Name ________________________________ ECON 186 Fall 2018

Final Exam

You have 150 minutes to finish the closed-book exam. You must show all of your work to get full credit. There are 7 problems with a total of 100 points. Good luck!

Problem 1: Taylor Approximation (6 points)

Find the third order Taylor approximation of

$$f(x) = x \ln x - x + 1$$

around $x = 1$.

$$\begin{aligned} f(x) &\approx f(1) + \frac{1}{2!}f'(1)(x-1) + f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3 \\ &= (1 \ln 1 - 1 + 1) + (\ln 1)(x-1) + \frac{1}{2}\left(\frac{1}{1}\right)(x-1)^2 + \frac{1}{6}\left(-\frac{1}{1^2}\right)(x-1)^3 \\ &= \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 \end{aligned}$$

Problem 2: Matrix (11 points)

Consider the following matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

a) Is A positive definite, negative definite or indefinite? Show your work. (2 points)
$|A_1| = -1 < 0$, $|A_2| = (-1)(1) - 1(1) = -2 < 0$, therefore, A is indefinite.

b) Calculate $|A|$, the determinant of A. (2 points). $|A| - 5$

c) Calculate A^{-1}, the inverse of A. (4 points).

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Problem 3: Probability (20 points)

Consider the following function

$$f(x) = \begin{cases} ax^2 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of a that makes $f(x)$ a valid probability density function. (4 points)

$$\int_0^3 ax^2 dx = 1 \Rightarrow a = \frac{1}{9}$$

For the rest parts of this problem, just use the value of a you have found in part a).

b) Write down the cumulative density function. (4 points)

$$F(x) = Pr(x \le b) = \int_{-\infty}^{b} ax^2 dx = \int_{-\infty}^{b} \frac{1}{9}x^2 dx = \begin{cases} 0 & x < 0 \\ \frac{1}{27}x^3 & 0 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

c) Find the expectation, $\mathrm{E}[X]$. (4 points)

$$\mathrm{E}[X] = \int_0^3 x\left(\frac{1}{9}x^2\right) dx = \frac{9}{4}$$

d) Now consider the probability density function of the random variable Y

$$g(y) = \begin{cases} 3e^{3y} & y < 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Var}[Y]$, the variance of Y, using the moment generating function. (4 points)

$$\Psi(t) = \mathrm{E}[e^{ty}] = \int_{-\infty}^{0} e^{ty}(3e^{3y})dx = \lim_{a \to -\infty} 3\int_a^0 e^{y(3+t)}dy = \lim_{a \to -\infty} \frac{3e^{y(3+t)}}{3+t}\bigg|_a^0 = \frac{3}{3+t}$$

$$\Psi'(t) = -\frac{3}{(3+t)^2} \qquad \Psi''(t) = \frac{6}{(3+t)^3}$$

Thus

$$\mathrm{E}[Y] = \Psi'(0) = -\frac{1}{3} \qquad \mathrm{E}[Y^2] = \Psi''(0) = \frac{6}{27} = \frac{2}{9}$$

$$\text{Var}[Y] = \text{E}[Y^2] - (\text{E}[Y])^2 = \frac{1}{9}$$

e) Assume that for random variable A and B, $\text{Var}[A] = 1$, $\text{Var}[B] = 2$, $\text{Cov}(A, B) = -1$. What is the value of $\text{Var}[A + 2B - 3]$. (4 points)

$$\text{Var}[A + 2B - 3] = \text{Var}[A] + 2^2\text{Var}[B] + 2(2)\text{Cov}(A, B) = 5$$

Problem 4: Statistical Inference (15 points)

(Based on a real research article)
Suppose you are a researcher trying to find the relation between political connections and firms' stock performance. Specifically, you are trying to determine the correlation between political connectedness and stock price. You run an ordinary least squares (OLS) regression on a cross-section of $n = 2,185$ politically connected firms and find a coefficient of price change $\hat{\beta} = -0.0924$ after their political connections end, with a sample variance of $s^2 = 2.61$. You want to test whether this effect is statistically different from 0 in both directions.

