ECON 186 Final Solutions

1)

$$L = -x_1^2 - x_1 x_2 - x_2^2 + \lambda_1 \left(-1 - x_1 + 2x_2\right) + \lambda_2 \left(2 - 2x_1 - x_2\right)$$

The Kuhn-Tucker Conditions are:

$$L_{\lambda_1} : -1 - x_1 + 2x_2 \geq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1 L_{\lambda_1} = 0$$

$$L_{\lambda_2} : 2 - 2x_1 - x_2 \geq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2 L_{\lambda_2} = 0$$

$$L_{x_1} : -2x_1 - x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$L_{x_2} : -2x_2 - x_1 + 2\lambda_1 - \lambda_2 = 0$$

Since $x_1, x_2 > 0$, we can just consider the 4 cases for λ_1 and λ_2.

Case 1: $\lambda_1 > 0, \lambda_2 > 0$

Then, $-1 - x_1 + 2x_2 = 0 \to x_1 = -1 + 2x_2$ and $2 - 2x_1 - x_2 = 0$. Combining them: $2 - 2\left(-1 + 2x_2\right) - x_2 = 4 - 5x_2 = 0 \to x_2^* = \frac{4}{5}$. Plugging back in, $x_1^* = -1 + \frac{8}{5} = \frac{3}{5}$. Now, we have to make sure λ_1 and λ_2 are ≥ 0. Plugging in x_1^* and x_2^* in to L_{x_1}, we get $-\frac{6}{5} - \frac{4}{5} - \lambda_1 - 2\lambda_2 = 0 \to \lambda_1 = -2 - 2\lambda_2$. Plugging into L_{x_2}, $-\frac{8}{5} - \frac{3}{5} + 2\left(-2 - 2\lambda_2\right) - \lambda_2 = -\frac{31}{5} - 5\lambda_2 = 0 \to \lambda_2 = -\frac{31}{25}$, which violates the condition that $\lambda_2 \geq 0$, so this cannot be a solution.

Case 2: $\lambda_1 > 0, \lambda_2 = 0$

Substituting $\lambda_2 = 0$ into $L_{x_1} : -2x_1 - x_2 - \lambda_1 = 0 \to \lambda_1 = -2x_1 - x_2$. Substituting into $L_{x_2} : -2x_2 - x_1 + 2\lambda_1 = 0 \to \lambda_1 = \frac{x_1 + 2x_2}{2}$. Then, $\lambda_1 = \lambda_1 \to -2x_1 - x_2 = \frac{x_1 + 2x_2}{2} \to -\frac{5}{2}x_1 = 2x_2 \to x_1 = -\frac{4}{5}x_2$. By complementary slackness, $-1 - x_1 + 2x_2 = 0$. Plugging into this constraint, we get $-1 + \frac{4}{5}x_2 + 2x_2 = -1 + \frac{14}{5}x_2 = 0 \to x_2^* = \frac{5}{14}$. Plugging back in, $x_1^* = -\frac{4}{14}$ and $\lambda_1^* = -2\left(-\frac{4}{15}\right) - \frac{5}{14} = \frac{3}{14}$. All the conditions are satisfied, so this is a candidate solution.

Case 3: $\lambda_1 = 0, \lambda_2 > 0$

Substituting $\lambda_1 = 0$ into $L_{x_1} : -2x_1 - x_2 - 2\lambda_2 = 0 \to \lambda_2 = -x_1 - \frac{x_2}{2}$. Substituting into $L_{x_2} : -2x_2 - x_1 - \lambda_2 = 0 \to \lambda_2 = -2x_2 - x_1$. Then, $\lambda_2 = \lambda_2 \to -x_1 - \frac{x_2}{2} = -2x_2 - x_1 \to x_2^* = 0$. By complementary slackness, $2 - 2x_1 - x_2 = 0$. Plugging into this constraint, we get

$2 - 2x_1 = 0 \to x_1^* = 1$. Plugging back in, $\lambda_2^* = -1 - 0 = -1$. This violates the constraint that $\lambda_1 \geq 0$, so this cannot be a solution.

