^1_{k,pm}(\alpha^2_{k,am}) > C^2_{k,am}(0) + C^2_{k,pm}(\alpha^2_{k,pm}) \\
& \quad C^1_{k,am}(M) + C^1_{k,pm}(\alpha^2_{k,pm}) < C^2_{k,am}(M) + C^2_{k,pm}(\alpha^2_{k,pm})
\end{align*}
\]

Then the coupled morning-evening commute problem will derive unique \( \alpha^2_{k,am} \) and \( \alpha^2_{k,pm} \), where

\[
\alpha^2_{k,pm} = \frac{W^3_{k,pm} + I^3_{k,pm} + E^3_{k,pm} - W^2_{k,pm} - E^2_{k,pm} - M \times \gamma^2_{k,pm}}{I^2_{k,pm} - \gamma^2_{k,pm}}
\]

Moreover, if we have \( C^3_{k,am} > C^2_{k,am}(\alpha^2_{k,am}) \), then \( f^a_{k,am} \), \( f^p_{k,pm} \), \( d^2_k \) will also be unique.

Proof.

From Theorem 1(d) we know that when the coupled morning-evening commute problem reaches an equilibrium, the sub-region from \( y_1 \) to \( y_2 \) also reaches an equilibrium, namely,

\[
\begin{align*}
\text{if} & \quad \eta^1_k - \bar{u}_k = 0, \quad \text{then} \quad d^1_k \geq 0 \\
\text{if} & \quad \eta^1_k - \bar{u}_k > 0, \quad \text{then} \quad d^1_k = 0 \quad \text{(also, the relevant} \quad h^2_k = 0)
\end{align*}
\]

Let’s consider \( \eta^1_k - \eta^2_k = (C^2_{k,am}(\alpha^2_{k,am}) + C^3_{k,pm}) - (C^2_{k,am}(\alpha^2_{k,am}) + C^2_{k,pm}(\alpha^2_{k,pm})) - C^3_{k,pm} - C^2_{k,pm}(\alpha^2_{k,pm}) \). Since \( \gamma^2_{k,pm} > \gamma^2_{k,pm} \), \( \eta^3_k - \eta^2_k \) is a linear decreasing function in terms of \( \alpha^2_{k,pm} \in [0, M] \). Thus, from (a) we have

\[
\begin{align*}
\alpha^2_{k,pm} = M \Rightarrow \quad (\eta^3_k - \eta^2_k)_{\text{min}} = C^3_{k,pm} - C^2_{k,pm}(M) \leq 0 \\
\alpha^2_{k,pm} = 0 \Rightarrow \quad (\eta^3_k - \eta^2_k)_{\text{max}} = C^3_{k,pm} - C^2_{k,pm}(0) \geq 0
\end{align*}
\]
We can conclude that there exists a unique $\alpha_k^{2,pm} \in [0, M]$ such that $\eta_k^1 = \eta_k^2$, and at the same time $\bar{C}_k^{2,pm}(\alpha_k^{2,pm}) = \bar{C}_k^{3,pm}$. Furthermore, from $\bar{C}_k^{2,pm}(\alpha_k^{2,pm}) = \bar{C}_k^{3,pm}$ we have

$$W_k^{3,pm} + I_k^{3,pm} + E_k^{3,pm} = W_k^{2,pm} + I_k^{2,pm} + M \times \gamma_k^{2,pm} + (\gamma_k^{2,pm} - \gamma_k^{2,pm}) \times \alpha_k^{2,pm}$$

which means that

$$\alpha_k^{2,pm} = \frac{W_k^{3,pm} + I_k^{3,pm} + E_k^{3,pm} - W_k^{2,pm} - I_k^{2,pm} - M \times \gamma_k^{2,pm}}{\gamma_k^{2,pm} - \gamma_k^{2,pm}} \in [0, M]$$

