In this paper, we focus on the cost-sharing problem for ride-sharing: determining how to allocate the total ride cost between the driver and the passengers. In particular, we focus on the scenario where drivers are also commuters with the goal of cost recovery. We identify the properties that a desirable cost-sharing mechanism should have and develop a general framework that can be used to create specific cost-sharing mechanisms. We propose specific mechanisms and analyze their relative advantages and disadvantages so that service providers can select a mechanism according to their different needs. In addition, we incorporate the value of time by allowing passengers to have inconvenience costs due to extra travel time caused by detours for picking up the passengers, and provide discount methods to compensate for these costs. We evaluate our approach using real traffic data from the downtown Los Angeles area. Our results show that each proposed mechanism has its unique advantages and that the discount methods can successfully reduce the number of no-passenger vehicles for a large ride-sharing system.

Key words: Cost-Sharing, Ride-Sharing, Mechanism Design

1. Introduction

Nowadays, due to continuous urbanization and growth in population, traffic congestion has become an important issue worldwide, especially in large cities. Schrank et al. (2019) estimates that (a) the average commuter endured a delay of 54 hours at peak periods in 2017, 34 hours longer than in 1982, and (b) the cost of congestion for every commuter in the United States is nearly $1080 a year, summing up to more than $179 billion. Meanwhile, according to Brown (2011), there is no public support for increased taxation to finance infrastructure capacity expansion. Therefore, transit agencies and cities have begun raising funds through alternative approaches such as congestion
pricing. As a result, travelers need to pay significant yearly fares, sometimes exceeding $1000. Needless to say, there is a need for innovative cost-efficient transportation modes that can improve transportation efficiency. Ride-sharing is one such innovative transportation mode that could help mitigate congestion increases, as it can tap into the significant amount of unused vehicular capacity in transportation networks. In this paper, we develop a method to distribute the cost fairly to all the participants of a ride-sharing operation to facilitate its adoption.

The emergence of shared mobility services such as Uber and Lyft has changed the travel behavior of individuals which has attracted researchers’ attention, with most of them focusing on policy-making [Sofi Dinesh et al. 2021, Gheorghiu and Delhomme 2018, Abrahamse and Keall 2012, Becker et al. 2017], passenger behavior [Lovejoy and Handy 2011, Galland et al. 2014, Bachmann et al. 2018], and system optimization [Di and Ban 2019, Zhong et al. 2020, Yu et al. 2019]. For example, Bachmann et al. (2018) discussed the intention for carpooling (ride-sharing) and proposed practical implications to facilitate carpooling. Di and Ban (2019) established an equilibrium framework to study the characteristics of shared mobility systems such as modal cost, system travel time, and deadhead miles. Zhong et al. (2020) studied the role of High Occupancy Vehicle (HOV) lanes in the morning, suggesting that HOV lanes help promote carpooling.

The increase of drivers providing professional ride-sharing services such as UberPool to earn income may generate more traffic within cities due to a number of factors, such as deadhead miles (Erhardt et al. 2019). Therefore, in this paper, we focus on a ride-sharing context where the drivers pick up passengers on their way to work, home, or while running errands, where the deadhead miles are minimal and the motivation for the driver is cost recovery rather than profit-oriented.

Two important research issues in the context of ride-sharing are: (a) how to determine the routes and schedules of the vehicles, including how to assign passengers to vehicles in the presence of conflicting objectives, such as maximizing the number of serviced passengers, or minimizing the operating cost; and (b) how to allocate the operating cost among the passengers. The first (optimization) and second (cost-sharing) problems are highly interrelated, because the optimized
vehicle routes determine the operating cost that needs to be shared. Conversely, the cost-sharing mechanism imposes constraints on how to optimize the routes; for instance, the cost shared by a passenger should not exceed the price initially quoted to them at any time during the operation.

The optimization problem has received considerable attention in the literature and is often solved as a vehicle routing problem with pickup and delivery (PDP) (see survey papers by Cordeau and Laporte 2007; Berbeglia et al. 2010). There has been some research to extend the PDP to consider special features of the ride-sharing problem. Agatz et al. (2011, 2012) developed an optimization-based approach to match drivers and riders in dynamic ride-sharing systems. Their results showed that even at relatively small participation rates, dynamic ride-sharing may have the potential to succeed with a sustainable ride-share population forming over time. Yan and Chen (2011) modeled the problem of multi-driver carpooling with pre-matching information and provided a Lagrangian relaxation-based algorithm. Wang et al. (2016) considered the problem of ride-sharing routing when there are special dedicated lanes such as HOV lanes to reduce the travel time when there are multiple individuals sharing a trip. In this case, there may be an incentive to take a detour to pick up extra passengers to qualify for driving in an HOV lane.

Despite the fact that the allocation of costs or savings to each participant is one crucial component of the ride-sharing system, the cost-sharing problem has largely been neglected in the literature. This paper aims to help fill this gap. For general cost-sharing problems in which the set of players and the cost function are both known and deterministic, Moulin mechanisms (Moulin 1999) and acyclic mechanisms (Mehta et al. 2009) are among the most studied families of cost-sharing mechanisms. They have built the foundation for designing truthful and approximately budget-balanced and economically efficient cost-sharing mechanisms for a variety of settings. However, these frameworks have not been applied to settings relevant to ride-sharing, such as the vehicle routing problem (VRP). Geisberger et al. (2009) were among the earliest authors to touch upon the cost allocation problem in the VRP context. They suggested evenly allocating the cost among passengers while proposing an effective algorithm for solving the vehicle routing problem.
et al. (2011) provided a ride-sharing mechanism based on parallel auctions that is adaptive to the individual preferences of the participants. This mechanism is incentive compatible, and allows a trade-off between reducing vehicle miles travelled and the probability of finding matches between drivers and passengers. Cheng et al. (2014) designed fare splitting mechanisms for last-mile ride-sharing which are budget balanced, individually rational and incentive compatible. More papers exist that study the cost-sharing problem from a pricing perspective rather than cost recovery. They usually appear in the context of taxi-sharing since taxi service providers are profit-oriented (Santos and Xavier 2015, Chen and Wang 2018, Sun et al. 2019, Bian and Liu 2019, Turan et al. 2020, Li et al. 2021). For instance, Santos and Xavier (2015) simulated the taxi-sharing problem with money as an incentive. Although their main focus is to minimize the total routing cost, they proposed a simple method to share the total cost, having the cost of each segment of the trip shared by whoever is in the vehicle during that segment. Sun et al. (2019) designed an optimal pricing scheme for ride-sharing platforms like Uber and Lyft under different matching strategies. They designed the optimal price so that the maximum acceptable passenger-driver distance is the same for both drivers and passengers, thereby ensuring passengers accept the price as well as the assigned driver. Chen and Wang (2018) proposed a pricing scheme for the last-mile transportation system with the goal of optimizing the social welfare. Bian and Liu (2019) designed a mechanism for first-mile ride-sharing considering personal requirements.

There exists some research that studies a context closer to our work but using different approaches. Liu and Li (2017) designed a pricing scheme that is able to attract more participants to a morning commute ride-sharing program. Asghari and Shahabi (2017) proposed an online truthful and individually rational mechanism for ride-sharing drivers based on Vickrey auctions. In the work by Furuhatata et al. (2015), the authors developed a cost allocation scheme for demand-responsive transport systems, called Proportional Online Cost Sharing (POCS). POCS satisfies a list of desirable properties, including online fairness, immediate response, and individual rationality. This mechanism combines the traditional incremental sharing and proportional sharing methods
while maintaining their respective advantages. However, since POCS was designed for a shuttle service with professional drivers, it has several limitations that make it problematic when applied directly to the ride-sharing context. The main problems exist in (1) how to allocate the driver’s direct travel cost; and (2) how to provide incentives to encourage passengers to declare their trip early. This is to facilitate better route planning of the ride-sharing operation. Furthermore, in the context of ride-sharing, the driver has their own unique origin, destination, and travel time limits. Therefore, the value of time should be taken into consideration, especially during peak hours. The above special features of ride-sharing make it impossible to directly apply the mechanism of Furuhata et al. (2015) to the ride-sharing problem.

In this paper, we design mechanisms that provide ride-sharing drivers ways to allocate their cost among passengers that incentivize both passengers and drivers to participate in the ride-sharing operation. We begin by developing three mechanisms that ignore driver or passenger time windows. The first mechanism we propose has the driver incurring none of the cost of the trip which could result in a passenger paying for the full cost of the trip if they are the only passenger in the vehicle. In the second mechanism, the driver shares the cost proportionally with the passengers. However, since the total number of additional passengers assigned to a driver is random at the time each passenger is given their initial quote, this mechanism could still generate high initial quotes for passengers, which is undesirable since passengers base their decision to accept or decline the trip based on the initial quote. That is, since we assume that passenger requests arrive dynamically, the passenger must immediately respond to the offer and only accept the offer if the initial quote is lower than the passenger’s willingness-to-pay level. The third mechanism incorporates a robust prediction of the number of passengers to address the high initial quote issue. We then extend these mechanisms to consider time windows. Compared to POCS by Furuhata et al. (2015) described above, (1) we consider the allocation of the drivers’ costs; (2) we integrate the drivers’ costs with passengers’ shared costs while ensuring the original five desirable properties of POCS still hold; (3) we define two new properties that do not penalize early passenger requests, and (4) we take into consideration the value of time.
The remainder of the paper is structured as follows. Section 2 describes the problem in detail. Section 3 presents a general solution framework followed by the presentation of the three mechanisms developed under this framework in Section 4. Section 5 extends the mechanisms to consider the value of time. Experimental results are presented in Section 6 and finally, in Section 7 we conclude our research and discuss possible future work. The proofs of all statements can be found in Appendices C and D.

2. Problem Description

In this section, we describe the ride-sharing problem and briefly introduce the POCS mechanism (Furuhata et al. 2015). Unlike services provided by companies like Uber and Lyft, our ride-sharing context does not involve a professional driver. We focus on the scenario where the drivers have their own origins and destinations. More formally, we have drivers who want to share their total costs with a limited number of passengers in exchange for their service. Each passenger who makes a request for the service has a demand (the distance between their origin and destination) and a willingness-to-pay level for the ride-sharing service. We assume passengers make requests dynamically but before the driver starts the operation. At the time of the request, a passenger is immediately given an initial quote and they also immediately decide whether to join the operation or not by comparing the initial quote with their willingness-to-pay level. At the end of the operation, passengers share the total cost of the trip, possibly also with the driver. Their shared costs may be lower than their initial quotes due to the matching of subsequent requests to the vehicle. Our goal is to design cost-sharing mechanisms that compute these initial quotes and shared costs for the passengers as new requests arrive to the system.

An important point is that we assume the drivers have the same spatial pattern as the passengers in origins and destinations. With this assumption in mind, different origin and destination patterns indeed affect the allocation of the costs among passengers and drivers in that certain cost-sharing mechanisms may generate favorable results for some passengers such as low shared
costs for passengers with high demand in certain patterns. Nevertheless, in this paper, we focus on designing mechanisms that are fair for all passengers regardless of the origin and destination pattern. To achieve this goal, we will aim to satisfy some desirable properties introduced in this section.

We first present some notation that we use throughout the paper. We let $t \in \mathbb{N}$ represent the running total number of passenger requests. We use $\Pi \subseteq \mathbb{N}$ to denote the set of passengers, and $\pi \in \Sigma(\Pi)$ to represent the order of submitted requests, which is a permutation of set $\Pi$. In order to facilitate understanding of the notations and the proofs throughout the paper, we use $k \in \{1, \ldots, t\}$ to index the order of passengers. Thus, $\pi(k)$ is the $k^{th}$ passenger who submits a request under submit order $\pi$. We let $\pi(0)$ represent the driver. The subscript $\{\pi(k), t\}$ means “for passenger $\pi(k)$ when $t$ passengers have requested service”, where $t \geq k$. We define $\alpha_{\pi(k)} \in \mathbb{R}^+$ as the demand (quantifying the direct distance) of passenger $\pi(k)$’s ride request. We let $F$ be the driver’s direct trip cost, $c_{\pi,t}$ the total cost to drive for $t$ requests under submit order $\pi$, and $c^d_{\pi,t} := c_{\pi,t} - F$ be the total detour cost for $t$ requests under submit order $\pi$. The willingness-to-pay level of passenger $\pi(k)$ is denoted by $W_{\pi(k)}$. For ease of reference, the lists of acronyms and notations used throughout the paper are provided in Appendices [A] and [B] respectively.

Next, we briefly introduce POCS, a mechanism that was developed for a shuttle service with a depot which is provided by operational companies to pickup and deliver passengers. In the shuttle service considered in POCS, the driver does not have their own origin and destination. Thus, the driver does not have their own direct trip cost $F$ considered in the total cost function for the passengers to share. The main idea of POCS is to put passengers into coalitions so that they share the same unit price within a coalition. Different coalitions may have different unit prices. A detailed summary is shown in Algorithm [1] when a new passenger $\pi(k)$ requests service, POCS calculates the marginal cost $c^m_{\pi(k)}$ for this passenger based on the selected routing strategy. This selected routing strategy outputs a vehicle route (optimal or not) specifying the sequence to pickup and deliver passengers who have already requested service. It then updates the coalition cost per $\alpha$.
value (CCPA) $c^a_{\pi(i,k)}$ for $i = 1, \ldots, k$. Finally, it computes the total shared cost $c^s_{\pi(l),k}$ for $l = 1, \ldots, k$ by first looking at the new CCPA values computed with the new passenger and then comparing them with the previous CCPA values (line 6 in Algorithm 1).