Case 4: $\lambda_1 = 0, \lambda_2 = 0$

Plugging into L_{x_1} and L_{x_2}, we get that $x_1^* = x_2^* = 0$ which violates the constraint $x_1 - 2x_2 < -1$, so this cannot be a solution.

So the unique solution to the problem is where $x_1^* = -\frac{4}{14}$, $x_2^* = \frac{5}{14}$, $\lambda_1^* = \frac{3}{14}$, $\lambda_2^* = 0$.

Point distribution: 2 points for Lagrangian setup, 5 points for Kuhn-Tucker conditions, 2 points for each case

2)

a)

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$$

METHOD 1
First, we form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 1 & 1 & 7 & 0 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

The first row operation is to get rid of the 1 in the second row of the first column by adding -1 times the first row to the second row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -4 & 5 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Next, divide the second row by -4

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Get rid of the -3 in the third row by adding 3 times the second row to the third row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & 1 \end{array}\right]$$

Multiply the third row by 4

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right]$$

Multiply the third row by 5/4 and add it to the second row

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right]$$

Multiply the second row by -5 and add it to the first row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -19 & 20 & -25 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right]$$

Multiply the third row by -2 and add it to the first row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -25 & 26 & -33 \\ 0 & 1 & 0 & 4 & -4 & 5 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}$$

METHOD 2

Step 1: Construct a matrix of minors

$$\begin{bmatrix} 25 & 4 & -3 \\ 26 & 4 & -3 \\ 33 & 5 & -4 \end{bmatrix}$$

Step 2: Construct a matrix of cofactors

$$\begin{bmatrix} 25 & -4 & -3 \\ -26 & 4 & 3 \\ 33 & -5 & -4 \end{bmatrix}$$

Step 3: Find the adjugate by taking the transpose of the cofactor matrix

$$\begin{bmatrix} 25 & -26 & 33 \\ -4 & 4 & -5 \\ -3 & 3 & -4 \end{bmatrix}$$

Step 4: Find the determinant of the original matrix A

$$\begin{vmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ -3 & 4 \end{vmatrix} - 5\begin{vmatrix} 1 & 7 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 4 + 21 - 5(4) + 2(-3) = -1$$

Step 5: Apply the formula

$$A^{-1} = \frac{1}{|A|} Adj\ A = -\begin{bmatrix} 25 & -26 & 33 \\ -4 & 4 & -5 \\ -3 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}$$

b)

$$\begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -25(1)+26(1)-33(0) & -25(5)+26(1)-33(-3) & -25(2)+26(7)-33(4) \\ 4(1)-4(1)+5(0) & 4(5)-4(1)+5(-3) & 4(2)-4(7)+4(5) \\ 3(1)-3(1)+4(0) & 3(5)-3(1)+4(-3) & 3(2)-3(7)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3)

a)

$$\int_0^{\frac{1}{4}} x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}}\Big|_0^{\frac{1}{4}} = 2\left(\frac{1}{4}\right)^{\frac{1}{2}} = 2\left(\frac{1}{2}\right) = 1$$

b)

$$E(X) = \int_0^{\frac{1}{4}} x\,(x)^{-\frac{1}{2}}\,dx = \int_0^{\frac{1}{4}} x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^{\frac{1}{4}} = \frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{2}{3}\left(\frac{1}{8}\right) = \frac{1}{12}$$

c)

$$E\left(X^2\right) = \int_0^{\frac{1}{4}} x^2 x^{-\frac{1}{2}} dx = \int_0^{\frac{1}{4}} x^{\frac{3}{2}} dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_0^{\frac{1}{4}} = \frac{2}{5}\left(\frac{1}{4}\right)^{\frac{5}{2}} = \frac{2}{5}\left(\frac{1}{32}\right) = \frac{1}{80}$$

$$\left(E\left(X\right)\right)^2 = \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

$$Var(X) = E\left(X^2\right) - \left(E\left(X\right)\right)^2 = \frac{1}{80} - \frac{1}{144} = \frac{9}{720} - \frac{5}{720} = \frac{4}{720} = \frac{1}{180}$$

d)