With $\alpha_k^{2,pm}$ fixed, $\bar{C}_k^{2,pm}(\alpha_k^{2,pm})$ is also fixed as a function of $\alpha_k^{2,pm}$. Let’s define a quadratic function $g_k^{am}(\alpha_k^{2,am})$ for each OD pair $k \in K$ as follows:

$$g_k^{am}(\alpha_k^{2,am}) = \eta_k^1 - \eta_k^2 = \bar{C}_k^{1,am}(\alpha_k^{2,am}) + \bar{C}_k^{1,pm}(\alpha_k^{2,pm}) - \bar{C}_k^{2,am}(\alpha_k^{2,am}) - \bar{C}_k^{2,pm}(\alpha_k^{2,pm}) = \gamma_k^{2,am} \left( \frac{\alpha_k^{2,am}}{\gamma_k^{2,am}} \right)^2 + \left( \frac{1}{\gamma_k^{2,am}} - \frac{1}{\gamma_k^{2,pm}} - M \gamma_k^{2,am} + \gamma_k^{2,am} \right) \alpha_k^{2,am} + \Omega_k^{2,am} - \frac{\Omega_k^{2,am} - \Omega_k^{2,pm} + \bar{C}_k^{1,pm}(\alpha_k^{2,pm}) - \bar{C}_k^{2,pm}(\alpha_k^{2,pm})}{4 \gamma_k^{2,am}^2} \alpha_k^{2,am}$$

$$\forall k \in K \text{ and associated } \bar{k}$$

From (b) we have

$$\begin{cases} 
M \gamma_k^{2,am} + \bar{E}_k^{2,am} - 2 \gamma_k^{2,am} - \gamma_k^{2,am} & \geq 0 \\
g_k^{am}(0) > 0 \\
g_k^{am}(M) < 0 
\end{cases}$$

From (c) we have

$$\begin{cases} 
\frac{M \gamma_k^{2,am} + \bar{E}_k^{2,am} - 2 \gamma_k^{2,am} - \gamma_k^{2,am}}{\gamma_k^{2,am}} & \leq M \\
g_k^{am}(0) < 0 \\
g_k^{am}(M) > 0 
\end{cases}$$

With any one of the conditions above, we can conclude that the quadratic equation $g_k^{am}(\alpha_k^{2,am}) = 0$ will have only one solution within $[0, M]$, which satisfies $\eta_k^1 = \eta_k^2$.

Similarly, consider $\eta_k^4 - \eta_k^2 = (\bar{C}_k^{4,am} + \bar{C}_k^{2,pm}(\alpha_k^{2,pm})) - (\bar{C}_k^{2,am}(\alpha_k^{2,am}) + \bar{C}_k^{2,pm}(\alpha_k^{2,pm})) = \bar{C}_k^{3,am} - \bar{C}_k^{3,am}(\alpha_k^{2,am})$. If $\bar{C}_k^{3,am} > \bar{C}_k^{2,am}(\alpha_k^{2,am})$, we have,

$$\eta_k^4 - \eta_k^2 = \bar{C}_k^{3,am} - \bar{C}_k^{2,am}(\alpha_k^{2,am}) > 0 \ \forall \alpha_k^{2,am} \in [0, M]$$

Since we have $\bar{C}_k^{3,pm} = \bar{C}_k^{2,pm}(\alpha_k^{2,pm})$ and $\bar{C}_k^{3,am} > \bar{C}_k^{2,am}(\alpha_k^{2,am})$, add them together we obtain that $\bar{C}_k^{3,am} + \bar{C}_k^{3,pm} > \bar{C}_k^{2,am}(\alpha_k^{2,am}) + \bar{C}_k^{2,pm}(\alpha_k^{2,pm})$, namely $\eta_k^5 > \eta_k^2$.