**Algorithm 1: The POCS Mechanism**

**Input**: Information for a new passenger $\pi(k)$: origin, destination and $\alpha_{\pi(k)}$

CCPA values of previous passengers: $c^a_{\pi(k_1,k_2)}$ for $1 \leq k_1 \leq k_2 \leq k - 1$

**Output**: $c^a_{\pi(i),k}$ for $i = 1, \ldots, k$

1. calculate $c_{\pi,k}$ and $c_{\pi,k-1}$ based on a selected routing strategy
2. $c^m_{\pi(k)} \leftarrow c_{\pi,k} - c_{\pi,k-1}$
3. for $i = 1, \ldots, k$ do
   4. $c^a_{\pi(i,k)} \leftarrow \frac{\sum_{j=i}^{k} c^m_{\pi(j)}}{\sum_{j=i}^{k} \alpha_{\pi(j)}}$
5. for $l = 1, \ldots, k$ do
   6. $c^s_{\pi(l),k} \leftarrow \alpha_{\pi(l)} \min_{1 \leq j \leq k} \max_{1 \leq i \leq j} c^a_{\pi(i,j)}$

We make the following two major assumptions.

**Assumption 1.** The total cost (for both the driver and the passengers) is non-decreasing in the number of requests. That is, for all number of requests $t$ and $t'$ with $t \leq t'$ and all submit orders $\pi$, $c_{\pi,t} \leq c_{\pi,t'}$.

**Assumption 2.** The total cost (for both the driver and the passengers) for any number of requests $t$ is independent of the submit order. That is, for all number of requests $t \geq 1$ and submit orders $\pi$ and $\pi'$ such that $\pi(1), \ldots, \pi(t)$ and $\pi'(1), \ldots, \pi'(t)$ contain the same set of passengers, $c_{\pi,t} = c_{\pi',t}$.

These two assumptions are satisfied, for example, when the total cost is defined as the minimum operating cost. These assumptions are closely related to the satisfaction of some desired properties. As originally described and defined in Furuhata et al. (2015), POCS satisfies the five desirable properties below under these two assumptions.
Definition 1. **Online Fairness Property.** The shared cost per $\alpha$ value of a passenger is never higher than those of passengers who submit their ride requests after them. That is, for all $k, k' \leq t$ and all submit orders $\pi$, it holds that

$$c^\pi_{\alpha(k_1),t}[k_1] \leq c^\pi_{\alpha(k_2),t}[k_2].$$

Definition 2. **Immediate Response Property.** Passengers are provided immediately after their ride request submissions with (ideally low) upper bounds on their shared costs. That is, for all $k, t_1$ and $t_2$ with $1 \leq k \leq t_1 \leq t_2$ and all submit orders $\pi$, it holds that

$$c^\pi_{\alpha(k),t_1} \geq c^\pi_{\alpha(k),t_2}.$$

Definition 3. **Individual Rationality Property.** The shared costs of passengers who accepted their quotes never exceed their willingness-to-pay when subsequent requests are made. That is, for all $k$ and $t$ with $1 \leq k \leq t$ and all submit orders $\pi$, we have

$$c^\pi_{\alpha(k),t} \leq W^\pi_{\alpha(k)}.$$

Definition 4. **Budget Balance Property.** The total cost equals the sum of the shared costs of all participants. That is, for all number of requests $t \geq 1$ and all submit orders $\pi$, we have

$$\sum_{j=0}^{t} c^\pi_{\alpha(j),t} = c^\pi_{\alpha,t}.$$

Definition 5. **Ex-Post Incentive Compatibility.** The best strategy of every passenger is to submit their ride requests truthfully. That is, given that all other passengers do not change their submit times and whether they accept or decline their quotes, a passenger cannot decrease their shared costs by delaying their ride request submissions. That is, for all $k, k' \leq t$ and all submit orders $\pi$ and $\pi'$ such that

$$\pi'(k) = \begin{cases} 
\pi(k + 1) & \text{if } k_1 \leq k < k_2, \\
\pi(k_1) & \text{if } k = k_2, \\
\pi(k) & \text{otherwise},
\end{cases}$$
we have $c_{\pi(k_1), t}^\alpha \leq c_{\pi(k_2), t}^\alpha$. In other words, consider any number of requests $t$, any submit order $\pi$, and any passenger $\pi(k)$. Assume that the passenger $\pi(k_1)$ delays their ride request and submits the $k_2$th instead of the $k_1$th ride request, with all other passenger requests in the same order. Then, the shared cost $c_{\pi(k_2), t}^\alpha$ for $t$ under the new submit order $\pi'$ should not be lower than the shared cost $c_{\pi(k_1), t}^\alpha$ for $t$ under the previous submit order $\pi$.

3. The Ride-Sharing Mechanism Framework

In this section, we introduce a general Ride-Sharing Mechanism Framework that satisfies certain desirable properties.

We first discuss the issues and challenges of directly applying POCS to our ride-sharing context. Although POCS satisfies the five desirable properties in Definitions 1–5, it is meant to be applied to a shuttle service, a different context from our ride-sharing operation. To adapt POCS to a ride-sharing operation, one could simply include the entire driver’s direct trip cost into the total cost for the passengers. In this simple mechanism, when $|\Pi| = 0$, the driver has no passengers and covers their own direct trip cost $F$. When $|\Pi| \geq 1$, the driver pays nothing. Then, we can apply POCS to $c_{\pi, t}$, the total cost to drive, to obtain the total shared cost $c_{\pi(k), t}^\alpha$ for each passenger.

It is easy to see that for this mechanism, all five desirable properties hold (which is proved in Hu et al. (2015)). However, the first passenger faces the chance of paying 100% of the driver’s direct trip cost $F$ if there ends up being only one passenger served, which may result in a high initial quote that could deter the first passenger from participating in the ride-sharing operation. To solve this issue, we need to strike a balance between compensating the driver and reducing the burden on the first passenger. We would also like to distribute the driver’s direct trip cost $F$ among the passengers proportionally to their $\alpha$ values. This is desirable because in the ideal case, where all passengers submit their requests at the same time and their origin and destination locations are known, this is arguably the most fair and natural way to distribute $F$ among the passengers. However, when we do not know how many passengers are going to submit requests, this property
can be difficult to satisfy. These issues and challenges serve as the motivation to develop our general mechanism framework for ride-sharing.

Let \( c_{\pi(k), t}^s \) and \( c_{\pi(k), t}^d \) be the total detour cost \( c_{\pi, t}^d \) and the direct trip cost of the driver \( F \) shared by passenger \( \pi(k) \) when \( t \) requests have been made under submit order \( \pi \), respectively. Let \( \beta_{\pi(k), t} \) be the fraction of \( F \) that will be covered by passenger \( \pi(k) \) when \( t \) requests have been made. We want to design a mechanism that satisfies the following properties:

- **Five Original Desirable Properties**: Online Fairness, Immediate Response, Individual Rationality, Budget Balance and Ex-Post Incentive Compatibility in Definitions[1]–[5].

- **Reduced Burden for the First Passenger Property**: the initial quote for the first passenger \( \pi(1) \) should not have a high \( \beta \) value. If \( \beta_{\pi(1), 1} \) is close or equal to 1, passenger \( \pi(1) \) may not have the incentive to join the ride-sharing operation.

- **Fairness in Sharing Driver’s Cost Property**: the final share of \( F \) paid by the passengers should be proportional to their \( \alpha \) values. Since we do not know in advance which passenger is going to be the last one, we require for all \( t \),

\[
\frac{\beta_{\pi(1), t}}{\alpha_{\pi(1)}} = \frac{\beta_{\pi(2), t}}{\alpha_{\pi(2)}} = \ldots = \frac{\beta_{\pi(t), t}}{\alpha_{\pi(t)}}.
\]  

(1)

POCS applied directly to the total cost \( c_{\pi, t} = c_{\pi, t}^d + F \) cannot guarantee that the five desirable properties and the Fairness in Sharing Driver’s Cost Property hold at the same time, because all passengers are not guaranteed to be in the same coalition.

We propose a new cost-sharing mechanism framework, the Ride-Sharing Mechanism Framework, that shares the total detour cost \( c_{\pi, t}^d \) and the driver’s trip cost \( F \) separately through 2 sub-mechanisms. We define the framework as follows: For all \( k, t \) with \( k \leq t \) and all submit orders \( \pi \), the total shared cost for passenger \( \pi(k) \) when \( t \) requests have been made under submit order \( \pi \) is

\[
c_{\pi(k), t}^s = c_{\pi(k), t}^s + c_{\pi(k), t}^d = c_{\pi(k), t}^s + \beta_{\pi(k), t} F
\]  

(2)

where the allocation of the total cost \( c_{\pi, t} \) consists of two parts: (a) \( c_{\pi(k), t}^s \) to share the total detour cost \( c_{\pi, t}^d \), which is calculated by applying POCS to the total detour cost \( c_{\pi, t}^d \) (see line 6 in Algorithm[1]); and (b) \( c_{\pi(k), t}^d := \beta_{\pi(k), t} F \) to share the driver’s direct trip cost \( F \), where the \( \beta \) values are...
determined by a specific sub-mechanism. In this framework, we require the $\beta$ values to satisfy two properties: the Fairness in Sharing Driver’s Cost property (see Equation (1)) and the Immediate Response property. In particular, the Immediate Response property requires the $\beta_{\pi(k),t}$ values to be non-increasing in $t$; that is, for all $k, t_1$ and $t_2$ with $k \leq t_1 \leq t_2$ and all submit orders $\pi$,

$$\beta_{\pi(k),t_2} \leq \beta_{\pi(k),t_1}.$$  

(3)

We consider specific methods of computing $\beta_{\pi(k),t}$ in Sections 4.1 – 4.2.

We next show that this Ride-Sharing Mechanism Framework satisfies the five original desirable properties except for the Budget Balance property. The Budget Balance property does not hold because the constraints on the $\beta$ values in the Ride-Sharing Mechanism Framework do not guarantee that all the $\beta$ values sum up to 1 for all $t$ and all submit orders $\pi$. We first introduce Proposition 1 which states that if $c_{\pi(k),t}^{s_1}$ and $c_{\pi(k),t}^{s_2}$ both satisfy the five original desirable properties, then $c_{\pi(k),t}^s$ must also satisfy the five original desirable properties. Note that $c_{\pi(k),t}^{s_1}$ and $c_{\pi(k),t}^{s_2}$ can be the shared costs from any two sub-mechanisms. Then, to show the Ride-Sharing Mechanism Framework satisfies the four properties, we only need to prove that, under (1) and (3), $c_{\pi(k),t}^{s_2}$ satisfies the four properties.

**PROPOSITION 1.** Provided that both $c_{\pi(k),t}^{s_1}$ and $c_{\pi(k),t}^{s_2}$ satisfy the Online Fairness, Immediate Response, Individual Rationality, Budget Balance and Ex-Post Incentive Compatibility properties, $c_{\pi(k),t}^s$ satisfies these five properties as well.

As previously mentioned, we can use the same approach as in Furuhata et al. (2015) to show $c_{\pi(k),t}^{s_1}$ satisfies all five desirable properties. Theorem 1 states that $c_{\pi(k),t}^{s_2}$ satisfies four of the five desirable properties.

**THEOREM 1.** Under the Ride-Sharing Mechanism Framework, $c_{\pi(k),t}^{s_2}$ satisfies the Online Fairness, Immediate Response, Individual Rationality, and Ex-Post Incentive Compatibility properties.
4. The Proposed Mechanisms

In this section, we propose three different mechanisms that fall under the Ride-Sharing Mechanism Framework and analyze their advantages and disadvantages. The proofs for all the theorems and propositions in this section are given in Appendix C.

4.1. Driver-out-of-Coalition Mechanism

In this mechanism, 100% of the driver’s direct trip cost $F$ is guaranteed to be transferred to the passengers and shared proportionally according to the passengers’ $\alpha$ values.

**Definition 6.** Under the Ride-Sharing Mechanism Framework, the **Driver-out-of-Coalition** mechanism specifies for all $k, t$ with $k \leq t$ and all submit orders $\pi$, the total shared cost for passenger $\pi(k)$ when $t$ requests have been made under submit order $\pi$ as

$$c_{\pi(k), t}^x = c_{\pi(k), t}^x + \beta_{\pi(k), t} F$$

where $\beta_{\pi(k), t} := \frac{\alpha_{\pi(k)}}{\sum_{i=1}^{t} \alpha_{\pi(i)}}$.

We can see that the $\beta$ values in the above definition satisfies (1) and (3). Therefore, this Driver-out-of-Coalition mechanism naturally satisfies four out of the five original desirable properties. We have the following theorem for the Budget Balance property.

**Theorem 2.** Under the Driver-out-of-Coalition mechanism, $c_{\pi(k), t}^x$ satisfies the Budget Balance property. That is, for all number of requests $t \geq 1$ and submit orders $\pi$, $\sum_{k=0}^{t} c_{\pi(k), t}^x = F$.

The advantages of this mechanism are: (a) all five original desirable properties hold; and (b) the Fairness in Sharing Driver’s Cost property holds. The disadvantage of this mechanism is that it fails to reduce the burden of $\pi(1)$. Proposition 2 below states that the Fairness in Sharing Driver’s Cost property and the Reduced Burden for the First Passenger property are contradictory under certain circumstances.

**Proposition 2.** When the driver’s direct trip cost $F$ is fully recovered by the passengers, the Fairness in Sharing Driver’s Cost property and the Reduced Burden for the First Passenger property cannot hold at the same time without breaking one of the five original desirable properties.
When there is more than one passenger and we know the maximum and minimum \( \alpha \) values before the mechanism runs, one possible solution to make all seven properties hold is to let \( \beta_{\pi(1),1} = \frac{\alpha_{\text{max}}}{\alpha_{\text{min}} + \alpha_{\text{max}}} \) and set the \( \beta \) values for all \( k \geq 1 \) and all \( t > 1 \) according to Definition 6. This way of determining \( \beta_{\pi(1),1} \) guarantees that \( \beta_{\pi(1),t} \leq \beta_{\pi(1),1} \) for all \( t > 1 \). Therefore, the Immediate Response property holds when the Reduced Burden of the First Passenger property and the Fairness in Sharing Driver’s Cost property hold. On the other hand, this approach may not sufficiently reduce the burden of \( \pi(1) \) if \( \frac{\alpha_{\text{max}}}{\alpha_{\text{min}} + \alpha_{\text{max}}} \) is close to 1.

