Math Review

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Derivatives

Expression representing the rate of change of a function with respect to an independent variable

How to compute:

\[
\frac{d}{dx} x^n = n x^{n-1}
\]

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

\[
\frac{d}{dx} \sin x = \cos x
\]

\[
\frac{d}{dx} e^x = e^x
\]
Rules to Remember: ① Product Rule ② Quotient Rule ③ Chain Rule

Also must know how to compute partial derivatives

Suppose we have the following function:
\[ f(x,y) = 4x \sin(y^2) - \frac{2x^2}{e^{y^2}} \]

What is \( \frac{df(x,y)}{dx} \), \( \frac{df(x,y)}{dy} \), \( \frac{d^2f(x,y)}{dydx} \)?

\[
\frac{df(x,y)}{dx} = 4\sin(y^2) - \frac{4x}{e^{y^2}} = 4\sin(y^2) - 4xe^{-y^2}
\]

\[
\frac{df(x,y)}{dy} = 8xy \cos(y^2) + x^2 e^{-y^2}
\]

\[
\frac{d^2f(x,y)}{dydx} = \frac{d}{dy} \left[ \frac{df(x,y)}{dx} \right] = \frac{d}{dy} \left[ 4\sin(y^2) - \frac{4x}{e^{y^2}} \right] = 8y \cos(y^2) + 2xe^{-y^2}
\]

Integration (a.k.a. The antiderivative)

From the fundamental theorem of calculus

\[
\int_a^b f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)
\]

This is the general way to compute an integral

Example) \[
\int_2^3 x^2 \, dx = \frac{1}{3} x^3 \bigg|_2^3 = \frac{1}{3} (27 - 8) = \frac{1}{3} (19)
\]

Note: \[
\frac{d}{dx} \left[ \frac{1}{3} x^3 \right] = x^2 \Rightarrow F'(x) = f(x)
\]
Evaluate the following integral
\[ \frac{\pi}{2} \int_{0}^{\frac{\pi}{2}} \cos^2 x \, dx \]

Recall \( \cos^2 x + \sin^2 x = 1 \)

\[ \frac{\pi}{2} \int_{0}^{\frac{\pi}{2}} \cos^2 x \, dx = \int_{0}^{\frac{\pi}{2}} 1 - \sin^2 x \, dx \]

Note: \( \cos(2x) = 1 - 2\sin^2 x \implies \frac{1}{2} \cos(2x) + \frac{1}{2} = 1 - \sin^2 x \)

\[ = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} \cos(2x) + \frac{1}{2} \, dx \]

\[ = \frac{1}{2} \left[ \left. \frac{1}{2} \sin(2x) + x \right|_{0}^{\frac{\pi}{2}} \right] \]

\[ = \frac{1}{2} \left[ \left. \frac{1}{2} \sin(2\pi) + \frac{\pi}{2} \right] - \left( \frac{1}{2} \sin(0) + 0 \right) \right] \]

\[ = \frac{1}{2} \left[ \frac{\pi}{2} \right] = \frac{\pi}{4} \]

Integration by parts

Protocol for integrating two functions at once

Let \( U(x) \) and \( V'(x) \) be functions. Then the integral of \( U(x)V'(x) \)

Can be solved using the following

\[ \int_{a}^{b} U(x)V'(x) \, dx = U(x)V(x) \bigg|_{a}^{b} - \int_{a}^{b} U'(x)V(x) \, dx \]
(Example) Evaluate the following integral

\[ \int_{0}^{\frac{\pi}{4}} (x+1)\sin x \, dx \]

Let \( u(x) = x + 1 \)  \( v(x) = \sin x \)

\[ u'(x) = 1 \quad v(x) = -\cos x \]

\[ \int_{0}^{\frac{\pi}{4}} (x+1)\sin x \, dx = \left. (x+1)(-\cos x) \right|_{0}^{\frac{\pi}{4}} - \int_{0}^{\frac{\pi}{4}} 1(-\cos x) \, dx \]

\[ = \left. (-x\cos x - \cos x) \right|_{0}^{\frac{\pi}{4}} + \int_{0}^{\frac{\pi}{4}} \cos x \, dx \]

\[ = -x\cos x - \cos x + \sin x \int_{0}^{\frac{\pi}{4}} \]

\[ = -\frac{\pi}{4} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4} + \left[ 0 + 1 + 0 \right] \]

\[ = -\frac{\pi}{4} \frac{\sqrt{2}}{2} + 1 = 1 - \frac{\pi\sqrt{2}}{8} \quad \text{or} \quad 1 - \frac{\pi}{4\sqrt{2}} \]
Complex Numbers

All complex numbers contain an imaginary component $i$. In complex field, anything goes $i = \sqrt{-1} \Rightarrow \sqrt{-4} = \pm 2i$

| Complex $z$ in polar coordinates $\sqrt{4} = \pm 2$
| \[ z = x + iy = re^{i\theta} \]
| Complex Conjugate
| \[ \bar{z} = x - iy = re^{-i\theta} \]