From the analysis above, we have that $\eta_k^1 = \eta_k^2 = \eta_k^3 < \min(\eta_k^4, \eta_k^5)$, and the equations $\eta_k^1 = \eta_k^2 = \eta_k^3$
have unique solution for $\alpha_k^{2, pm}$ and $\alpha_k^{2, am}$. From Theorem 1(d) we have that $d_k^4 = d_k^5 = 0$, which means that no one will choose be ridesourcing passengers in the morning. Solve the following equation system combining Equation (8) in Section 3.4 and Equation (9) in Section 3.5:

$$
\begin{align*}
\begin{cases}
  d_k &= \sum_{j=1}^{5} d_k^j \\
  \alpha_k^{am} &= \frac{d_k^2 + d_k^3}{d_k + \epsilon} \\
  \alpha_k^{pm} &= \frac{d_k^2 + d_k^4}{d_k^1 + \epsilon}
\end{cases}
\end{align*}
$$

we have,

$$
\begin{align*}
  d_k^1 &= \frac{d_k - \alpha_k^{am} \epsilon}{1 + \alpha_k^{am}} \geq 0 \\
  d_k^2 &= \frac{\alpha_k^{pm} (d_k + \epsilon)}{1 + \alpha_k^{am}} \geq 0 \\
  d_k^3 &= \frac{(\alpha_k^{am} - \alpha_k^{pm})(d_k + \epsilon)}{1 + \alpha_k^{am}} \geq 0
\end{align*}
$$

Since $\alpha_k^{2, pm}$ and $\alpha_k^{2, am}$ are unique, here $d_k^1, d_k^2, d_k^3, d_k^4$ and $d_k^5$ are all fixed, which means that travelers’ mode choice is fixed. From Theorem 1(b) and Theorem 1(c) we know that the morning route choice problem and the evening route choice problem are both equivalent to a traditional traffic equilibrium problem. As a result, the morning arc flows $f_{am}^a$ and the evening arc flows $f_{pm}^a$ are unique. □

Remark:

(1) Condition (a) means that the evening ridesourcing passengers’ cost is no smaller than the basic evening rideshare passengers’ cost, and is no larger than the evening rideshare passengers’ cost when all evening rideshare vehicles are at capacity;

(2) Condition (b) indicates that: i) when there is one passenger in each morning rideshare vehicle on average, namely $\alpha_k^{2, am} = 1$, the summation of morning rideshare price and morning rideshare passengers’ inconvenience cost is bigger than the morning (rideshare) drivers’ inconvenience cost; ii) given $\alpha_k^{2, pm}$, when $\alpha_k^{2, am} = 0$, the cost of mode combination 1 (am driver + pm driver) is larger than the cost of mode combination 2 (am rideshare + pm rideshare); iii) given $\alpha_k^{2, pm}$, when $\alpha_k^{2, am} = M$, the cost of mode combination 1 (am driver + pm driver) is smaller than that of mode combination 2 (am rideshare + pm rideshare). Condition (c) can be explained similarly;

(3) The assumption $I_k^{2, pm} > \gamma_k^{2, pm}$ means that the rideshare passenger inconvenience is larger than the conversion factor of rideshare under-capacity to surcharge in the evening. Similar conditions under which the coupled morning-evening commute model will have unique $\alpha_k^{2, am}, \alpha_k^{2, pm}, f_{am}^a, f_{pm}^a$ and $d_k^1$ can be derived when $I_k^{2, pm} \leq \gamma_k^{2, pm}$;

(4) If $\tilde{C}_k^{3, am} > \tilde{C}_k^{2, am}(\alpha_k^{2, am})$, no travelers will choose to be ridesourcing passengers in the morning since it is too expensive. Thus, mode combinations 4 (am ridesourcing + pm rideshare) and 5 (am ridesourcing + pm ridesourcing) will not be used by the travelers, namely $d_k^4 = d_k^5 = 0$. The travel demands of mode combinations 1 (am driver + pm driver), 2 (am rideshare + pm rideshare), 3 (am rideshare + pm ridesourcing), namely $d_k^1, d_k^2, d_k^3$, can be represented using $\alpha_k^{2, am}$ and $\alpha_k^{2, pm}$ as in the proof.