$$\psi(t) = E\left(e^{ty}\right) = 3\int_{-\infty}^{0} e^{ty}e^{3y} dx = 3\int_{-\infty}^{0} e^{y(3+t)} dy = \begin{matrix} lim \\ a \to -\infty \end{matrix}\; 3\int_a^0 e^{y(3+t)} dy$$

$$= \begin{matrix} lim \\ a \to -\infty \end{matrix}\; \frac{3e^{y(3+t)}}{3+t}\Big|_a^0 = \begin{matrix} lim \\ a \to -\infty \end{matrix}\; \frac{3}{3+t}\left(e^{0(3+t)} - e^{a(3+t)}\right) = \frac{3}{3+t}$$

$$\psi^{'}(t) = -\frac{3}{(3+t)^2}$$

$$\psi^{''}(t) = \frac{2\,(3)\,(3+t)}{(3+t)^4} = \frac{6}{(3+t)^3}$$

$$\psi^{'}(0) = -\frac{1}{3} = E(Y)$$

$$\psi''(0) = \frac{6}{27} = \frac{2}{9} = E\left(Y^2\right)$$

$$Var(Y) = E\left(Y^2\right) - (E(Y))^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

e)

$$Var(3X + 4Y + 1) = 9Var(X) + 16Var(Y) + 2(3)(4)Cov(X, Y)$$

$$= 9\left(\frac{1}{180}\right) + 16\left(\frac{1}{9}\right) + 2(3)(4)(2) = \frac{9}{180} + \frac{320}{180} + \frac{8640}{180} = \frac{8969}{180} \approx 49.828$$

4)

a)

$$dU = U_x dx + U_y dy = (10x + 3y)\,dx + (3x + 4y)\,dy$$

b)

$$d^2U = d(dU) = d(U_x dx + U_y dy) = d(U_x dx) + d(U_y dy)$$

$$= U_{xx}dx^2 + U_{yx}dxdy + U_{xy}dydx + U_{yy}dy^2 = U_{xx}dx^2 + 2U_{xy}dxdy + U_{yy}dy^2$$

$$= 10dx^2 + 2(3)\,dxdy + 4dy^2 = 10dx^2 + 6dxdy + 4dy^2$$

c)

$$\begin{vmatrix} 10 & 3 \\ 3 & 4 \end{vmatrix} = 40 - 9 = 31$$

d) The first leading principal minor is $|10| > 0$ and the second leading principal minor is $31 > 0$, so d^2U is positive definite.

5)

a)

$$L = x_1^{0.4}x_2^{0.5} + \lambda(108 - 3x_1 - 4x_2)$$

b)

$$\frac{\partial L}{\partial x_1} = 0.4x_1^{-0.6}x_2^{0.5} - 3\lambda = 0 \to \lambda = \frac{0.4x_2^{0.5}}{3x_1^{0.6}}$$

$$\frac{\partial L}{\partial x_2} = 0.5x_1^{0.4}x_2^{-0.5} - 4\lambda = 0 \to \lambda = \frac{0.5x_1^{0.4}}{4x_2^{0.5}}$$

$$\frac{\partial L}{\partial \lambda} = 108 - 3x_1 - 4x_2 = 0$$

Setting the $\lambda' s$ equal,

$$\frac{0.4x_2^{0.5}}{3x_1^{0.6}} = \frac{0.5x_1^{0.4}}{4x_2^{0.5}} \rightarrow \frac{16}{10}x_2 = \frac{15}{10}x_1 \rightarrow x_1 = \frac{16}{15}x_2$$

Plugging into the budget constraint:

$$108 - 3\left(\frac{16}{15}x_2\right) - 4x_2 = 0 \rightarrow 108 - \frac{48}{15}x_2 - \frac{60}{42}x_2 = 0 \rightarrow 108 = \frac{108}{15}x_2 \rightarrow x_2^* = 15$$