### 4.2. Driver-in-Coalition Mechanism

In this mechanism, we do not enforce that the driver’s direct trip cost \( F \) is fully recovered by the passengers. Instead, we have the driver share a portion of the direct trip cost \( F \). We let \( \alpha_{\pi(0)} \) be the value quantifying the driver’s demand.

**Definition 7.** Under the Ride-Sharing Mechanism Framework, the Driver-in-Coalition mechanism specifies for all \( k \) and \( t \) with \( 0 \leq k \leq t \) and all submit orders \( \pi \), the total shared cost for passenger \( \pi(k) \) when \( t \) requests have been made under submit order \( \pi \) as

\[
c^s_{\pi(k),t} = c^s_{\pi(k),t} + \beta_{\pi(k),t} F
\]

where \( \beta_{\pi(k),t} := \frac{\alpha_{\pi(k)}}{\sum_{i=0}^{t} \alpha_{\pi(i)}} \).

The analysis is the same as the analysis in Section 4.1, so the Driver-in-Coalition mechanism satisfies the five original desirable properties. Note that the Driver-in-Coalition mechanism also satisfies the Budget Balance property. In this case, the sum of all the passenger costs is less than the total cost since the driver will pay a share of their direct cost \( F \). However, since budget balance is defined for all participants (driver and passengers), it holds for this mechanism. The difference between this mechanism and the Driver-out-of-Coalition mechanism is the involvement of the driver in paying \( F \). By including the driver into the coalition for covering \( F \), the initial quote for the first passenger \( \pi(1) \) becomes more reasonable in that \( \beta_{\pi(1),1} \) is bounded above by \( \frac{\alpha_{\pi(1)}}{\alpha_{\pi(0)} + \alpha_{\pi(1)}} \).
a result, $\beta_{\pi(1),1}$ is high only when $\pi(1)$’s $\alpha$ value is too high compared to the driver’s $\alpha$ value. The advantages of the Driver-in-Coalition mechanism are: (a) all five original desirable properties hold; (b) the Fairness in Sharing Driver’s Cost property holds; and (c) it provides some reasonable reduction on the burden of the first passenger $\pi(1)$. The disadvantage of the Driver-in-Coalition mechanism is that the driver will have to cover some portion of $F$ no matter how large $|\Pi|$ is.

### 4.3. Passengers Predicting Mechanism

One way to reduce the initial quote of the first passenger is to estimate how many additional passengers will be sharing this trip and use this estimate to proportionally share $F$ in the quote. That is, in this mechanism, we deal with the uncertainty of the total number of passengers and the total $\alpha$ values of the passengers using prediction. We estimate the total $\alpha$ value in order to normalize the share of $F$ that is allocated to each passenger and reduce the initial quote for the first passenger. We use a robust optimization based approach to bound the total $\alpha$ value so that the desirable properties hold. We first introduce our proposed mechanism and then describe in detail our robust estimation method.

The total $\alpha$ value is formally defined in below:

**Definition 8.** For all number of requests $t$ and all submit orders $\pi$, the total $\alpha$ value for all passengers when $t$ requests have been made under submit order $\pi$ is:

$$A := \sum_{i=1}^{t} \alpha_{\pi(i)} .$$

Now, we formally define the Passengers Prediction mechanism.

**Definition 9.** Under the Ride-Sharing Mechanisms Framework, the **Passengers Prediction** mechanism specifies for all $k$, $t$ with $k \leq t$ and all submit orders $\pi$, the total shared cost for passenger $\pi(k)$ when $t$ requests have been made under submit order $\pi$ as

$$c_{\pi(k),t}^s = c_{\pi(k),t}^{s1} + \beta_{\pi(k),t} F$$
where

\[ \beta_{\pi(k),t} = \frac{\alpha_{\pi(k)}}{A}. \] (4)

We let \( \tilde{A} \) represent the estimate of \( A \) using the proposed robust method in Section 4.3.1.

It can be seen that \( \beta_{\pi(k),t} \) defined in Definition 9 are proportional to the passengers’ \( \alpha \) values and are non-increasing in \( t \). Therefore, the Passengers Predicting mechanism satisfies the requirements of the Ride-Sharing Mechanism Framework, thus satisfying the Online Fairness, Individual Rationality, Immediate Response and Ex-Post Incentive Compatibility properties. The advantage of this mechanism is that it satisfies all of the properties listed in Section 3 except for the Budget Balance property. However, this mechanism will be closer to satisfying the Budget Balance property as the prediction accuracy increases.

### 4.3.1. Robust Optimization Method for Total \( \alpha \) Value Estimation

We are interested in estimating the quantiles of the distribution of the sum of \( \alpha \) values associated with passengers that arrived on or prior to some given time \( e \in \mathbb{R}_+ \), after which no more requests are accepted. Since both passenger arrival times and \( \alpha \) values are uncertain, even if the distributions associated with these quantities are perfectly known, this is a hard problem in applied probability which, to the best of our knowledge, does not have an analytical solution. If the distributions of \( \alpha \) values and inter-arrival times are perfectly known, asymptotically accurate estimates for the quantiles can be obtained by simulation. Unfortunately, estimating these distributions accurately is often not possible due to lack of data. Thus, we propose an approach inspired by modern robust queuing theory \cite{Bandi2015} to obtain estimates for the quantiles of the distribution of the sum of \( \alpha \) values of those passengers that arrived on or prior to time \( e \). Their approach is originally used to analyze the performances of queuing systems and is now well established in the context of robust queuing theory \cite{Mamani2017, Bandi2018, Whitt2019, He2019}. Our approach yields estimates that are robust to ambiguity in the distributions of the uncertain parameters. Note that our problem scope, proof technique, and applications are all different.
Our proposed approach proceeds as follows. First, we design uncertainty sets for the passenger interarrival times and for the $\alpha$ values. The sets are parameterized by $\Gamma$ values and $\tau$ values which are chosen by the users so that the uncertain parameters are guaranteed to materialize within the uncertainty sets with certain probability. Second, we formulate an optimization problem whose optimal objective value we refer to as “robust sum of $\alpha$ values” and corresponds to an estimate of the desired quantile of the sum of $\alpha$ values. Then, we show that the optimal objective value of this problem can be easily computed in closed form.

Let $T_i \in \mathbb{R}_+, i = 1, 2, \ldots, n$ denote the (uncertain) interarrival time between the $i^{th}$ passenger and the $(i-1)^{th}$ passenger. Thus, the sequence $\{T_1, T_2, \ldots, T_n\}$ contains the interarrival times of all passengers. Suppose that $\lambda$ is the arrival rate of passengers in the system. Similarly, let $\alpha_i \in \mathbb{R}_+, i = 1, 2, \ldots, n$ denote the (uncertain) $\alpha$ value of the $i^{th}$ passenger to arrive and let $\bar{\alpha}$ denote the expectation of the distribution of $\alpha$ values. The parameter $n$ here is a number large enough to simulate the population of the passengers, and by the Central Limit Theorem, larger $n$ yields a smaller standard deviation. The choice of $n$ depends on $e$.

We propose to adapt the model of uncertainty from [Bandi et al. (2015)] which is used to bound partial sums over the interarrival times and the $\alpha$ values within a given time period. We restrict the $\alpha$ values to lie in the uncertainty set adapted

$$
\mathcal{U}_a := \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n : -\Gamma_a \leq \frac{\sum_{i=1}^i \alpha_i - i\bar{\alpha}}{(i)\tau_a} \leq \Gamma_a, \quad \forall i = 1, \ldots, n \right\},
$$

(5)

where $\Gamma_a$ is a budget of uncertainty parameter and $\tau_a \in (1, 2]$ is a parameter modeling heavy-tailed probability distributions. Accordingly, we restrict the interarrival times to lie in the set

$$
\mathcal{U}_t := \left\{ (T_1, \ldots, T_n) \in \mathbb{R}_+^n : -\Gamma_t \leq \frac{\sum_{i=1}^i T_i - i\bar{T}}{(i)\tau_t} \leq \Gamma_t, \quad \forall i = 1, \ldots, n \right\},
$$

(6)

where $\Gamma_t$ is a budget of uncertainty parameter and $\tau_t \in (1, 2]$ is a parameter modeling heavy-tailed probability distributions. The parameters $\Gamma_a$ and $\Gamma_t$ are chosen to guarantee that with some prescribed probability $p$, $\alpha := (\alpha_1, \ldots, \alpha_n)$ and $T := (T_1, \ldots, T_n)$ will materialize in these regions. For example, if the $\alpha$ values are normally distributed, and we are looking to estimate the $p = 95^{th}$
quantile of the sum of $\alpha$ values of passengers that arrived before time $t$, then, we can set $\tau_a = 2$ and $\Gamma_a = 1.64$, corresponding to the 95th percentile of the truncated normal distribution since our $\alpha$ values are non-negative.

We define the robust sum of $\alpha$ values as the maximum value that the sum of all $\alpha$'s associated with passengers that arrived prior to time $e$ can take, subject to $\alpha$ and $T$ both lying in their respective uncertainty sets. In other words, it is the optimal value of the optimization problem

$$\max \sum_{i=1}^{n} \alpha_i \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell \leq e\right)$$

s.t. $\alpha \in U_a, T \in U_t$. 

(7)

The following proposition and theorem then state that this problem admits an analytical solution for the specific choices of uncertainty sets in (5) and (6).

**Proposition 3.** An optimal solution $(\alpha^*, T^*)$ to Problem (7) is given by:

$$T_i^* := \max \left(0, \frac{i}{\lambda} - \Gamma_t(i)\frac{1}{\tau_t} - \sum_{\ell=1}^{i-1} T_\ell^*\right) \quad \forall i = 1, \ldots, n$$

and

$$\alpha_i^* = i\bar{\alpha} + \Gamma_a(i)\frac{1}{\tau_a} - \sum_{\ell=1}^{i-1} \alpha_\ell^* \quad \forall i = 1, \ldots, n.$$ 

(8)

(9)

**Theorem 3.** The analytical optimal solution to Problem (7) with $U_a$ and $U_t$ defined as in (5) and (6), respectively, is given by

$$A^* = \Gamma_a \cdot \sqrt{i^*} + i^*\bar{\alpha},$$

(10)

where $i^*$ is defined as the non-negative integer (or the nearest rounded-down integer) solution of the equation $e = \frac{i^*}{\lambda} - \Gamma_t(i^*)\frac{1}{\tau_t}$.

To better understand how the closed form solution is calculated, we use $\tau_t = 2$ as an example, then we have $i^* = \left\lfloor \frac{i^*^2 + \sqrt{i^*^4 + 4\mu^2e^2}}{2\mu^2} \right\rfloor$ where $\mu = \frac{1}{\lambda}$. The $A^*$ is then obtained since $\Gamma_a$ and $\bar{\alpha}$ are known.
4.4. An Example of the Proposed Mechanisms

In this section, we present a simple example on how the proposed mechanisms in Sections 4.1, 4.2, and 4.3 calculate the shared costs. As shown in Figure 1, suppose there are four passengers and one driver with all origins on the left and all destinations on the right. The dotted line suggests a possible ride-sharing tour. The \( \alpha \) value for the driver is \( \alpha_{\pi(0)} = 6 \) while the \( \alpha \) values for the passengers are \( \alpha_{\pi(1)} = \alpha_{\pi(3)} = 4 \), \( \alpha_{\pi(2)} = \alpha_{\pi(4)} = 2 \). We assume that these \( \alpha \) values are from a truncated normal distribution with known expectation and standard deviation, and that the passenger predicting mechanism estimates the total \( \alpha \) value to be 14, which is calculated according to Theorem 3 with \( \Gamma_{a} = 1.64 \), \( i^{*} = 4 \), and \( \bar{\alpha} = 2.68 \). The \( \beta \) values calculated as passenger requests come in are shown in Table 1. The table is divided into 3 different parts, illustrating the trajectories of \( \beta_{\pi(k),t} \) for each mechanism. The columns are the time steps when \( t \) requests are made, each row represents the trajectory of \( \beta \) values for a particular passenger. Next we illustrate how the \( \beta \) values are calculated. For instance, in the Driver-Out-of-Coalition mechanism, passenger \( \pi(1) \) starts with a \( \beta \) value of 1 and drops to \( 0.67 = \frac{4}{4+2} \) when 2 passenger requests are made and continues to drop to \( 0.33 = \frac{4}{4+2+4+2} \) when all 4 passenger requests are made. In the Driver-in-Coalition mechanism, passenger 1 starts with a \( \beta \) value of \( 0.4 = \frac{4}{4+6} \) and ends with \( 0.22 = \frac{4}{6+4+2+4+2} \). Finally, in the Passengers Predicting mechanism, the \( \beta \) value for passenger 1 stays the same at \( 0.29 = \frac{4}{14} \).
Table 1  The Trajectories of $\beta_{n(k),t}$ under the Different Mechanisms

<table>
<thead>
<tr>
<th>Passenger</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t = 4$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t = 4$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.67</td>
<td>0.4</td>
<td>0.33</td>
<td>0.4</td>
<td>0.33</td>
<td>0.25</td>
<td>0.22</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>2</td>
<td>0.33</td>
<td>0.2</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.125</td>
<td>0.11</td>
<td>0.14</td>
<td>0.14</td>
<td>0.14</td>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>0.33</td>
<td>0.25</td>
<td>0.22</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
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<td>0.14</td>
<td>0.14</td>
<td></td>
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</tr>
</tbody>
</table>