Under different parameter settings, there could be more scenarios for which mode combinations will exist in the system. Similar to Theorem 2 and Theorem 3, we can also derive the conditions under which the coupled morning-evening commute problem will have a unique solution.
5 Computational Results

In this section, we present computational experiments illustrating the benefits of having a model that combines the morning and evening commute simultaneously. We performed the sensitivity analysis of three parameters: (1) $\{I_{k}^{2;am}, I_{k}^{2;pm}\}_{k \in K}$: the inconvenience for each rideshare passenger; (2) $\{\gamma_{k}^{2;am}, \gamma_{k}^{2;pm}\}_{k \in K}$: the conversion factors of rideshare under-capacity to surcharge; (3) $\{E_{k}^{3;am}, E_{k}^{3;pm}\}_{k \in K}$: unit payments for each ridesourcing passenger. In addition, the outputs of the coupled model and the decoupled morning (or evening) commute model are compared. Formulated as a (mixed) complementarity problem, the proposed model is solved using the PATH solver (Ferris and Munson, 1998) coded in AMPL (Fourer et al., 2003).

The well-studied Sioux-Falls network is used to test the model and the solution approach. We follow the settings used in Ben (2020), including the geometry, travel demand for each OD pair, and parameters of the BPR function for each arc. We selected five nodes (1, 2, 4, 7, 9) as origins and another five nodes (13, 19, 20, 23, 24) as destinations. To increase the congestion level of the network, we use ten times the travel demands of Ben (2020). Furthermore, we set the travel demand to be a small value (i.e., ten) if it is zero in order to keep the complementarity problem square. Similar to Ban et al. (2019), for each OD pair, twenty shortest paths (in terms of free flow travel time) are selected for the analysis. The parameters of the travel modes are determined based on the following guidelines: (1) unit inconvenience for rideshare passengers is not less than unit inconvenience for ridesourcing passengers; (2) waiting time for rideshare passengers is not less than that for ridesourcing passengers. Also, the parameters are chosen to satisfy Theorem 2 or Theorem 3, in order to ensure a unique solution. The parameters in the base case can be found in Table 3 in the Appendix. The conversion factor of time to money, $\psi$, is set to be one dollar per minute.

5.1 Sensitivity analysis

The results of rideshare prices and travelers’ mode choices when changing $I^{2;pm}$ are shown in Table 4 and Fig. 4, respectively. As demonstrated in Fig. 4(b), when we increase $I^{2;pm}$, and at the same time keep all the rest of the parameters fixed to the values given in Table 4, rideshare passengers switch to ridesourcing passengers during the evening commute. At the same time, the number of drivers almost remains the same. This means that there are fewer rideshare passengers in each vehicle, which leads to higher payment for each passenger, as can be seen in Table 4.

<table>
<thead>
<tr>
<th>$I^{2;pm}$</th>
<th>2.3</th>
<th>2.4</th>
<th>2.5</th>
<th>2.6</th>
<th>2.9</th>
<th>3.2</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rideshare Price</td>
<td>0.500</td>
<td>0.509</td>
<td>0.522</td>
<td>0.533</td>
<td>0.563</td>
<td>0.587</td>
<td>0.606</td>
</tr>
</tbody>
</table>
The sensitivity analysis of $\gamma_{2;pm}$ is shown in Table 5 and Fig. 5. As described in Section 3.5, $\gamma_{2;am}$ and $\gamma_{2;pm}$ are actually related to the upper bound of the rideshare prices in the morning and evening, respectively. That is, the larger $\gamma_{2;am}$ or $\gamma_{2;pm}$ value gives a higher upper bound on the morning or evening rideshare price. This explains the results illustrated in Table 5, when $\gamma_{2;pm}$ is increased, the rideshare price becomes higher, due to the higher upper bound of the rideshare price. The results of the travelers’ mode choices when changing $\gamma_{2;pm}$ can be seen in Fig. 5. Fig. 5(b) shows that when we increase $\gamma_{2;pm}$ (the evening rideshare price is increased at the same time), the number of rideshare passengers continues to decrease in the evening, switching to first drivers then ridesourcing passengers. When $\gamma_{2;pm}$ is larger than 0.7, the number of rideshare passengers starts to decrease rapidly. Most of them switch to ridesourcing passengers, while the number of drivers slightly decreases since the market for rideshare decreases significantly and does not need so many drivers. In addition, we can see a coupled morning-evening effect in Fig. 5(a). Since the number of drivers should be equal in the morning and in the evening, the change of morning drivers coincides with that of evening drivers: it first increases then decreases. As a result, the number of morning rideshare passengers first decreases then increases, due to the conservation of total flows for morning trip demands and evening trip demands.