a) Set up the null and alternative hypotheses. (3 points)

$$H_0 : \hat{\beta} = 0 \qquad H_1 : \hat{\beta} \neq 0$$

b) Calculate the t-statistic and perform a t-test at the 5% significance level. (4 points)

$$|t| = \left| \frac{\sqrt{n}(-0.0924 - 0)}{\sqrt{2.61}} \right| = 2.66 > t_c = 1.960$$

So, H_0 was rejected at the 5% significance level

c) Approximate the p-value using the t-table on the last page of exam paper. (4 points)
Since

$$t_c^{99\%} < |t| < t_c^{99.8\%}$$

we have the p-value greater than 0.2% but less than 1%.

d) Construct a 99% confidence interval. Can you reject the null hypothesis at the 99% confidence level? (4 points)

$$(\overline{x} - t_c \frac{s}{\sqrt{n}}, \overline{x} + t_c \frac{s}{\sqrt{n}}) = (-0.2394, -0.0031163)$$

Since 0 is *not* in this interval, we reject H_0 with 99% confidence.

Problem 5: Constrained Optimization I (16 points)

Find the values of x_1 and x_2 that maximize the function $C(x_1, x_2) = -x_1^2 - x_1 x_2 - 2x_2^2$, subject to $x_1 - 2x_2 \leq -1$ and $x_1 + x_2 \geq -2$. For this question you don't need to check for second-order condition.

a) Write the Lagrangian (4 points). Define the Lagrangian

$$L = -x_1^2 - x_1 x_2 - 2x_2^2 + \lambda_1(-1 - x_1 + 2x_2) + \lambda_2(x_1 + x_2 + 2)$$

b) Show the first-order conditions (including complementary slackness condition, if any) (4 points). F.O.C. implies that

$$\frac{\partial L}{\partial x_1} = -2x_1 - x_2 - \lambda_1 + \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_2} = -x_1 - 4x_2 + 2\lambda_1 + \lambda_2 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = -1 - x_1 + 2x_2 \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_1 \frac{\partial L}{\partial \lambda_1} = 0$$
$$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + 2 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_2 \frac{\partial L}{\partial \lambda_2} = 0$$

c) Solve the conditions. (8 points).

Case 1 $\lambda_1 = \lambda_2 = 0$ Will solve $x_1 = x_2 = 0$, but $0 - 2 \times 0 > -1$ violates the first constraint.

Case 2 $\lambda_1 > 0$, $\lambda_2 > 0$ Will solve $x_1 = -\frac{5}{3}$, $x_2 = -\frac{1}{3}$, $\lambda_1 = \frac{2}{9}$, but $\lambda_2 = -\frac{31}{9} < 0$.

Case 3 $\lambda_1 > 0$, $\lambda_2 = 0$ Will solve $x_1 = -\frac{3}{8}$, $x_2 = \frac{5}{16}$, $\lambda_1 = \frac{7}{16}$, which is a candidate solution.

Case 4 $\lambda_1 = 0$, $\lambda_2 > 0$ Will solve $x_1 = -\frac{3}{2}$, $x_2 = -\frac{1}{2}$, but $\lambda_2 = -\frac{7}{2} < 0$. Therefore, the only local max is $(x_1^*, x_2^*) = \left(-\frac{3}{8}, \frac{5}{16}\right)$

Problem 6: Constrained Optimization II (16 points)

Elaine gains utility by consuming two goods, x_1 and x_2. Her utility function is given by

$$u(x_1, x_2) = \left(x_1^{0.5} + x_2^{0.5}\right)^2$$

The price for x_1 is \$1 and the price for x_2 is \$2. She has a total of \$100. Assume that she would spend *all* of her money on buying non-negative number of x_1 and x_2.

a) If Elaine seeks to maximize their utility, write the Lagrangian function for this constrained optimization problem. (4 points)

$$L = \left(x_1^{0.5} + x_2^{0.5}\right)^2 + \lambda(100 - x_1 - 2x_2)$$

b) Show the first-order conditions. (4 points)