5. Ride-Sharing with Time Constraints

In this section, we discuss how to extend our mechanisms when drivers and passengers all have time constraints. In other words, ride-sharing drivers now have a limit on how much extra time they are willing to spend driving in order to provide the ride-sharing service. Additionally, ride-sharing passengers would like to spend as little time as possible in the vehicle. The motivation for incorporating time constraints is that passengers and drivers all have time limits for their trips. By constraining the time spent in the vehicle, the detour is constrained from a time perspective and, indirectly, by the distance. By quantifying the time limits (into costs) and having willingness-to-pay levels, the detour is constrained from a cost perspective. The main difference between the detour and inconvenience costs is that the detour cost is a base cost that all passengers incur based on total detour miles. The inconvenience cost is based on the extra inconvenience that a new passenger may place on existing passengers and is based on the value of time for each passenger. The detour costs do not deter from adding passengers to the vehicle if it makes sense in terms of willingness-to-pay levels, while the inconvenience costs deter from adding too many passengers when the time sensitivities of the passengers on-board are high. For example, if a passenger has a high willingness-to-pay level but is very time sensitive, the detour cost alone will not deter the passenger from accepting the trip, but their high inconvenience costs may prevent other later arriving requests from being added to the vehicle since they would have to compensate the passenger for inconveniencing them. The proofs for all the theorems in this section are given in Appendix D.
First, we introduce the additional notation we use in this section. We denote by $T_{t}^{\text{tot}}$ the total operation time period when $t$ requests are made. Note that this is the total travel time needed to finish the operation according to the $t$ passenger requests received. We denote by $T_{\pi(k)}$ the maximum length of time passenger $\pi(k)$ can spend in the vehicle and $L_{\pi(k),t}$ as the in-vehicle time of passenger $\pi(k)$ when $t$ requests are made. Let $c_{\pi(k),t}^{\text{ic}}$ be the inconvenience cost of passenger $\pi(k)$ when $t$ requests are made ($c_{\pi(1),1}^{\text{ic}} = 0$), and $\Delta c_{\pi(k),t}^{\text{ic}} := c_{\pi(k),t}^{\text{ic}} - c_{\pi(k),t-1}^{\text{ic}}$ be the marginal inconvenience cost of passenger $\pi(k)$ when $t$ requests are made. Extending the framework in Section 3, we let $c_{\pi(k),t}^{\text{dis}}$ represent the discount amount provided to passenger $\pi(k)$ when $t$ requests are made and $c_{\pi(k_1,k_2)}^{\text{da}}$ represent the coalition discount cost per $\alpha$ value of passengers $\pi(k_1), \ldots, \pi(k_2)$. We let $c_{\pi(k),t}^{\text{ss}}$ be the cost of discounts shared by passenger $\pi(k)$ under a discount method when $t$ requests are made.

In our extension of the Ride-Sharing Mechanism Framework in Section 3, passengers receive a discount for the time-based inconvenience caused by other passengers. We have two basic rules when inserting a new passenger $\pi(k)$ with $k \geq 1$: (a) $T_{t}^{\text{tot}}$ does not exceed $T_{\pi(0)}$; and (b) $c_{\pi(k),t}^{\text{ic}}$ the initial quote price of passenger $\pi(k)$, does not exceed $W_{\pi(k)}$. If either one of the rules is violated, passenger $\pi(k)$ will not join the ride-sharing operation. Notice that if no discount is provided in this time constrained scenario, this is a trivial extension from the previous mechanisms: the driver simply rejects any new passenger that causes the in-vehicle time to exceed $T_{\pi(0)}$. Therefore, we study the more interesting case where we allow discounts to be provided. To better illustrate the procedure, see the process flow diagram in Figure 2. From Figure 2 we can see that the two processes marked in yellow are the ones we need to design. We need to design a way to provide discounts to passengers to compensate for their inconvenience costs brought by passenger $\pi(k)$’s service request. We also do the same for the new passenger $\pi(k)$. In order to determine the discount, we need to measure how passengers value their time spent in the vehicle. We use a non-decreasing convex function $f_{\pi(k)}(L_{\pi(k),t})$ to quantify the inconvenience cost for passenger $\pi(k)$. Figure 3 illustrates two possibilities of the inconvenience cost function. On the left, we have a piecewise linear function $f_{\pi(k)}(L_{\pi(k),t}) = \max\{L_{\pi(k),t} - 4, 0\}$ for $0 \leq L_{\pi(k),t} \leq 7$. On the right, we have an exponential function $f_{\pi(k)}(L_{\pi(k),t}) = \frac{1}{10}L_{\pi(k),t}^2$ for $0 \leq L_{\pi(k),t} \leq 7$. 
When providing discounts, we need to consider 2 different cases: 1) The direct trip cost of the driver compensated; 2) The driver may pay a portion of his direct trip cost.

Let \( c_k \) denote the total operation time.

For each request with \( k \) passengers, we need to design our cost sharing mechanisms for, we need to come up with a way to provide discount to a passenger when his or her actual time spent in the operation exceeds his or her willingness to spend on the vehicle before reaching his or her destination.

We next introduce our discount providing solutions: the Basic Discount Method and the Inconvenience Cost Based Discount Method. Each method outputs the discount cost \( c^\Delta_{\pi(k),t} \), which serves as another cost component in the general Ride-Sharing Mechanism Framework when calculating the total shared cost \( c^\pi_{\pi(k),t} \). In other words, Equation (2) becomes:

\[
    c^\pi_{\pi(k),t} = c^{s_1}_{\pi(k),t} + c^{s_2}_{\pi(k),t} + c^\Delta_{\pi(k),t}.
\]  

(11)
We first formally define the discount amount provided to an existing passenger as well as the new passenger whenever there is a new service request.

**Definition 10.** For all $m, k, t$ with $1 \leq m < k \leq t$ and all submit orders $\pi$, the discount amount provided to passenger $\pi(m)$ for $k$ requests is:

$$
c_{\pi(m),k}^{\text{dis}} \coloneqq \min \left( 0, \left( c_{\pi(m),k-1}^{\pi_1} + c_{\pi(m),k-1}^{\pi_2} \right) - \left( c_{\pi(m),k}^{\pi_1} + c_{\pi(m),k}^{\pi_2} \right) - \Delta c_{\pi(m),k}^{\text{IC}} \right).
$$

(12)

### 5.1. Basic Discount Method

In this method, the driver does not pay any portion of the discount. Instead, all the burden falls on the new passenger whose request may increase the existing passengers' inconvenience costs. Therefore, no discount is provided to this new passenger at the time of their request. The procedure is described in Algorithm 2.

**Algorithm 2: The Basic Discount Method**

**Input:** Information for a new passenger $\pi(k)$: origin, destination, $c_{\pi(k),t}^{\text{IC}}$ and $\alpha_{\pi(k)}$

Information for previous passengers: $c_{\pi(i),k}^{\pi_j}$ for $i = 1, \ldots, k-1$ and $j = 1, 2, 3$

**Output:** $c_{\pi(i),k}^{\pi_j}$ for $i = 1, \ldots, k$

1. $c_{\pi(k),k}^{\pi_3} \leftarrow 0$

2. for $m = 1, \ldots, k - 1$ do

3. if $\Delta c_{\pi(m),k}^{\text{IC}} > 0$ then

4. $c_{\pi(m),k}^{\pi_3} \leftarrow c_{\pi(m),k-1}^{\pi_3} + c_{\pi(m),k}^{\text{dis}}$

5. $c_{\pi(k),k}^{\pi_3} \leftarrow c_{\pi(k),k}^{\pi_3} - c_{\pi(m),k}^{\text{dis}}$

6. $c_{\pi(k),k}^{\pi} \leftarrow c_{\pi(k),k}^{\pi_1} + c_{\pi(k),k}^{\pi_2} + c_{\pi(k),k}^{\pi_3}$

We next examine if this discount method satisfies the properties listed in Section 3. The Fairness in Sharing Driver’s Cost and the Reduced Burden for the First Passenger properties hold because the discount component $c_{\pi(k),t}^{\pi_3}$ is independent from $c_{\pi(k),t}^{\pi_2}$. For the five original desirable properties,
since the discount method serves as another additive component under the Ride-Sharing Mechanism Framework, based on Theorem 1, we only need to examine if the $c_{s3}^{\pi(k),t}$ values generated by this discount method satisfy the five original desirable properties. As shown in the theorem below, the Basic Discount Method satisfies three of the five original desirable properties, with the Online Fairness and the Ex-Post Incentive Compatibility properties no longer holding.

**Theorem 4.** Under the Basic Discount Method, $c_{s3}^{\pi(k),t}$ satisfies the Immediate Response, the Individual Rationality, and the Budget Balance properties.

The advantages of this discount method are: (a) the Fairness in Sharing Driver’s Cost and the Reduced Burden for the First Passenger properties still hold; and (b) the cost is easy to calculate and passengers are not responsible for the inconvenience costs that are not caused by them. The disadvantage of this discount method is that the Online Fairness and the Ex-Post Incentive Compatibility properties are lost. This can be seen through the following counterexamples.

For the Online Fairness property, suppose at $t = 2$, both passengers $\pi(1)$ and $\pi(2)$ have no inconvenience costs and $\alpha_{\pi(1)} = \alpha_{\pi(2)}$. In addition, suppose when passenger $\pi(3)$ requests service at $t = 3$, both passengers’ in-vehicle time increase and their inconvenience costs go from 0 to positive values $f_{\pi(1)}(L_{\pi(1),3})$ and $f_{\pi(2)}(L_{\pi(2),3})$ respectively. If $f_{\pi(1)}(L_{\pi(1),3}) < f_{\pi(2)}(L_{\pi(2),3})$, then $\Delta c_{s3}^{\pi(1),3} < \Delta c_{s3}^{\pi(2),3}$, and this results in $c_{s3}^{\pi(1),3} > c_{s3}^{\pi(2),3}$ assuming that their total shared costs before receiving any discount decrease by the same amount. Therefore we have $c_{s3}^{\pi(1),3} > c_{s3}^{\pi(2),3}$, which implies $\frac{c_{s3}^{\pi(1),3}}{\alpha_{\pi(1)}} > \frac{c_{s3}^{\pi(2),3}}{\alpha_{\pi(2)}}$, thus contradicting the Online Fairness property.

For the Ex-Post Incentive Compatibility property, suppose passengers $\pi(3)$ and $\pi(4)$ share the same origin and destination (this also means $\alpha_{\pi(3)} = \alpha_{\pi(4)}$). Suppose the request of $\pi(3)$ increases the inconvenience costs for the existing passengers and so $c_{s3}^{\pi(3),3} > 0$. Since $\pi(4)$ has the same origin and destination as $\pi(3)$, the participation of this passenger will not increase in-vehicle times for passengers $\pi(1), \pi(2)$ and $\pi(3)$. Thus, $c_{s3}^{\pi(4),4} = 0$. Therefore, it is beneficial for $\pi(3)$ to delay their request submission until right after $\pi(4)$’s. Under this new submit order $\pi'$, $\pi'(4) = \pi(3), \pi'(3) = \pi(4)$ and $c_{s3}^{\pi'(4),4} = 0 < c_{s3}^{\pi(3),3}$. 
5.2. Inconvenience Cost Based Discount Method

Similarly, in the Inconvenience Cost Based Discount Method, the driver is not responsible for providing discounts. All the inconvenience costs are shared by all the passengers in a way that is similar to POCS, namely, they form into coalitions to share the total inconvenience cost. They then obtain their discounts based on their inconvenience costs. In other words, $c^{s3}_{\pi(k),t}$ consists of two parts: (a) the amount of the total inconvenience cost passenger $\pi(k)$ accounts for; and (b) the discounts provided to passenger $\pi(k)$ to compensate for the extra in-vehicle time.

Formally, for all $k$ and $t$ and all submit orders $\pi$ with $k \leq t$, the cost of passenger $\pi(k)$ calculated by the Inconvenience Cost Based Discount Method at $t$ under submit order $\pi$ is:

$$c^{s3}_{\pi(k),t} := \alpha_{\pi(k)} \min_{k \leq j \leq 1 \leq i \leq j} c^{da}_{\pi(i,j)} + (-c^{ic}_{\pi(k),t})$$

(13)

where $c^{da}_{\pi(i,j)}$ is the coalition discount cost per $\alpha$ value (CDPA) of passengers $\pi(i), \ldots, \pi(j)$ at $t$ ($i \leq j \leq t$) under submit order $\pi$. The CDPA value is calculated in the same manner as the CCPA value in POCS except with the inconvenience costs $c^{ic}_{\pi(k),t}$ which results in $c^{da}_{\pi(i,j)} = \frac{\sum_{l=1}^{i-1} c^{ic}_{\pi(l),t} - \sum_{l=1}^{j-1} c^{ic}_{\pi(l),t}}{\sum_{l=1}^{j} \alpha_{\pi(l)}}$.

Suppose the new passenger is $\pi(k)$ who requests service at $t = k$. At the time of the request, the procedure of this discount method is shown in Algorithm 3.

We next examine if this discount method satisfies the properties described in Section 3. The Fairness in Sharing Driver’s Cost and the Reduced Burden for the First Passenger properties hold because the discount component $c^{s3}_{\pi(k),t}$ is independent from $c^{s2}_{\pi(k),t}$. For the five original desirable properties, since the discount method serves as another additive component under the Ride-Sharing Mechanism Framework, based on Theorem 1 we only need to examine if the $c^{s3}_{\pi(k),t}$ values generated by this discount method satisfy the five original desirable properties. The first part of $c^{s3}_{\pi(k),t}$ can be shown to satisfy the five original desirable properties using the same arguments as those for POCS: the total inconvenience cost for all the passengers satisfies Assumptions 1 and 2 and the CDPA values and the first part of (13) are basically the same as that in POCS. So, we can focus our attention on the second part, $c^{ic}_{\pi(k),t}$. As shown in the theorem below, $c^{s3}_{\pi(k),t}$ satisfies four of the five original desirable properties, with only the Online Fairness property not holding. As a result, the Inconvenience Cost Based Discount Method satisfies the same properties as well.
Algorithm 3: The Inconvenience Cost Based Discount Method

**Input:** Information for a new passenger \( \pi(k) \): origin, destination, \( c^{ic}_{\pi(k),t} \) and \( \alpha_{\pi(k)} \)

CDPA values of previous passengers: \( c^{da}_{\pi(k_1,k_2)} \) for \( 1 \leq k_1 \leq k_2 \leq k - 1 \)

**Output:** \( c^a_{\pi(i),k} \) for \( i = 1, \ldots, k \)

1. for \( i = 1, \ldots, k \) do
2. calculate \( c^{ic}_{\pi(i),k} \) based on a selected routing strategy
3. \( c^{da}_{\pi(i,k)} \leftarrow \frac{\sum_{j=1}^{k} c^{ic}_{\pi(j)} - \sum_{j=1}^{i} c^{ic}_{\pi(j)}}{\sum_{j=1}^{i} \alpha_{\pi(j)}} \)
4. for \( l = 1, \ldots, k \) do
5. \( c^a_{\pi(l),k} \leftarrow \alpha_{\pi(l)} \min_{1 \leq j \leq k} \max_{1 \leq i \leq j} c^{da}_{\pi(i,j)} + \left( -c^{ic}_{\pi(k),t} \right) \)
6. \( c^a_{\pi(k),k} \leftarrow c^a_{\pi(k),k} + c^a_{\pi(k),k} + c^a_{\pi(k),k} \)

**Theorem 5.** Under the Inconvenience Cost Based Discount Method, \( c^a_{\pi(k),t} \) satisfies the Immediate Response, Individual Rationality, Budget Balance and Ex-Post Incentive Compatibility properties.

