Name ____________________________________ ECON 186 Fall 2018

Midterm Solution

You have 90 minutes to finish the closed-book exam. You must show all of your work to get full credit. There are 6 problems with a total of 100 points. Good luck!

Problem 1

Consider the national-income model

$$Y = C + I_0 + G$$
$$C = a + b(Y - T_0) \qquad (a > 0, 0 < b < 1)$$
$$G = gY \qquad (0 < g < 1)$$

a) Identify the endogenous variables (2 points).
The endogenous variables are Y (national income), C (consumption), and G (government spending)

b) Write a system of equations in the form $Ax = d$ (4 points).

$$A=\begin{bmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -g & 0 & 1 \end{bmatrix}, x=\begin{bmatrix} Y \\ C \\ G \end{bmatrix}, d=\begin{bmatrix} I_0 \\ a - bT_0 \\ 0 \end{bmatrix}$$

c) Calculate the equilibrium values of Y, C and G, using Cramer's Rule (14 points).

$$|A|=\begin{vmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -g & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -b & 0 \\ -g & 1 \end{vmatrix} - \begin{vmatrix} -b & 1 \\ -g & 0 \end{vmatrix} = 1 - b - g$$

$$|A_1| = \begin{vmatrix} I_0 & -1 & -1 \\ a - bT_0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I_0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} a - bT_0 & 0 \\ -g & 1 \end{vmatrix} - \begin{vmatrix} a - bT_0 & 1 \\ 0 & 0 \end{vmatrix} = I_0 + a - bT_0$$

$$|A_2| = \begin{vmatrix} 1 & I_0 & -1 \\ -b & a - bT_0 & 0 \\ -g & 0 & 1 \end{vmatrix} = \begin{vmatrix} a - bT_0 & 0 \\ 0 & 0 \end{vmatrix} - I_0 \begin{vmatrix} -b & 0 \\ -g & 1 \end{vmatrix} - \begin{vmatrix} -b & a - bT_0 \\ -g & 0 \end{vmatrix} = a - bT_0 + bI_0 -$$

$g(a - bT_0) = (a - bT_0)(1 - g) + bI_0$

$$|A_3| = \begin{vmatrix} 1 & -1 & I_0 \\ -b & 1 & a - bT_0 \\ -g & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & a - bT_0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} -b & a - bT_0 \\ -g & 0 \end{vmatrix} + I_0 \begin{vmatrix} -b & 1 \\ -g & 0 \end{vmatrix} = g(a - bT_0) + gI_0 =$$

$g(a - bT_0 + I_0)$

So, applying Cramer's Rule, we get

$Y^* = \frac{a - bT_0 + I_0}{1 - b - g}$

$C^* = \frac{(a - bT_0)(1 - g) + bI_0}{1 - b - g}$

$G^* = \frac{g(a - bT_0 + I_0)}{1 - b - g}$

Problem 2

Consider the following matrix

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

a) Reduce the matrix A to Row Echelon Form (12 points).

First, swap row 1 and 2 (note that this is not strictly necessary, but having a 1 in the upper left corner makes the arithmetic easier).

$$\begin{bmatrix} 1 & 2 & -2 \\ 4 & -1 & 2 \\ 5 & -3 & 1 \end{bmatrix}$$

Remove the 1st column by adding -4 and -5 times the first row to the second and third rows, respectively.

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -9 & 10 \\ 0 & -13 & 11 \end{bmatrix}$$

Eliminate the -13 by adding $-\frac{13}{9}$ times the second row to the third row.

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -9 & 10 \\ 0 & 0 & -\frac{31}{9} \end{bmatrix}$$

b) What is the rank of A (3 points)?
3

c) Is this matrix singular? Why(3 points)?
No, the matrix is non-singular since the matrix has dimension 3×3 and also has rank 3.

Problem 3

Consider the following matrix

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$$

a) Calculate the determinant of A (4 points).
$|A|$= 1(1)(4) + 5(7)(0) + 1(-3)(2) - 2(1)(0) - 5(1)(4) - 1(7)(-3)= -1

b) Find the inverse of A (10 points).
Method 1.
First, we form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 1 & 1 & 7 & 0 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

The first row operation is to get rid of the 1 in the second row of the first column by adding -1 times the first row to the second row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -4 & 5 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Next, divide the second row by -4

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Get rid of the -3 in the third row by adding 3 times the second row to the third row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & 1 \end{array}\right]$$

Multiply the third row by 4

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 3 & \text{-}3 & 4 \end{array}\right]$$

Multiply the third row by 5/4 and add it to the second row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & \text{-}4 & 5 \\ 0 & 0 & 1 & 3 & \text{-}3 & 4 \end{array}\right]$$