Table 5. Results of rideshare prices when changing $\gamma_{2;pm}$.

<table>
<thead>
<tr>
<th>$\gamma_{2;pm}$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.75</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rideshare Price</td>
<td>0.500</td>
<td>0.801</td>
<td>1.132</td>
<td>1.488</td>
<td>1.867</td>
<td>2.275</td>
<td>2.516</td>
<td>2.773</td>
<td>3.343</td>
</tr>
</tbody>
</table>
The results of how rideshare prices and travelers’ mode choices change when changing $E_{3;pm}$ are shown in Table 6 and Fig. 6, respectively. As illustrated in Fig. 6(b), when $E_{3;pm}$ becomes higher, ridesourcing passengers switch to drivers and even much more rideshare passengers. Consequently, there are more rideshare passengers in each vehicle, which leads to a lower rideshare price for each passenger, as can be seen in Table 6.

**Table 6. Results of rideshare prices when changing $E_{3;pm}$.**

<table>
<thead>
<tr>
<th>$E_{3;pm}$</th>
<th>4.2</th>
<th>4.3</th>
<th>4.4</th>
<th>4.5</th>
<th>4.6</th>
<th>4.7</th>
<th>4.8</th>
<th>4.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rideshare Price</td>
<td>0.606</td>
<td>0.600</td>
<td>0.594</td>
<td>0.588</td>
<td>0.582</td>
<td>0.576</td>
<td>0.573</td>
<td>0.573</td>
</tr>
</tbody>
</table>

Figure 5. Results of travelers’ mode choices when changing $\gamma_{2;pm}$.

Figure 6. Results of travelers’ mode choices when changing $E_{3;pm}$.
5.2 Comparison with a decoupled modeling approach

For a set of OD demands, we developed a coupled morning-evening commute model (referred as the coupled model hereinafter for short) to simultaneously determine the route and mode choice for the am commute trip and the route and mode choice for the pm commute trip constraining that a traveler if chooses to be a driver he/she must be a driver both in the am and pm. In this section, we compare the equilibrium solution from the coupled model with a decoupled modelling approach.

Let the decoupled morning model be one that solves both the route and mode choice for the am given a set of OD demands and the decoupled evening model be one that solves both the route and mode choice for the pm given a set of OD demands. Identifying an equilibrium solution to these two problems separately and then combining them together would most likely violate the constraint that a traveler if chooses to be a driver he/she must be a driver both in the am and pm. Thus, the two decoupled models must be linked in some manner.

Before presenting our approach for linking the models, we present two other models. Let the constrained decoupled morning model be one that solves the route choice and partial mode choice (rideshare, and e-hailing passengers) for the am given a set of demands and set of drivers for each OD pair and the constrained decoupled evening model be one that solves the route choice and partial mode choice (rideshare, and e-hailing passengers) for the pm given a set of demands and drivers for each OD pair.

In this paper, for a decoupled approach we assume a traveler uses one of the decoupled models to determine whether they become a driver or not. Based on this assumption, we present two decoupled solution approaches.