The advantage of this discount method is that the Fairness in Sharing Driver’s Cost and the Reduced Burden for the First Passenger properties still hold. The disadvantages of this discount method are: (a) the Online Fairness property does not hold. The same counter-example as in Section 5.1 can be used to show that the property does not hold; and (b) passengers who have high tolerance for in-vehicle time may not get any discounts while being responsible for a portion of the total inconvenience cost.

### 6. Experimental Results

In this section, we present the experiments we conducted to test the performance of the proposed mechanisms in Sections 4 and 5. The experiments consist of two parts. First, we compared the mechanisms in sharing the driver’s direct trip cost \( F \) from Section 4 using simulations on a randomly generated data set. Our results show that the robust Passenger Prediction mechanism best balances the driver and passenger costs. Second, we compared our discount methods from Section 5. The
second part is performed on a large data set that is based on travel demand of an area around downtown Los Angeles.

6.1. Results of Proposed Mechanisms without Discount

When testing the mechanisms in sharing the driver’s direct trip cost $F$, we are interested in (1) illustrating the desired properties, and (2) testing how they perform on the unsatisfied properties. We assumed each vehicle has four passengers, each with a unique origin and destination. We assumed that the cost per mile is $1, and that the $\alpha$ value is the same as the direct distance. We tested the mechanisms on a 40 by 40 grid where OD pairs are randomly generated. We chose a clustered spatial pattern in which the origins were generated randomly within a cluster of size 10 by 10 at the bottom left corner of the grid and the destinations were generated randomly within a cluster of size 10 by 10 at the top right of the grid. Figure 4 shows the scatter plot of the 5 OD pairs (1 driver and 4 passengers) in a sample instance as well as the distribution of distances. We chose this spatial pattern because ride-sharing operations in daily life usually occur during a driver’s commute to or from work, and business (industrial) areas and residential areas are usually clustered. We performed 100 replications. In each replication, because the number of passengers is small per vehicle, the optimal routing cost can be easily determined through enumeration. We used this optimal routing cost as the total cost to be shared. Then, we used the three proposed mechanisms to allocate a portion of the total cost to the passengers, namely the Driver-out-of-Coalition (DooC), the Driver-in-Coalition (DiC), and the Passenger Prediction (PP) mechanisms. Notice that for the PP mechanism, the mean and standard deviation of the $\alpha$ values are required; we estimated them using 10,000 random samples of the origin and destination distributions.

Table 2 compares the average performance of the three mechanisms. The first two rows are the total cost of the operation and the driver’s direct trip cost $F$, respectively. Note these two values are not impacted by the different mechanisms and depend on how the origin and destinations are generated. The mechanisms differ in how they allocate the total cost among the driver and passengers. The third row shows the average final shared cost per $\alpha$ value among all the passengers.
The fourth row shows the percentage of the absolute error of the budget balance violation: note that the PP mechanism does not necessarily satisfy the Budget Balance property. The fifth row shows how much of the driver’s direct cost $F$ is recovered. As we have shown, the DooC mechanism results in a high initial quote for the first passenger. The last row shows the percentage reduction of this initial quote, comparing the initial quote of the first passenger in the corresponding mechanism with that of the DooC mechanism.

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>DooC</th>
<th>DiC</th>
<th>PP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost of the Operation ($)</td>
<td>69.61</td>
<td>69.61</td>
<td>69.61</td>
</tr>
<tr>
<td>Driver’s Direct Trip Cost ($)</td>
<td>42.46</td>
<td>42.46</td>
<td>42.46</td>
</tr>
<tr>
<td>Average Passenger Cost ($)</td>
<td>17.40</td>
<td>15.26</td>
<td>17.17</td>
</tr>
<tr>
<td>% of Absolute Budget Balance Error</td>
<td>0</td>
<td>0</td>
<td>2.2</td>
</tr>
<tr>
<td>% of Driver’s Cost Recovered</td>
<td>100</td>
<td>80.01</td>
<td>97.79</td>
</tr>
<tr>
<td>% of Reduced Burden for the First Passenger</td>
<td>0</td>
<td>39.91</td>
<td>60.05</td>
</tr>
</tbody>
</table>

As we can see from Table 2, the DiC mechanism produced the lowest average passenger cost because the driver is included in sharing the direct cost $F$. The DooC mechanism recovered all of $F$
but had the highest burden for the first passenger. The PP mechanism had an overall balanced performance in that it reduced the burden for the first passenger significantly and recovered most of the drivers’ costs while maintaining the second lowest average passenger cost. Because of this, we chose to test the PP mechanism for sharing $F$ when comparing the discount methods on our large data set.

### 6.2. Results of Proposed Mechanisms with Discount

When comparing the discount methods, we note that if the number of passengers that the driver picks up on their way is small, the in-vehicle time may not be large and result in no or little inconvenience cost. Therefore, fewer discounts are generated within the system, which makes this case less interesting to study. Taking this into consideration, a more appropriate experiment setting for testing the discount methods is to have multiple drivers and lots of passengers in a system and observe how different discount methods differ in assigning passengers to drivers. As a result, we extend our basic experimental setting in Section 6.1 to a larger one that involves hundreds of drivers and passengers. In this way, a driver may pickup multiple passengers on the way and the inconvenience costs may be high so that we can test the discount methods. Whenever a passenger requests service, the corresponding cost-sharing mechanism is executed for each driver which
provides to the passenger different initial quotes to choose from. We assume in the multi-driver context, the passenger chooses the driver with the smallest initial quote. Note that in this multi-driver context, Assumption 2 is difficult to satisfy, which the Ex-Post Incentive Compatibility property relies upon. The reason is that every time a new passenger request is submitted, the multi-driver multi-passenger system needs to be re-optimized in order to guarantee that Assumption 2 holds. However, when the number of passengers and drivers is large, re-optimization is a time consuming approach for routing and an initial quote needs to be provided at the time a passenger makes a request, making it difficult to satisfy. As a result, in this set of experiments, we use a heuristic approach (cheapest insertion) instead that does not guarantee optimal vehicle routes but can provide efficient solutions with limited computation time. This choice results in the loss of the Ex-Post Incentive Compatibility property. Therefore, in addition to comparing the performance of the different discount methods, we also investigated “how much” Ex-Post Incentive Compatibility we lose when we use a routing approach that violates Assumption 2.

We tested the discount methods on a data set with road sensor data, provided by Los Angeles Metro and archived by researchers at the University of Southern California. They developed the Archived Data Management System (ADMS) that collects, archives, and integrates a variety of transportation data sets from Los Angeles, Orange, San Bernardino, Riverside, and Ventura Counties. ADMS includes access to real-time traffic data with 9500 highway and arterial loop detectors providing data on traffic volume and speed approximately every 1 minute. We selected a region within Los Angeles County that includes 33 sensors on 7 freeways: I-5, I-10, I-105, I-110, I-710, SR-60, and SR-101 (see Figure 5a). The red dots are the locations of the sensors.

Based on the traffic flow data, we generated an origin-destination (OD) matrix of demand in which each cell represents the vehicle counts traveling from one (origin) sensor to another (destination) sensor during rush hours. We used the algorithm of Ma and Qian (2018) to estimate the OD matrix of demand from the traffic volume and speed. Based on this demand matrix, we generated an OD probability matrix in which each cell represents the percentage of trips travelling from one
(origin) cluster to another (destination) cluster. A distribution of the distances between the clusters are shown in Figure 5b. One-third of the trips are within 2.55 to 4.55 miles and another third of the trips are within 7.55 to 10.55 miles. We randomly selected an OD by sampling according to the OD probability matrix. Because the sensors are on the freeway which represent the cluster center of locations rather than the specific location ODs, we randomly generated origins and destinations within three miles of the cluster centers. In summary, we (1) used historical data from sensors on traffic volume and speed to estimate OD demand between sensors; (2) converted the OD-sensor matrix to an OD probability matrix; (3) randomly selected a pair of OD sensors from the probability matrix for a passenger or a driver, and (4) randomly selected a location within 3 miles of the origin sensor (and the destination sensor) to get the specific OD locations.

Our simulation settings are as follows. We assumed that the average speed of all the drivers is 36 miles per hour. Each passenger has a different sensitivity towards in-vehicle time modeled with different inconvenience cost functions when their in-vehicle time exceeds their direct travel time. The slopes of these linear functions were generated randomly between 0 and 1. For example, the inconvenience cost function of passenger $\pi(1)$ could be $f_{\pi(1)}(L_{\pi(1)}, t) = 0.6 \max\{L_{\pi(1)}, t - 4, 0\}$ where 4 is the travel time if passenger $\pi(1)$ travels directly to the destination. The maximum in-vehicle time $T_{\pi(k)}$ for passengers and drivers was set to be either 1.5 or 2 times their direct travel time. We evaluated the system with 1000 passenger requests and 300 or 500 ride-sharing drivers, and a willingness-to-pay-level of 1.5, 2 and 3 times ($W$-factor) the passengers’ direct cost. Note that the direct cost refers to the cost a passenger generates when travelling directly to their destinations in their own vehicles. This is not to be confused with the direct travel cost when a passenger takes a taxi which typically will be much higher. The complete set of evaluated scenarios is shown in Table 3 where the $T$ in the “Time Limit” column represents the direct travel time of passengers and drivers.

To examine the loss of Ex-Post Incentive Compatibility due to using a heuristic routing approach, for every instance we used the setting of Scenario 1 and tested the mechanism when no discount
Table 3  Simulation Settings for the Different Scenarios

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Number of Requests</th>
<th>Number of Drivers</th>
<th>Time Limit (min)</th>
<th>W-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1000</td>
<td>300</td>
<td>1.5T</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
<td>300</td>
<td>2T</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>500</td>
<td>1.5T</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>300</td>
<td>1.5T</td>
<td>1.5</td>
</tr>
<tr>
<td>5</td>
<td>1000</td>
<td>300</td>
<td>1.5T</td>
<td>3</td>
</tr>
</tbody>
</table>

is included. We delayed the first passenger’s request to become the 250\textsuperscript{th}, 500\textsuperscript{th}, 750\textsuperscript{th} and 1000\textsuperscript{th} passenger request. This resulted in altogether 100 \times 4 = 400 samples. The results are shown in Table 4. Each column represents the delayed requests, meaning that the first passenger has delayed to become the 250\textsuperscript{th}, 500\textsuperscript{th}, 750\textsuperscript{th} or the last in the submit order. The first three rows are the percentage of the samples in which the passengers’ final shared costs are better off, not changed and worse, the fourth row is the percentage of the samples in which the passengers are not served since the delayed passengers may be given a high initial quote that they are not willing to pay. The last row shows the average change in final price per \(\alpha\) value. Note that a positive price change means that the behavior of delaying one’s request submission time on average results in a higher final price. We can see that although there may be cases where the passenger is better off by delaying their request, on average the passenger is worse off. Furthermore, the possibility of the passengers rejecting their initial quote is higher when they delay their request submission time. These results suggest that the mechanisms tested in this experiment, although they do not satisfy the Ex-Post Incentive Compatibility property when using an insertion based routing approach, still work well on average to discourage passengers in delaying their request time in hopes of getting a better final price.

We next present the results of comparing the discount methods. We performed 100 replications. Table 5 contains the detailed results for Scenario 1. The first row presents the average direct trip cost per vehicle, which is independent of the mechanism used. The second row contains the total cost of the operation per vehicle. The third row shows the shared cost per passenger which is
averaged over all the passengers served in one instance. The fourth row presents the average cost paid by the driver. The fifth row shows the percentage of passenger requests that are satisfied by the ride-sharing system. The last row shows the number of drivers that did not pick up any passengers. We compare the Inconvenience Cost Based Discount method (ICBD) and the Basic Discount Method with the case where no discounts are provided.