Multiply the second row by -5 and add it to the first row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \text{-}19 & 20 & \text{-}25 \\ 0 & 1 & 0 & 4 & \text{-}4 & 5 \\ 0 & 0 & 1 & 3 & \text{-}3 & 4 \end{array}\right]$$

Multiply the third row by -2 and add it to the first row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \text{-}25 & 26 & \text{-}33 \\ 0 & 1 & 0 & 4 & \text{-}4 & 5 \\ 0 & 0 & 1 & 3 & \text{-}3 & 4 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \text{-}25 & 26 & \text{-}33 \\ 4 & \text{-}4 & 5 \\ 3 & \text{-}3 & 4 \end{bmatrix}$$

Method 2

Step 1: Construct a matrix of minors

$$\begin{bmatrix} 25 & 4 & \text{-}3 \\ 26 & 4 & \text{-}3 \\ 33 & 5 & \text{-}4 \end{bmatrix}$$

Step 2: Construct a matrix of cofactors

$$\begin{bmatrix} 25 & \text{-}4 & \text{-}3 \\ \text{-}26 & 4 & 3 \\ 33 & \text{-}5 & \text{-}4 \end{bmatrix}$$

Step 3: Find the adjugate by taking the transpose of the cofactor matrix

$$\begin{bmatrix} 25 & \text{-}26 & 33 \\ \text{-}4 & 4 & \text{-}5 \\ \text{-}3 & 3 & \text{-}4 \end{bmatrix}$$

Step 4: Find the determinant of the original matrix A

$$\begin{vmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ -3 & 4 \end{vmatrix} - 5\begin{vmatrix} 1 & 7 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 4 + 21 \text{ -}5(4) + 2(\text{-}3) = \text{-}1$$

Step 5: Apply the formula

$$A^{-1} = \frac{1}{|A|}\text{AdjA} = -\begin{bmatrix} 25 & -26 & 33 \\ -4 & 4 & -5 \\ -3 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}$$

c) Confirm that the inverse you find in part b) is actually the inverse by showing that $A^{-1}A = I$ (6 points)?

$$\begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -25(1)+26(1)-33(0) & -25(5)+26(1)-33(-3) & -25(2)+26(7)-33(4) \\ 4(1)-4(1)+5(0) & 4(5)-4(1)+5(-3) & 4(2)-4(7)+4(5) \\ 3(1)-3(1)+4(0) & 3(5)-3(1)+4(-3) & 3(2)-3(7)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 4

Find the derivatives for the following functions

a) $y(x) = \left(6x^{\frac{1}{2}} + 2x\right)\ln(x^4 - x^{-1})$ (5 points).

$\frac{dy}{dx} = (6x^{\frac{1}{2}} + 2x)\frac{4x^3+x^{-2}}{x^4-x^{-1}} + (3x^{-\frac{1}{2}} + 2)ln(x^4 - x^{-1})$

b) $U(c) = [\alpha c^\sigma + \beta(1-c)^\sigma]^{\frac{1}{\sigma}}$ (9 points).

$\frac{dU(c)}{dc} = \frac{1}{\sigma}[\alpha c^\sigma + \beta(1-c)^\sigma]^{\frac{1}{\sigma}-1} \times [\alpha\sigma c^{\sigma-1} + \beta\sigma(1-c)^{\sigma-1}(-1)]$

$= [\alpha c^\sigma + \beta(1-c)^\sigma]^{\frac{1}{\sigma}-1} \times [\alpha c^{\sigma-1} - \beta(1-c)^{\sigma-1}]$

Problem 5

Find the partial derivative $\frac{\partial z}{\partial x}$ and total derivative $\frac{dz}{dx}$ of the function $z(x,y) = \frac{3e^{-2x}+2y}{x}$, where $y = -x^2 + 2x + 14$ (15 points).

$z(x,y) = \frac{3e^{-2x}}{x} + 2yx^{-1} \Rightarrow \frac{\partial z}{\partial x} = \frac{3e^{-2x}(-2)x - 3e^{-2x}}{x^2} - 2yx^{-2} = \frac{-6e^{-2x}x - 3e^{-2x}}{x^2} - 2yx^{-2}$

$z(x,y) = \frac{3e^{-2x}+2(-x^2+2x+14)}{x} = \frac{3e^{-2x}}{x} - 2x + 4 + 28x^{-1} \Rightarrow \frac{dz}{dx} = \frac{-6e^{-2x}x - 3e^{-2x}}{x^2} - 2 - 28x^{-2}$

Problem 6

Consider the system of equations

$$\begin{aligned} u &= y - z \\ v &= x + z^2 \\ w &= x - y^2 + 2yz \end{aligned}$$

Find the Jacobian (determinant of the Jacobian matrix) of the system (13 points).

The Jacobian is

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 2z \\ 1 & -2y+2z & 2y \end{vmatrix} = 0$$