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 2\left(x_1^{0.5} + x_2^{0.5}\right)\left(\frac{1}{2}x_1^{-0.5}\right) - \lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 2\left(x_1^{0.5} + x_2^{0.5}\right)\left(\frac{1}{2}x_2^{-0.5}\right) - 2\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 100 - x_1 - 2x_2 = 0
\end{aligned}$$

c) Solve for the optimal values of x_1 and x_2 (i.e., x_1^* and x_2^*), respectively. (4 points)
Eliminating λ by combining the first two conditions,

$$\frac{2\left(x_1^{0.5} + x_2^{0.5}\right)\left(\frac{1}{2}x_1^{-0.5}\right)}{2\left(x_1^{0.5} + x_2^{0.5}\right)\left(\frac{1}{2}x_2^{-0.5}\right)} = \frac{1}{2} \qquad \Rightarrow \qquad \frac{x_1}{x_2} = 4$$

Plugging into the budget constraint and we have

$$(x_1^*, x_2^*) = \left(\frac{200}{3}, \frac{50}{3}\right)$$

d) Show that x_1^* and x_2^* are the values that maximize utility using the second-order sufficient conditions. (4 points)

The bordered Hessian matrix is given by

$$\bar{H} = \begin{bmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -\frac{1}{2}x_1^{-1.5}x_2^{0.5} & \frac{1}{2}x_1^{-0.5}x_2^{-0.5} \\ 2 & \frac{1}{2}x_1^{-0.5}x_2^{-0.5} & -\frac{1}{2}x_1^{0.5}x_2^{-1.5} \end{bmatrix}$$

Note that there are $n = 2$ endogenous variables and $m = 1$ constraint, thus we only need to check for the last $n - m = 1$ leading principal minor, which is and the Hessian (determinant of Hessian matrix) is

$$|\bar{H}_3| = |\bar{H}| = 2x_1^{-0.5}x_2^{-0.5} + 2x_1^{-1.5}x_2^{0.5} + \frac{1}{2}x_1^{0.5}x_2^{-1.5} > 0$$

Therefore, the solution $(x_1^*, x_2^*) = \left(\frac{200}{3}, \frac{50}{3}\right)$ is a local maximization.

Problem 7: Ordinary Differential Equations (16 points)

Solve the following differential equations, i.e, solve for the explicit solution $y(x)$.

a) $xy' - 2y - x^2 = 0$ (8 points)

Rewrite it as

$$y^{'} - \frac{2}{x}y = x$$

Multiplying the integration factor $\mu(x) = \int -\frac{2}{x}dx = \frac{1}{x^2}$

$$\frac{1}{x^2}y^{'} - \frac{2}{x^3}y = \frac{1}{x}$$

$$\left(\frac{1}{x^2}y\right)^{'} = \frac{1}{x}$$

Taking integration of both sides

$$\int \left(\frac{1}{x^2}y\right)^{'} dx = \int \frac{1}{x}dx$$

$$\frac{1}{x^2}y = ln\,|x| + c$$

which gives

$$y = x^2 ln\,|x| + cx^2$$

b) $\frac{dy}{dx} = \frac{x+1}{y-3}, \quad y(1) = 0 \qquad$ (8 points)
(Hint: the solution for $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$)
This is a separable equation, rewrite it as

$$(y-3)dy = (x+1)dx$$

Taking integral

$$\int (y-3)dy = \int (x+1)dx$$

$$\frac{1}{2}y^2 - 3y = \frac{1}{2}x^2 + x + C$$

that is,

$$y^2 - 6y = x^2 + 2x + C$$

Note that $y(1) = 0$, plugging in $(x, y) = (1, 0)$ into this equation and we can get $C = -3$. Therefore,

$$y^2 - 6y = x^2 + 2x - 3$$

To acquire the explicit form, rewrite the equation as

$$y^2 - 6y - (x^2 + 2x - 3) = 0$$

Using the quadratic formula,

$$y = \frac{6 \pm \sqrt{(-6)^2 + 4(x^2 + 2x - 3)}}{2} = 3 \pm \sqrt{x^2 + 2x + 6}$$

Using the condition $(x, y) = (1, 0)$ again, we can see that

$$0 = 3 \pm \sqrt{1^2 + 2(1) + 6} = 3 \pm 3$$

therefore, we take the "$-$" sign, and the solution is

$$y = 3 - \sqrt{x^2 + 2x + 6}$$

t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										