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>No Discount</th>
<th>ICBD</th>
<th>Basic Discount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver’s Direct Trip Cost ($)</td>
<td>7.33</td>
<td>7.33</td>
<td>7.33</td>
</tr>
<tr>
<td>Total Operation Cost per Vehicle ($)</td>
<td>9.54</td>
<td>9.82</td>
<td>9.65</td>
</tr>
<tr>
<td>Shared Cost Per Passenger ($)</td>
<td>3.10</td>
<td>3.33</td>
<td>3.19</td>
</tr>
<tr>
<td>Shared Cost Per Driver ($)</td>
<td>2.72</td>
<td>2.48</td>
<td>2.48</td>
</tr>
<tr>
<td>% of Requests Served</td>
<td>74.67</td>
<td>71.86</td>
<td>75.76</td>
</tr>
<tr>
<td># of No-Passenger Vehicles</td>
<td>87.34</td>
<td>46.23</td>
<td>62.03</td>
</tr>
</tbody>
</table>

Comparing the discount methods in Scenario 1, we find that both methods reduce the shared costs for the drivers. The ICBD method results in the highest shared cost per passenger as well as the total operation cost per vehicle. The higher shared cost per passenger also causes more passengers to reject their quotes, resulting in fewer served passengers. Combined together with its low number of no-passenger vehicles, we can see that the ICBD method reduces the driver’s cost by spreading the passengers among more vehicles so that more drivers have some form of cost recovery. The Basic Discount method, on the other hand, reduces the driver’s cost by bringing more

Table 4  Effect of Loss of Ex-Post Incentive Compatibility Property

<table>
<thead>
<tr>
<th>Delay Slots</th>
<th>250&lt;sup&gt;th&lt;/sup&gt;</th>
<th>500&lt;sup&gt;th&lt;/sup&gt;</th>
<th>750&lt;sup&gt;th&lt;/sup&gt;</th>
<th>1000&lt;sup&gt;th&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Better Off</td>
<td>10.0</td>
<td>6.0</td>
<td>5.0</td>
<td>3.0</td>
</tr>
<tr>
<td>% No Change</td>
<td>20.0</td>
<td>12.0</td>
<td>11.0</td>
<td>9.0</td>
</tr>
<tr>
<td>% Worse</td>
<td>56.0</td>
<td>58.0</td>
<td>46.0</td>
<td>43.0</td>
</tr>
<tr>
<td>% Unserved</td>
<td>14.0</td>
<td>24.0</td>
<td>38.0</td>
<td>45.0</td>
</tr>
<tr>
<td>% of Average Price Change</td>
<td>9.8</td>
<td>22.4</td>
<td>26.3</td>
<td>38.5</td>
</tr>
</tbody>
</table>
passengers in the operation and using fewer vehicles (more no-passenger vehicles) compared to the ICBD method. Its increase in the passengers’ shared cost is not as much as the ICBD method which attracts more passengers to the operation.

Table 6 presents the results for Scenario 2, where the time limit is higher. When increasing the time limit, the total operation cost per vehicle increases compared to Scenario 1 since a higher time limit allows the drivers to pick up more passengers during their trip. This in turn helps the drivers to recover more of their direct cost. As a result, the number of passengers served in the system increases and the number of no-passenger vehicles decreases. Table 7 presents the results for Scenario 3, where the number of drivers is higher. When increasing the number of drivers in the system, the total operation cost per driver decreases because more vehicles are empty and the average number of detours served by a single vehicle is lower. Also, an increase in the number of drivers allows for an increase in the number of passengers served since there is better opportunity for more efficient matching and there is a higher possibility of lower initial quotes. This is because when passengers need to provide discounts for already on-board passengers, their initial quotes are lower when joining a no-passenger vehicle. Consequently, the drivers’ shared cost increases.

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>No Discount</th>
<th>ICBD</th>
<th>Basic Discount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver’s Direct Trip Cost ($)</td>
<td>7.30</td>
<td>7.30</td>
<td>7.30</td>
</tr>
<tr>
<td>Total Operation Cost per Vehicle ($)</td>
<td>11.27</td>
<td>11.87</td>
<td>11.35</td>
</tr>
<tr>
<td>Shared Cost Per Passenger ($)</td>
<td>3.10</td>
<td>3.40</td>
<td>3.15</td>
</tr>
<tr>
<td>Shared Cost Per Driver ($)</td>
<td>2.75</td>
<td>2.43</td>
<td>2.42</td>
</tr>
<tr>
<td>% of Requests Served</td>
<td>91.89</td>
<td>86.40</td>
<td>90.95</td>
</tr>
<tr>
<td># of No-Passenger Vehicles</td>
<td>85.7</td>
<td>29.49</td>
<td>51.54</td>
</tr>
</tbody>
</table>

We are also interested in how the willingness-to-pay-level affects the ride-sharing operation system. Comparing Scenarios 1, 4, and 5 where only the willingness-to-pay levels are different, we examine the discount methods based on these four performance indicators: shared cost per passenger, shared cost per driver, percentage of requests served and the number of no-passenger vehicles.
Table 7  Average Performance Measures for the Discount Methods in Scenario 3

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>No Discount</th>
<th>ICBD</th>
<th>Basic Discount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver’s Direct Trip Cost ($)</td>
<td>7.33</td>
<td>7.33</td>
<td>7.33</td>
</tr>
<tr>
<td>Total Operation Cost per Vehicle ($)</td>
<td>8.74</td>
<td>9.17</td>
<td>8.88</td>
</tr>
<tr>
<td>Shared Cost Per Passenger ($)</td>
<td>3.16</td>
<td>3.46</td>
<td>3.28</td>
</tr>
<tr>
<td>Shared Cost Per Driver ($)</td>
<td>3.70</td>
<td>3.24</td>
<td>3.38</td>
</tr>
<tr>
<td>% of Requests Served</td>
<td>90.89</td>
<td>90.67</td>
<td>91.63</td>
</tr>
<tr>
<td># of No-Passenger Vehicles</td>
<td>208.29</td>
<td>115.97</td>
<td>115.00</td>
</tr>
</tbody>
</table>

The results are shown in Figures 6 and 7. The horizontal axis in each figure represents the $W$-factor while the vertical axis represents the performance indicator. As we can observe from the figures, the increase in passengers’ willingness-to-pay-level increases the shared cost of passengers for both discount methods since the passengers are willing to pay more. Moreover, this also allows more passengers in the system which decreases the drivers’ cost. The ICBD method benefits the most because this method has fewer no-passenger vehicles which spreads the passengers across the vehicles more efficiently.

7. Conclusions

In this paper, we studied the problem of designing cost allocation mechanisms in a ride-sharing context. We first explored the desirable properties under our ride-sharing scenario and explained why mechanisms from the existing literature do not apply in our setting. We next proposed a
general mechanism framework that satisfies the Online Fairness, Immediate Response, Individual Rationality and Ex-Post Incentive Compatibility properties. This framework is flexible and can be adapted to serve different purposes. Based on this general framework, we then proposed three different mechanisms, each with its advantages and disadvantages.

Next, we incorporated time constraints with the previously proposed mechanisms and analyzed the performances of the proposed methods. Again, the choice of the proposed method to deal with time constraints depends on the preferences of the driver. Finally, we compared our proposed mechanisms using simulations. We first compared the sub-mechanisms in Section 4 in a small data set and the results supported our theoretical analysis. The performance of the Passengers Predicting mechanism was more balanced compared to the Driver-out-of-Coalition and the Driver-in-Coalition mechanisms. We next compared the discount methods in a large data set that is closer to reality. The results show that both discount methods can reduce the driver’s cost by spreading the same number of passengers onto more vehicles. The Basic Discount method in general outperformed the Inconvenience Cost Based Discount method in that it had lower shared cost per passenger and higher percentage of requests served with a competitive driver cost. However, the Inconvenience Cost Based Discount method led to a more distributed system with fewer no-passenger vehicles and higher reduction in the driver’s cost when the passengers had a high willingness-to-pay-level.

Finally, we discuss some possible directions for future research. One possible direction could be to consider the case where requests come in during the operation. Routing optimization is not the focus in this paper but it would be an interesting research question to investigate the interactions.
between routing and cost-sharing. Another direction of future research could be the question of how to place drivers so that they will reach the maximum service level within a particular region. Lastly, a deeper study of the interactions between cost-sharing methods and driver-passenger matching would provide a more holistic view of the ride-sharing system.

References


A. Appendix: List of Acronyms

CCPA  Coalition Cost per $\alpha$ Value
CDPA  Coalition Discount Cost per $\alpha$ Value
DiC   Driver-in-Coalition
DooC  Driver-out-of-Coalition
HOV   High Occupancy Vehicle
ICBD  Inconvenience Cost Based Discount
PDP   Pickup and Delivery Problem
POCS  Proportional Online Cost Sharing
PP    Passengers Predicting
RSMF  Ride-Sharing Mechanism Framework
VRP   Vehicle Routing Problem
B. Appendix: List of Notations

\( t \) The running total of passenger requests.

\( \Pi \) Set of passengers.

\( \pi \) A permutation of set \( \Pi \) which represents a specific order of submitted requests.

\( \pi(k) \) The passenger who submits the \( k \)th ride request under submit order \( \pi \). Especially, \( \pi(0) \) represents the driver.

\( \alpha_{\pi(k)} \) The demand of passenger \( \pi(k) \)’s ride request.

\( A \) The total \( \alpha \) value for all passengers who submitted their requests.

\( F \) The driver’s direct trip cost.

\( F_{\pi,t} \) The driver’s direct trip cost recovered at \( t \) under submit order \( \pi \).

\( c_{\pi,t} \) The total cost to drive for \( t \) requests under submit order \( \pi \).

\( c_{d_{\pi,t}} \) The total detour cost for \( t \) requests under submit order \( \pi \).

\( c_{p_{\pi,t}} \) The sum of the shared cost paid by all passengers in submit order \( \pi \) for \( t \) requests.

\( c_{ic_{\pi(k),t}} \) The inconvenience cost of passenger \( \pi(k) \) at \( t \).

\( c_{dis_{\pi(k),t}} \) The discount amount provided to passenger \( \pi(k) \) at \( t \).

\( W_{\pi(k)} \) The willingness-to-pay level of passenger \( \pi(k) \).

\( c_{m_{\pi(k)}} \) The marginal cost for \( \pi(k) \) joining the operation.

\( c_{a_{\pi(k_1,k_2)}} \) The coalition cost per \( \alpha \) Value from passenger \( \pi(k_1) \) to \( \pi(k_2) \).

\( c_{da_{\pi(k_1,k_2)}} \) The coalition discount cost per \( \alpha \) Value from passenger \( \pi(k_1) \) to \( \pi(k_2) \).

\( c_{s_{\pi(k),t}} \) The total shared cost paid by passenger \( \pi(k) \) when \( t \) requests are made.

\( c_{s_{1_{\pi(k),t}}} \) The shared cost of the total detour paid by passenger \( \pi(k) \) at \( t \).

\( c_{s_{2_{\pi(k),t}}} \) The shared cost of \( F \) paid by passenger \( \pi(k) \) at \( t \).

\( c_{s_{3_{\pi(k),t}}} \) The shared cost of the discount paid by passenger \( \pi(k) \) at \( t \).

\( \beta_{\pi(k),t} \) The fraction of \( F \) covered by passenger \( \pi(k) \) at \( t \).
C. Appendix: Proofs for Sections 3 and 4

Proof for Proposition 1: For the Online Fairness property, we need to show that for all $k_1, k_2, t$ with $1 \leq k_1 \leq k_2 \leq t$ and submit orders $\pi$:

$$\frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \leq \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}.$$ 

We know that both $c^s_{\pi(k_1),t}$ and $c^s_{\pi(k_2),t}$ satisfy the Online Fairness property. That is, for all $k_1, k_2, t$ with $1 \leq k_1 \leq k_2 \leq t$ and submit orders $\pi$,

$$\frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \leq \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} \text{ and } \frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \leq \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}.$$ 

Therefore,

$$\frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} = \frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} + \frac{c^s_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \leq \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} + \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} = \frac{c^s_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}.$$ 

Thus, the Online Fairness property is satisfied.

For the Immediate Response property, we need to prove that for all $k, t_1, t_2$ with $1 \leq k \leq t_1 \leq t_2$ and submit orders $\pi$:

$$c^s_{\pi(k),t_1} \geq c^s_{\pi(k),t_2}.$$ 

We know that both $c^s_{\pi(k),t_1}$ and $c^s_{\pi(k),t_2}$ satisfy the Immediate Response property. That is, for all $k, t_1, t_2$ with $1 \leq k \leq t_1 \leq t_2$ and submit orders $\pi$,

$$c^s_{\pi(k),t_1} \geq c^s_{\pi(k),t_2} \text{ and } c^s_{\pi(k),t_1} \geq c^s_{\pi(k),t_2}.$$ 

Therefore,

$$c^s_{\pi(k),t_1} = c^s_{\pi(k),t_1} + c^s_{\pi(k),t_1} \geq c^s_{\pi(k),t_2} + c^s_{\pi(k),t_2} = c^s_{\pi(k),t_2}.$$ 

Thus, the Immediate Response property is satisfied.

For the Individual Rationality property, we need to prove that for all $k, t$ with $1 \leq k \leq t$ and submit orders $\pi$:

$$c^s_{\pi(k),t} \leq W_{\pi(k)},$$ 

for passengers who accepted their initial quotes.
Since we have proven that $c_{\pi(k),t}^s$ satisfies the Immediate Response property, we have:

$$c_{\pi(k),k}^s \geq c_{\pi(k),t}^s.$$  

For passengers who accepted their initial quotes, we have:

$$c_{\pi(k),k}^s \leq W_{\pi(k)}.$$  

Therefore, we have for all $k, t$ with $1 \leq k \leq t$ and submit orders $\pi$:

$$c_{\pi(k),t}^s \leq W_{\pi(k)}.$$  

Thus, the Individual Rationality property is satisfied.

For the Budget Balance property, we need to prove that for all number of requests $t \geq 1$ and submit orders $\pi$:

$$\sum_{j=0}^{t} c_{\pi(j),t}^s = c_{\pi,t}^p.$$  

We know that both $c_{\pi(k),t}^s_1$ and $c_{\pi(k),t}^s_2$ satisfy the Budget Balance property. That is, for all number of requests $t \geq 1$ and submit orders $\pi$:

$$\sum_{j=0}^{t} c_{\pi(j),t}^s_1 = c_{\pi,t}^d \quad \text{and} \quad \sum_{j=0}^{t} c_{\pi(j),t}^s_1 + c_{\pi(j),t}^s_2 = F_{\pi,t},$$

where $F_{\pi,t}$ is the amount of the driver’s direct trip cost recovered when $t$ requests are made under submit order $\pi$. We then have:

$$\sum_{j=0}^{t} c_{\pi(j),t}^s = \sum_{j=0}^{t} (c_{\pi(j),t}^s_1 + c_{\pi(j),t}^s_2) = c_{\pi,t}^d + F_{\pi,t} = c_{\pi,t}^p.$$  

Thus, the Budget Balance property is satisfied.