Note
\[ z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 \]
and
\[ z \cdot \bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2 \]
Therefore
\[ r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2} \]

Polar coordinates can be expressed in sines and cosines
\[ z = r\cos(\theta) + ir\sin(\theta) = re^{i\theta} \]
\[ \Rightarrow e^{i\theta} = \cos(\theta) + isin(\theta) \]
\[ \text{Euler's formula} \]
\[ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{r \sin(\theta)}{r \cos(\theta)} = \frac{x}{y} \]
Differential Equations

Equation that relates some function with its derivatives

Ordinary - vs. Partial -

only one independent variable

Partial - More than one independent variable:

e.g.) \( \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 U(x,t)}{\partial t^2} \)

Wave equation. Eq. 2.1 in book

Guess and Check Method

Procedure

1. Collect variables of same kind and move to same side
2. Guess a solution
   (usually \( e^{\alpha x} \) where \( \alpha \) is a constant. Sometimes we guess a linear function (\( f(x) = mx + b \) for \( \frac{\partial^2 y}{\partial x^2} = 0 \))
3. Solve for the constant \( \alpha \)
4. Construct the general solution
   \( y(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} \)
5. Solve for the coefficients using the boundary conditions
Example: find a solution to the following differential equation with the given boundary conditions

\[ \frac{d^2y(x)}{dx^2} = y'(x) = -5y(x) \quad y(0) = 1 \]

\[ y' = -5y \]

1. \( y'(x) + 5y(x) = 0 \)
2. Guess \( y(x) = Ce^{\alpha x} \)

\[ y'(x) = C\alpha e^{\alpha x} \]

3. Plug in to solve for \( \alpha \)

\[ C\alpha e^{\alpha x} + 5Ce^{\alpha x} = 0 \]

\[ Ce^{\alpha x}(\alpha + 5) = 0 \]

\( e^{\alpha x} \) can never be zero. \( C \) can be zero but that solution is trivial.

\[ \Rightarrow \alpha + 5 = 0 \quad \Rightarrow \quad \alpha = -5 \]

4. Construct the general solution

\[ y(x) = Ce^{-5x} \]

5. Solve for coefficients using boundary conditions \( y(0) = 1 \)

\[ Ce^{-5(0)} = 1 \quad \Rightarrow \quad C = 1 \]

\[ \therefore \text{Solution is } y(x) = e^{-5x} \]
Example: Solve the following

\[ \frac{d^2 x(t)}{dt^2} + \omega^2 x(t) = 0 \quad \quad x(0) = A \]
\[ x'(0) = 0 \]

1. done already \[ \frac{d^2 x(t)}{dt^2} + \omega^2 x(t) = 0 \]

2. Guess \[ x(t) = Ce^{\alpha t} \Rightarrow x'(t) = C\alpha e^{\alpha t} \Rightarrow x''(t) = C\alpha^2 e^{\alpha t} \]

3. Solve for \( \alpha \)

\[ C\alpha^2 e^{\alpha t} + \omega^2 Ce^{\alpha t} = 0 \]
\[ Ce^{\alpha t} (\alpha^2 + \omega^2) = 0 \]
\[ \alpha^2 + \omega^2 = 0 \quad \Rightarrow \quad \alpha^2 = -\omega^2 \quad \Rightarrow \quad \alpha = \pm i\omega \]

4. Construct the general solution

\[ x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \]

5. Solve for \( C_1 \) and \( C_2 \)

\[ x(0) = A \quad \text{and} \quad x'(0) = 0 \]
\[ x(0) = C_1 e^{i\omega(0)} + C_2 e^{-i\omega(0)} = A \]
\[ C_1 + C_2 = A \]
\[ x'(t) = C_1 i\omega e^{i\omega t} - C_2 i\omega e^{-i\omega t} \]
\[ x'(0) = C_1 i\omega e^{i\omega(0)} - C_2 i\omega e^{-i\omega(0)} = 0 \]
\[ i\omega (C_1 - C_2) = 0 \quad \Rightarrow \quad C_1 = C_2 \]

\[ 2C_1 = A \]
\[ C_1 = \frac{A}{2} \]
\[ C_2 = \frac{A}{2} \]

\[ x(t) = \frac{A}{2} e^{i\omega t} + \frac{A}{2} e^{-i\omega t} \]

\[ x(t) = A cos(\omega t) + \frac{A}{2} i sin(\omega t) + \frac{A}{2} i sin(\omega t) - \frac{A}{2} i sin(\omega t) \]

\[ x(t) = A \cos(\omega t) \]
This type of solution is ideal for Euler's formula
\[ e^{\pm \text{i} \omega t} = \cos \omega t \pm \text{i} \sin \omega t \]

\[ \Rightarrow X(t) \text{ can be written as} \]
\[ X(t) = C_3 \cos \omega t + iC_4 \sin \omega t \]
and \[ X'(t) = -\omega C_3 \sin \omega t + i\omega C_4 \cos \omega t \]