Solution D1

Step 1: Identify an equilibrium solution to the decoupled morning model to determine for each OD pair the set of routes and mode choice (drivers, rideshare passengers, and e-hailing passengers) for the am only.

Step 2: Given the set of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the constrained evening model to determine for each OD the set of routes and mode choice (rideshare passengers, and e-hailing passengers) for the pm only.

Solution D2

Step 1: Identify an equilibrium solution to the decoupled evening model to determine for each OD pair the set of routes and mode choice (drivers, rideshare passengers, and e-hailing passengers) for the pm only.

Step 2: Given the set of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the constrained morning model to determine for each OD the set of routes and mode choice (rideshare passengers, and e-hailing passengers) for the am only.

We next compare the equilibrium solution from the coupled model against the decoupled model. The decoupled model is based on solution D2, and similar results can be found for solution D1. The main quantities for comparison include Vehicle Miles Traveled (VMT), and number of each type of travelers.

We use the same parameters as those in Table 3, except that $I_{3}^{pm}$ is 3.5 dollars. Recall $I_{3}^{am}$ is 2.3, the rideshare inconvenience cost is higher during the evening commute than in the morning commute.
We assume that individuals will use the higher cost parameters to determine their mode choice in a
decoupled model. Thus, in the decoupled model, since the rideshare inconvenience cost is higher in the
evening and all other parameters are the same, an individual will determine whether or not to be a driver
using the evening parameter settings.

The comparison between the two models is shown in Table 7. The decoupled model overestimates the
number of drivers by 25.7% and the VMT by 8.7% compared with the coupled model because the coupled
model is capable of capturing the mode switches between morning and evening, which leads to fewer
drivers and less VMT in the system. In this case, 31.5% of the morning rideshare passengers switch to
ridesourcing service in the evening because of the higher inconvenience cost of ridesharing during the
evening commute, due to, for example, some individuals needing to pick up their children at their after
school activities, making the use of rideshare service during the evening less convenient. The decoupled
model cannot capture this effect and most likely will predict that the traveler will drive to work, thus
causing the overestimation of the number of drivers under this situation.

<table>
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6 Conclusions and Future Research

In this study, we include both ridesourcing and ridesharing as travel modes and integrate morning and
evening commute trips in a general network equilibrium modeling framework. The model is formulated as
a (mixed) complementarity problem. We prove the solution existence for the proposed model, and show
its properties under an equilibrium state. Then we provides the conditions under which the solution will
be unique, and compare the proposed coupled model with a decoupled model. The proposed model is
evaluated in the Sioux-Falls network. The results show that the proposed coupled morning-evening model
is effective in capturing the mode switches between morning and evening, compared with a decoupled
morning (evening) commute model. In particular, our numerical examples show that modeling morning
and evening commutes separately tends to overestimate the number of drivers and total vehicle miles
traveled (VMT) in the network when accounting for travelers’ capabilities for mode switching. For
example, the coupled model produces 25.7% fewer drivers and 8.7% less VMT in the system compared
with the decoupled model when the inconvenience cost due to ridesharing is higher during the evening
commute than in the morning commute. This is due to the fact that the coupled model can capture the
switching to ridesourcing passengers in the evening commute by the rideshare passengers in the morning
commute. A decoupled model cannot capture this effect and most likely will predict that these travelers
will drive to work. With a coupled model, transportation planners can better understand the modal
choice dynamics and provide appropriate incentives to increase vehicle occupancy and reduce VMT.
Further research could focus on including other realistic elements in this modeling framework: deployment of High Occupancy Vehicle (HOV) lanes and rideshare pick-up and drop-off locations, to just name a few. These have the potential of integrating ridesourcing and rideshare services seamlessly and more effectively, which could reduce solo driving, and consequently lessen traffic demand, congestion, and VMT.

Acknowledgements

We acknowledge METRANS for their kind support of this research.

References


Appendix. Table 3. Parameters in the base case.

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