For the Ex-Post Incentive Compatibility property, we need to prove that for all $k_1, k_2, t$ with $1 \leq k_1 \leq k_2 \leq t$ and submit orders $\pi, \pi'$ with

$$\pi'(k) = \begin{cases} 
\pi(k + 1) & \text{if } k_1 \leq k < k_2, \\
\pi(k_1) & \text{if } k = k_2, \\
\pi(k) & \text{otherwise}, 
\end{cases}$$

We have:

$$c_{\pi(k),t}^s \leq W_{\pi(k)}.$$
Therefore, as any compatibility property. That is, fix any submit orders \(\pi, \pi'\) that satisfy the conditions above, as well as any \(k_1, k_2, t\) with \(1 \leq k_1 \leq k_2 \leq t\); We have:

\[
\pi(k_1), t \leq \pi(k_2), t \quad \text{and} \quad \pi(k_1), t \leq \pi(k_2), t.
\]

Therefore,

\[
\pi(k_1), t \leq \pi(k_2), t + \pi(k_1), t \leq \pi(k_2), t = \pi(k_2), t.
\]

Thus, the Ex-Post Incentive Compatibility property is satisfied and the proposition is proven. \(\square\)

**Proof for Theorem 1**: For the Online Fairness property, we need to prove that \(\frac{c_{\pi(k_1), t}}{c_{\pi(k_2), t}} \leq \frac{c_{\pi(k_2), t}}{c_{\pi(k_2), t}}\). By \([1]\), we have \(\frac{\beta_{\pi(k_1), t} = \alpha_{\pi(k_1)}}{\beta_{\pi(k_2), t} = \alpha_{\pi(k_2)}}\) or equivalently, \(\beta_{\pi(k_1), t}^F = \beta_{\pi(k_2), t}^F\), which implies \(\frac{c_{\pi(k_1), t}}{c_{\pi(k_2), t}} = \frac{c_{\pi(k_2), t}}{c_{\pi(k_2), t}}\).

For the Immediate Response property, we need to prove that for all \(k, t_1, t_2\) with \(1 \leq k \leq t_1 \leq t_2\) and submit orders \(\pi: c_{\pi(k), t_1} \geq c_{\pi(k), t_2}\). Directly by \([3]\), we have \(\beta_{\pi(k), t_1} F \geq \beta_{\pi(k), t_2} F\) which implies \(c_{\pi(k), t_1} \geq c_{\pi(k), t_2}\).

For the Individual Rationality property, we need to prove that for all \(k, t\) with \(1 \leq k \leq t\) and submit orders \(\pi: c_{\pi(k), t} \leq W_{\pi(k)}\). By satisfying the Immediate Response property, we know that \(c_{\pi(k), t} \leq c_{\pi(k), k}\). We also know that \(c_{\pi(k), t} \leq W_{\pi(k)}\) since the passenger accepted the fare quote and so the property is satisfied.

For the Ex-Post Incentive Compatibility property, we need to prove that for all \(k_1, k_2, t\) with \(1 \leq k_1 \leq k_2 \leq t\) and submit orders \(\pi, \pi'\) with

\[
\pi'(k) = \begin{cases} 
\pi(k + 1) & \text{if } k_1 \leq k < k_2, \\
\pi(k_1) & \text{if } k = k_2, \\
\pi(k) & \text{otherwise,}
\end{cases}
\]

we have \(c_{\pi(k_1), t} \leq c_{\pi(k_2), t}\). By \([1]\), for both submit orders \(\pi\) and \(\pi'\), the \(\beta\) values are proportional to their \(\alpha\) values; that is:

\[
\frac{\beta_{\pi(1), t}}{\alpha_{\pi(1)}} = \ldots = \frac{\beta_{\pi(k_1-1), t}}{\alpha_{\pi(k_1-1)}} = \frac{\beta_{\pi(k_1), t}}{\alpha_{\pi(k_1)}} \quad \text{and} \quad \frac{\beta_{\pi'(1), t}}{\alpha_{\pi'(1)}} = \ldots = \frac{\beta_{\pi'(k_2-1), t}}{\alpha_{\pi'(k_2-1)}} = \frac{\beta_{\pi'(k_2), t}}{\alpha_{\pi'(k_2)}}.
\]
Since we have
\[
\pi'(k) = \begin{cases} 
\pi(k+1) & \text{if } k_1 \leq k < k_2, \\
\pi(k_1) & \text{if } k = k_2, \\
\pi(k) & \text{otherwise,}
\end{cases}
\]
we also have \(\alpha_{\pi'(k)} = \alpha_{\pi(k)}\) for \(k < k_1\) and \(\alpha_{\pi'(k_2)} = \alpha_{\pi(k_1)}\). Therefore,
\[
\frac{\beta_{\pi(1),t}}{\beta_{\pi'(1),t}} = \cdots = \frac{\beta_{\pi(k_1-1),t}}{\beta_{\pi'(k_2-1),t}} = \frac{\beta_{\pi(k_1),t}}{\beta_{\pi'(k_2),t}}.
\] (14)

The \(\beta\) values for passengers \(\pi(k) = \pi'(k)\) for \(k < k_1\) will be determined the same way: \(\beta_{\pi(k),t} = \beta_{\pi'(k),t}\) for \(k < k_1\). Then, combined with (14), we have \(\beta_{\pi(k_1),t} = \beta_{\pi'(k_2),t}\) which implies \(\beta_{\pi(k_1),t} F = \beta_{\pi'(k_2),t} F\), and so \(c_{\pi(k_1),t}^2 = c_{\pi'(k_2),t}^2\).

**Proof for Theorem 2:** Follows immediately from Definition 6, we have:
\[
\sum_{k=0}^{t} c_{\pi(k),t}^2 = \sum_{k=1}^{t} c_{\pi'(k),t}^2 = F \sum_{k=1}^{t} \beta_{\pi(k),t} = F \sum_{k=1}^{t} \frac{\alpha_{\pi(k)}}{\sum_{k=1}^{t} \alpha_{\pi(k)}} = F.
\] □

**Proof for Proposition 2:** Note that we only consider the case where \(|\Pi| \geq 2\) since \(\beta_{\pi(1),1} = 1\) when \(|\Pi| = 1\) and the Reduced Burden of the First Passenger property is always lost. Let the first passenger \(\pi(1)\) request a ride at \(t = 1\). Suppose both the Fairness in Sharing Driver’s Cost property and the Reduced Burden for the First Passenger property hold. Since we also have that \(F\) is fully recovered by the passengers, thus,
\[
\beta_{\pi(1),1} < 1
\]
\[
\frac{\beta_{\pi(1),t}}{\alpha_{\pi(1)}} = \frac{\beta_{\pi(2),t}}{\alpha_{\pi(2)}} = \cdots = \frac{\beta_{\pi(t),t}}{\alpha_{\pi(t)}}
\]
\[
\sum_{i=1}^{t} \beta_{\pi(i),t} = 1 \quad \forall t > 1.
\]

These equations together lead to
\[
\beta_{\pi(1),t} = \frac{\alpha_{\pi(1)}}{\sum_{i=1}^{t} \alpha_{\pi(i)}} \quad \forall t.\]
Since the $\alpha$ values of the passengers can take arbitrary positive values, it is possible that when the second passenger submits their request,

$$\beta_{\pi(1),2} = \frac{\alpha_{\pi(1)}}{\alpha_{\pi(1)} + \alpha_{\pi(2)}} > \beta_{\pi(1),1},$$

which leads to the loss of the Immediate Response property. For example, when $\beta_{\pi(1),1} = \frac{4}{5} < 1$ and $\frac{\alpha_{\pi(1)}}{\alpha_{\pi(2)}} = 5$, then $\beta_{\pi(1),2} = \frac{5}{6} > \frac{4}{5} = \beta_{\pi(1),1}$. Thus we have proven that when the Fairness in Sharing Driver’s Cost and the Reduced Burden for the First Passenger properties hold, one of the five original desirable properties is lost. □

**Proof for Proposition 3** The proof is in two parts. First, we show that for any fixed $\alpha$ feasible in Problem (7), $T^*$ results in an objective value no smaller than that attained by any other feasible $T$. Second, we show that for any fixed $T$ feasible in Problem (7), $\alpha^*$ results in an objective value no smaller than that attained by any other feasible $\alpha$.

Fix $\tilde{\alpha} \in U_\alpha$ in Problem (7). Then, Problem (7) reduces to

$$\max \sum_{i=1}^{n} \tilde{\alpha}_i \left( \sum_{\ell=1}^{i} T_\ell \leq e \right)$$

s.t. $T \in U_t$. (15)

We show that $T^*$ defined in the premise of the proposition is optimal in Problem (15). First, we show that $T^*$ is feasible in Problem (15). According to Equation (8), for all $i \in \{1, \ldots, n\}$, we have

$$\sum_{\ell=1}^{i} T^*_\ell = T^*_i + \sum_{\ell=1}^{i-1} T^*_\ell = \sum_{\ell=1}^{i-1} T^*_\ell + \max \left( 0, \frac{i}{\lambda} - \Gamma_i(i) \frac{1}{\tau} - \sum_{\ell=1}^{i-1} T^*_\ell \right)$$

$$\quad = \max \left( \sum_{\ell=1}^{i-1} T^*_\ell \frac{i}{\lambda} - \Gamma_i(i) \frac{1}{\tau} \right).$$

We now show by induction that $\sum_{\ell=1}^{i} T^*_\ell$ satisfies the constraints in the definition of $U_t$, i.e., that $T^* \in U_t$. We first show the base case. Since $\Gamma_i$ is positive, we have

$$\frac{1}{\lambda} - \Gamma_i(1) \frac{i}{\pi} \leq T^*_i = \max \left( 0, \frac{1}{\lambda} - \Gamma_i(1) \frac{i}{\pi} \right) \leq \frac{1}{\lambda} + \Gamma_i(1) \frac{i}{\pi}.$$
Next, we show the induction step. Fix $i \in \{1, \ldots, n\}$ and suppose that
\[
\frac{i-1}{\lambda} - \Gamma_{t}(i-1)^{\frac{1}{\tau}} \leq \sum_{\ell=1}^{i-1} T_{\ell}^{*} \leq \frac{i-1}{\lambda} + \Gamma_{t}(i-1)^{\frac{1}{\tau}}.
\] (17)

Then,
\[
\sum_{\ell=1}^{i} T_{\ell}^{*} = \max \left( \sum_{\ell=1}^{i-1} T_{\ell}^{*}, \frac{i}{\lambda} - \Gamma_{t}(i)^{\frac{1}{\tau}} \right),
\] (18)
and there are two cases. If $\sum_{\ell=1}^{i-1} T_{\ell}^{*} \leq \frac{i}{\lambda} - \Gamma_{t}(i)^{\frac{1}{\tau}}$, then it immediately follows that
\[
\frac{i}{\lambda} - \Gamma_{t}(i)^{\frac{1}{\tau}} = \sum_{\ell=1}^{i} T_{\ell}^{*} \leq \frac{i}{\lambda} + \Gamma_{t}(i)^{\frac{1}{\tau}}
\]

On the other hand, if $\sum_{\ell=1}^{i-1} T_{\ell}^{*} > \frac{i}{\lambda} - \Gamma_{t}(i)^{\frac{1}{\tau}}$, we then have
\[
\frac{i}{\lambda} - \Gamma_{t}(n)^{\frac{1}{\tau}} < \sum_{\ell=1}^{i-1} T_{\ell}^{*} \leq \frac{i-1}{\lambda} + \Gamma_{t}(i-1)^{\frac{1}{\tau}} < \frac{i}{\lambda} + \Gamma_{t}(i)^{\frac{1}{\tau}}.
\]

where the second inequality comes from inequality (17) and the last inequality holds due to the fact that $f(i) = \frac{i}{\lambda} + \Gamma_{t}(i)^{\frac{1}{\tau}}$ is increasing in $i$. Thus we have proven that $T^{*}$ by Equation (8) is a feasible solution to Problem (15). We then show that no other feasible $T$ will result in a better objective value than the $T^{*}$ constructed above. For any $T$ that is within the uncertainty set $U_{t}$, we have
\[
-\Gamma_{t}(i)^{\frac{1}{\tau}} + \frac{i}{\lambda} \leq \sum_{\ell=1}^{i} T_{\ell} \leq \Gamma_{t}(i)^{\frac{1}{\tau}} + \frac{i}{\lambda} \quad \forall i = 1, \ldots, n.
\]

Suppose there exists a feasible $T^{'} \in U_{t}$ that yields a strictly better objective value. Define $i^{*}$ and $i^{'}$ as
\[
i^{*} := \left\{ \max_{i=1,\ldots,n} i : \left( \sum_{\ell=1}^{i} T_{\ell}^{*} \leq e \right) = 1 \right\}
\]
\[
i^{'} := \left\{ \max_{i=1,\ldots,n} i : \left( \sum_{\ell=1}^{i} T_{\ell}^{'} \leq e \right) = 1 \right\}.
\] (19)

Then the existence of $T^{'}$ implies that $i^{'} \geq i^{*} + 1$ which in turn results in
\[
\sum_{\ell=1}^{i^{*}} T_{\ell}^{*} \leq e \text{ and } \sum_{\ell=1}^{i^{*}+1} T_{\ell}^{*} > e,
\]
\[
\sum_{\ell=1}^{i^{*}} T_{\ell}^{'} \leq e \text{ and } \sum_{\ell=1}^{i^{*}+1} T_{\ell}^{'} \leq e.
\]
We then have

\[ \sum_{\ell=1}^{i^*+1} T'_\ell \leq e < \sum_{\ell=1}^{i^*+1} T^*_\ell. \]

From Equation (18), \( T^* \) is constructed such that

\[ \sum_{i=1}^{i^*+1} T^*_\ell = \max \left( \sum_{i=1}^{i^*} T^*_\ell, \frac{i^*+1}{\lambda} - \Gamma_i(i^*+1) \frac{1}{\tau} \right). \]

By the definition of \( i^* \), we know that \( \sum_{i=1}^{i^*+1} T^*_\ell > \sum_{i=1}^{i^*} T^*_\ell \), so we have

\[ \sum_{\ell=1}^{i^*+1} T'_\ell < \frac{i^*+1}{\lambda} - \Gamma_i(i^*+1) \frac{1}{\tau} = \sum_{\ell=1}^{i^*+1} T^*_\ell, \]

which contradicts with \( T' \) being feasible.