(5) Solve for \( C_3 \) and \( C_4 \) when \( x(0) = A \) and \( x'(0) = 0 \)

\[ x(0) = C_3 \cos \omega (0) + iC_4 \sin \omega (0) = A \]

\[ C_3 = A \]

\[ x'(0) = -\omega C_3 \sin \omega (0) + i\omega C_4 \cos \omega (0) = 0 \]

\[ i\omega C_4 = 0 \]
\[ \text{i can never be 0} \]
\[ \Rightarrow C_4 = 0 \]

\[ \therefore X(t) = A \cos \omega t \]
Separation of Variables

Procedure to Solve differential Equations with more than one independent variable (PDE)

1. Move all like Variables to the Same Side
   
   \[ f(t) \frac{dg(x)}{dx} = h(x) \frac{dm(t)}{dt} \]
   
   \[ \downarrow \]

   \[ \frac{f(t)}{dm(t)} = \frac{h(x)}{dg(x)} \]

2. Integrate both sides

   \[ \int \frac{f(t)}{dm(t)} \, dt = \int \frac{h(x)}{dg(x)} \, dx \]

   \[ \downarrow \]

   \[ \int \frac{f(t)}{dm(t)} \, dt = \frac{F(t)}{m(t)} + C \]

3. Solve for C using boundary conditions
e.g.) Solve \( P(y) \frac{dy}{dx} = Q(x) \)

Where \( P(y) = y^2 \) \& \( Q(x) = \frac{1}{x} \)

Find the general solution

\[
y^2 \frac{dy}{dx} = \frac{1}{x}
\]

① Move like variables to same side

\[
y^2 \, dy = \frac{dx}{x}
\]

② Integrate both sides

\[
\int y^2 \, dy = \int \frac{dx}{x}
\]

\[
\frac{1}{3} y^3 + C_1 = \ln x + C_2
\]

Let \( C = C_2 - C_1 \)

\[
\frac{1}{3} y^3 = \ln x + C
\]

This is the general solution
Example Find the Solution to the following differential Eq. with the given boundary Condition

\[ x^2 \frac{dy}{dx} = y^2 \]

where \( x = 1 \)
\[ y = 3 \]

1. \[ \frac{dy}{y^2} = \frac{dx}{x^2} \]

2. \[ \int \frac{1}{y} dy = \int \frac{1}{x^2} dx \]
\[ \frac{-1}{y} + C_1 = \frac{-1}{x} + C_2 \]
\[ \Rightarrow \frac{-1}{y} = \frac{-1}{x} + C \]

3. \[ \frac{-1}{3} = -1 + C \]
\[ \Rightarrow C = \frac{2}{3} \]
\[ \therefore \frac{-1}{y} = \frac{-1}{x} + \frac{2}{3} \]
\[ \therefore y = \frac{3 - 2x}{3x} \]
Matrices

Suppose $A$ and $B$ are matrices such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$$

Addition of matrices

$$A + B = \begin{pmatrix} a + x & b + y \\ c + s & d + t \end{pmatrix}$$

Matrix Multiplication

$$AB = \begin{pmatrix} ax + bs & ay + bt \\ cx + ds & cy + dt \end{pmatrix}$$

Scalar Multiplication

$$KA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \text{ where } K \text{ is a constant (Scalar)}$$

Note: When multiplying two matrices, the number of columns of the first matrix must equal the number of rows of the second matrix.

P.S. The product will have a dimension that equals the

$\begin{array}{c}
\text{# of rows} \times \text{# columns} \\
\text{1st matrix} \times \text{2nd matrix}
\end{array}$

e.g.) $(\begin{pmatrix} a \\ b \\ c \end{pmatrix})(\begin{pmatrix} de & ef \\ bd & be & bf \\ cd & ce & cf \end{pmatrix}) < dimension \text{ Rows } = 3 \text{ ? } 3 \times 3$

$\text{Mat}$$

$(\begin{pmatrix} d & e & f \end{pmatrix})(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = da + be + cf$ \leftarrow dimension \text{ Row } = 1 \text{ ? } 1 \times 1 \rightarrow \text{ Number}$
Suppose \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

The determinant of \( A \) is: \( \det(A) = ad - bc \)

The transpose of \( A \) is: \( A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \)

Suppose \( A = \begin{pmatrix} a+i & b \\ c+i & d \end{pmatrix} \)

The complex conjugate is \( A^* = \begin{pmatrix} a-i & b \\ c-i & d \end{pmatrix} = \bar{A} \)

The conjugate transpose is \( A^{\#} = \begin{pmatrix} a-i & c-i \\ b & d \end{pmatrix} \)

If a matrix is equal to its complex conjugate transpose, then that matrix is a Hermitian Matrix

*** All Hermitian Matrices (operators) have real eigenvalues (observables)