Similarly, we show that \( \alpha^* \) constructed in Proposition 3 is an optimal solution for Problem (7).

Fix \( \tilde{T} \in U_\alpha \) in Problem (7). Then, Problem (7) reduces to

\[
\max \sum_{i=1}^{n} \alpha_i \mathbb{I} \left( \sum_{\ell=1}^{i} \tilde{T}_\ell \leq e \right) \\
s.t. \quad \alpha \in U_\alpha.
\]

(20)

First, we show that \( \alpha^* \) defined in the premise of the proposition is feasible in Problem (20).

According to Equation (9), for all \( i = 1, \ldots, n \), we have

\[ \alpha^*_i = \bar{\alpha} + \Gamma_a \left( \frac{1}{\tau_a} - (i-1) \frac{1}{\tau_a} \right) \geq 0 \]

and

\[ \sum_{\ell=1}^{i} \alpha^*_\ell = i\bar{\alpha} + \Gamma_a(i) \frac{1}{\tau_a} \]

which is within the uncertainty set \( U_\alpha \) given by Equation (5). Thus \( \alpha^* \) constructed by Equation (9) is a feasible solution to Problem (20). We next show that no other feasible \( \alpha \) can achieve a strictly higher objective value than the \( \alpha^* \) constructed. Assume there exists a \( \alpha' \) that is feasible and achieves a strictly better objective value than our constructed \( \alpha^* \). Then we have

\[ \sum_{i=1}^{i^*} \alpha'_i > \sum_{i=1}^{i^*} \alpha^*_i = i^*\bar{\alpha} + \Gamma_a(i^*) \frac{1}{\tau_a}, \]

which contradicts with the assumption that \( \alpha' \) is feasible. \( \square \)
Proof for Theorem 3. In order to facilitate the proofs, we first observe that for the case of general convex uncertainty sets, our problem can be formulated as the following mixed-integer convex program

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \alpha_i y_i \\
\text{s.t.} & \quad y \in \{0, 1\}^n, \quad \alpha \in \mathcal{U}_{\alpha}, T \in \mathcal{U}_t \\
\end{align*}
\]

\[
\sum_{\ell=1}^{i} T_{\ell} - e \leq M(1 - y_i) \quad \forall i = 1, \ldots, n,
\]

where \(M\) is a “big-M” constant. Note that this problem is an MILP if the uncertainty sets are both polyhedral. In this formulation, we use the auxiliary binary variables \(y_i\) for \(i = 1, \ldots, n\). The big-\(M\) constraints in Problem (21) ensure that, at an optimal solution, \(y_i = 1\) if and only if \(\sum_{\ell=1}^{i} T_{\ell} \leq e\) so that the \(i\)th passenger has arrived on or before time \(e\). We have the following proposition stating that these two formulations are equivalent.

PROPOSITION 4. Problems (7) and (21) are equivalent. In particular, the two problems have the same optimal objective value and any feasible solution to Problem (7) (resp. (21)) can be used to construct a feasible solution to Problem (21) (resp. (7)) with the same cost.

Next we formally start the proof for Theorem 3. By Proposition 3, we have the optimal solutions \(T^*\) and \(\alpha^*\) ready to compute \(A^*\). According to Proposition 4, \(i^*\) is uniquely defined by \(T^*\). To solve for \(i^*\), we follow Definition (19) and we have:

\[
\sum_{\ell=1}^{i^*} T_{\ell} \leq e \Rightarrow -\Gamma_i(i^*) \frac{1}{\lambda} + \frac{i^*}{\lambda} \leq e.
\]

By letting \(-\Gamma_i(i^*) \frac{1}{\lambda} + \frac{i^*}{\lambda} = e\), we can solve for \(i^*\). This is exactly the same as the procedures provided in the Theorem. With \(i^*\) calculated, we shall have:

\[
A^* = \sum_{\ell=1}^{i^*} \alpha^*_\ell = i^* \bar{\alpha} + \Gamma_a(i^*) \frac{1}{\lambda}.
\]
Proof for Proposition 4: Let \((\alpha^*, T^*)\) be feasible in Problem (7) and define \(y^*\) through
\[
y^*_i := \mathbb{I} \left( \sum_{\ell=1}^i T^*_\ell \leq e \right).
\]
We show that the triple \((\alpha^*, T^*, y^*)\) is feasible in Problem (21) with the same cost. Fix \(i' \in \{1, \ldots, n\}\). If \(\sum_{\ell=1}^{i'} T^*_\ell \leq e\), then, \(y^*_i = 1\) and it follows that
\[
\sum_{\ell=1}^{i'} T^*_\ell - e \leq 0 = M(1 - y^*_i).
\]
If, on the other hand, \(\sum_{\ell=1}^{i'} T^*_\ell > e\), then, \(y^*_i = 0\) and, for \(M\) sufficiently large, it holds that
\[
\sum_{\ell=1}^{i'} T^*_\ell - e \leq M = M(1 - y^*_i).
\]
Since the choice of \(i'\) was arbitrary, we conclude that \((y^*, \alpha^*, T^*)\) is feasible in Problem (21). Moreover, by definition of \(y^*\), it holds that
\[
\sum_{i=1}^n \alpha^*_i y^*_i = \sum_{i=1}^n \alpha^*_i \mathbb{I} \left( \sum_{\ell=1}^i T^*_\ell \leq e \right).
\]
We have constructed a feasible solution to Problem (21) that attains the same cost as that attained by \((\alpha^*, T^*)\) in Problem (7). Since the choice of \((\alpha^*, T^*)\) was arbitrary, Problem (21) upper bounds Problem (7). For the converse, let \((y^*, \alpha^*, T^*)\) be optimal in Problem (21). Since \(U_a \subseteq \mathbb{R}_+^n\), we may assume without loss of generality that
\[
y^*_i = 1 \quad \forall i \text{ such that } \sum_{\ell=1}^i T^*_\ell \leq e.
\]
Indeed, if there exists some \(i'\) such that \(\sum_{\ell=1}^{i'} T^*_\ell \leq e\) and \(y^*_i = 0\), we can always increase \(y^*_i\) to 1 and remain feasible and optimal, since the objective value will not decrease in the process. From the feasibility of \(y^*\) in Problem (21), it must hold that \(y^*_i = 0\) for all \(i\) such that \(\sum_{\ell=1}^i T^*_\ell > e\). We conclude that \(y^*_i = \mathbb{I} \left( \sum_{\ell=1}^i T^*_\ell \leq e \right)\). Therefore, \((\alpha^*, T^*)\) is feasible in Problem (7) and attains the same cost as that attained by \((y^*, \alpha^*, T^*)\) in Problem (21). We conclude that the two problems are equivalent. \(\square\)
D. Appendix: Proofs for Theorems in Section 5

Proof of Theorem 4: for the Immediate Response property, we need to prove that for all $k, t_1, t_2$ with $1 \leq k \leq t_1 \leq t_2$ and submit orders $\pi$: $c_{\pi(k), t_1}^{s_3} \geq c_{\pi(k), t_2}^{s_3}$.

Directly by Algorithm 2, we have that passenger $\pi(k)$ has a non-negative $c_{\pi(k), k}^{s_3}$ for all $1 \leq k \leq t$ and that $c_{\pi(k), t}^{\text{dis}} \leq 0$. Since $c_{\pi(k), t}^{s_3} = c_{\pi(k), t-1}^{s_3} + c_{\pi(k), t}^{\text{dis}}$ for all $k < t$, we have

$$c_{\pi(k), t_2}^{s_3} = c_{\pi(k), t_1}^{s_3} + \sum_{t=t_1+1}^{t_2} c_{\pi(k), t}^{\text{dis}} \leq c_{\pi(k), t_1}^{s_3} .$$

For the Individual Rationality property, we need to prove that for all $k, t$ with $1 \leq k \leq t$ and submit orders $\pi$: $c_{\pi(k), t}^{s_3} \leq W_{\pi(k)}$. By satisfying the Immediate Response property, we know that $c_{\pi(k), t}^{s_3} \leq c_{\pi(k), k}^{s_3}$. We also know that $c_{\pi(k), k}^{s_3} \leq W_{\pi(k)}$ since the passenger accepted the fare quote and so the property is satisfied.

For the Budget Balance property, we need to prove that for all number of requests $t \geq 1$ and submit orders $\pi$, $\sum_{i=1}^{t} c_{\pi(i), t}^{s_3} = 0$ (since the driver is not participating in sharing the discount, the summation starts from $i = 1$ instead of $i = 0$). This means that the discounts provided to the passengers are generated within the system.

Directly by Algorithm 2, we have that all the discounts provided to the existing passengers when a new passenger requests service are covered by the new passenger. This means that:

$$\sum_{i=1}^{t} c_{\pi(i), t}^{s_3} = \sum_{i=1}^{t} \left( c_{\pi(i), i}^{s_3} + \sum_{j=i+1}^{t} c_{\pi(i), j}^{\text{dis}} \right)$$

$$= \sum_{i=1}^{t} c_{\pi(i), i}^{s_3} + \sum_{j=i+1}^{t} \sum_{i=1}^{j} c_{\pi(i), j}^{\text{dis}}$$

$$= \sum_{i=1}^{t} c_{\pi(i), i}^{s_3} + \sum_{j=2}^{t} \sum_{i=1}^{j-1} c_{\pi(i), j}^{\text{dis}}$$

$$= \sum_{i=2}^{t} c_{\pi(i), i}^{s_3} + \sum_{j=2}^{t} \sum_{i=1}^{j-1} c_{\pi(i), j}^{\text{dis}}$$

$$= \sum_{i=2}^{t} c_{\pi(i), i}^{s_3} - \sum_{j=2}^{t} c_{\pi(j), j}^{s_3}$$

$$= 0 ,$$
where the fourth equality holds because there are no existing passengers when the first passenger $\pi(1)$ requests service, thus we have $c_{\pi(1),1}^{s_3} = 0$. □

Proof of Theorem 5: for the Immediate Response property, we have that the first part of $c_{\pi(k),t}^{s_3}$ already satisfies this property. For the second part, since $c_{\pi(k),t}^{ic} = f_{\pi(k)}(T_{t}^{\text{tot}})$, and $f$ is a non-decreasing convex function, we have that $-c_{\pi(k),t}^{ic} \geq -c_{\pi(k),t}^{ic}$ because $t_1 \geq t_2$. And this leads to:

$$
c_{\pi(k),t}^{s_3} = \alpha_{\pi(k)} \min_{k \leq j \leq t_1} \max_{1 \leq i \leq j} c_{\pi(i,j)}^{da} + \left(-c_{\pi(k),t_1}^{ic}\right),
$$

$$
\geq \alpha_{\pi(k)} \min_{k \leq j \leq t_2} \max_{1 \leq i \leq j} c_{\pi(i,j)}^{da} + \left(-c_{\pi(k),t_2}^{ic}\right),
$$

$$
= c_{\pi(k),t_2}^{s_3}.
$$

For the Individual Rationality property, we need to prove that for all $k, t$ with $1 \leq k \leq t$ and submit orders $\pi$: $c_{\pi(k),t}^{s_3} \leq W_{\pi(k)}$. By satisfying the Immediate Response property, we know that $c_{\pi(k),t}^{s_3} \leq c_{\pi(k),k}^{s_3}$. We also know that $c_{\pi(k),k} \leq W_{\pi(k)}$ since the passenger accepted the fare quote and so the property is satisfied.

For the Budget Balance property, since $\alpha_{\pi(k)} \min_{k \leq j \leq t_1} \max_{1 \leq i \leq j} c_{\pi(i,j)}^{da}$ satisfies the Budget Balance property, then we have

$$
\sum_{k=1}^{t} \alpha_{\pi(k)} \min_{k \leq j \leq t} \max_{1 \leq i \leq j} c_{\pi(i,j)}^{da} = \sum_{k=1}^{t} c_{\pi(k),t}^{ic},
$$

since the driver is not participating in sharing the discount, the summation starts from $k = 1$ instead of $k = 0$. In other words, the summation of the value for passengers in the same coalition is the marginal inconvenience cost for that coalition. Summing this value over all the passengers equals to summing the marginal inconvenience cost for all the coalitions which equals to the total inconvenience cost. Then, we have:

$$
\sum_{i=1}^{t} c_{\pi(i),t}^{s_3} = \sum_{k=1}^{t} c_{\pi(k),t}^{ic} - \sum_{k=1}^{t} c_{\pi(k),t}^{ic} = 0.
$$

For the Ex-Post Incentive Compatibility property, recall that the total inconvenience cost for all the passengers satisfy Assumptions 1 and 2. This means that $\sum_{k=1}^{t} c_{\pi(k),t}^{ic}$ is independent of
the submit order and is non-decreasing in $t$. And since $\pi(k_1)$ and $\pi'(k_2)$ both refer to the same passenger, then we have $c^{ic}_{\pi(k_1),t} = c^{ic}_{\pi'(k_2),t}$ which implies $-c^{ic}_{\pi(k_1),t} \leq -c^{ic}_{\pi'(k_2),t}$. Combined with the fact that the first part of $c^{s3}_{\pi(k),t}$ satisfies this property, we have:

$$
c^{s3}_{\pi(k_1),t} = \alpha_{\pi(k_1)} \min_{k_1 \leq j \leq t_1} \max_{1 \leq i \leq j} c^{da}_{\pi(i,j)} + (-c^{ic}_{\pi(k_1),t})
\leq \alpha_{\pi'(k_2)} \min_{k_2 \leq j \leq t_1} \max_{1 \leq i \leq j} c^{da}_{\pi(i,j)} + (-c^{ic}_{\pi'(k_2),t})
= c^{s3}_{\pi'(k_2),t}.
\square
$$