Plugging back into the marginal rate of substitution

$$x_1^* = 16$$

c)

$$L_{11} = \frac{4}{10}\left(-\frac{6}{10}\right)x_1^{-1.6}x_2^{0.5} = -\frac{6}{25}x_1^{-1.6}x_2^{0.5}$$

$$L_{12} = \frac{4}{10}\frac{1}{2}x_1^{-0.6}x_2^{-0.5} = L_{21} = \frac{1}{5}x_1^{-0.6}x_2^{-0.5}$$

$$L_{22} = \frac{1}{2}\left(-\frac{1}{2}\right)x_1^{0.4}x_2^{-1.5} = -\frac{1}{4}x_1^{0.4}x_2^{-1.5}$$

$$g_1 = 3,\ g_2 = 4$$

Since x_1 and x_2 are always positive, the value of these variables will not change the sign of the bordered hessian, so we can form the bordered hessian with just the coefficients in order to find the sign. Forming the bordered Hessian with x_1 and x_2 and plugging in $x_1^* = 16$ and $x_2^* = 15$ is the correct way to find the exact value of the bordered Hessian, which will be positive.

$$\begin{vmatrix} 0 & 3 & 4 \\ 3 & -\frac{6}{25} & \frac{1}{5} \\ 4 & \frac{1}{5} & -\frac{1}{4} \end{vmatrix} = -3\begin{vmatrix} 3 & \frac{1}{5} \\ 4 & -\frac{1}{4} \end{vmatrix} + 4\begin{vmatrix} 3 & -\frac{6}{25} \\ 4 & \frac{1}{5} \end{vmatrix}$$

$$= -3\left(-\frac{3}{4} - \frac{4}{5}\right) + 4\left(\frac{3}{5} + \frac{24}{25}\right) > 0$$

Since the bordered Hessian is greater than 0, d^2u is negative definite, which means that $u\left(x_1^*, x_2^*\right)$ is a maximum.

6)

a) First, put the differential equation in the standard form

$$y' - \left(\frac{2}{x}\right)y = x$$

So, the integrating factor is

$$\mu(t) = e^{-\int \frac{2}{x}dx} = e^{-2ln|x|} = e^{ln\,x^{-2}} = \frac{1}{x^2}$$

Applying the formula

$$y(x) = \frac{\int \frac{1}{x^2}xdx + c}{\frac{1}{x^2}} = \frac{ln\,|x| + c}{\frac{1}{x^2}} = x^2\left(ln\,|x| + c\right)$$

b) This is a separable differential equation, so first put it in the proper form

$$(2y-4)\,dy = \left(3x^2+4x-4\right)dx$$

$$\int (2y-4)\,dy = \int \left(3x^2+4x-4\right)dx$$

$$y^2-4y = x^3+2x^2-4x+c$$

Next, let's apply the initial condition $y(1)=3$

$$(3)^2-4(3) = 1^3+2(1)^2-4(1)+c \to c=-2$$

So the implicit particular solution to the initial value problem is then

$$y^2-4y = x^3+2x^2-4x-2$$

To find the explicit solution, first rewrite as

$$y^2-4y-\left(x^3+2x^2-4x-2\right) = y^2-4y+\left(-x^3-2x^2+4x+2\right)=0$$

Use the quadratic formula

$$y(x) = \frac{4\pm\sqrt{16-4(1)\left(-x^3-2x^2+4x+2\right)}}{2}$$

$$= \frac{4\pm\sqrt{16+4\left(x^3+2x^2-4x-2\right)}}{2}$$

$$= 2\pm\sqrt{4+x^3+2x^2-4x-2}$$

Now, to figure out which one of the signs it should be, we must reapply the initial value

$$3 = y(1) = 2\pm\sqrt{1+2-4+2} = 2\pm 1 = 3,1$$

So, the "+" sign must be correct for our solution. So the explicit solution is

$$y(x) = 2+\sqrt{x^3+2x^2-4x+2}$$

7)

$$f(x) = (x+1)^{\frac{1}{2}} \qquad f(0)=1$$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} \qquad f'(0)=\frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}} \qquad f''(0)=-\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}} \qquad f'''(0)=\frac{3}{8}$$

$$f(x) = f(0) = f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$\sqrt{x+1} \approx 1+\frac{x}{2}-\frac{1}{4}\frac{1}{2!}x^2+\frac{3}{8}\frac{x^3}{3!} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$$

8)

a)

$$H_0 : \mu = 0$$
$$H_1 : \mu \neq 0$$

b)

$$t = \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{s} = \frac{\sqrt{31}\,(2.3 - 0)}{5} = 2.56$$

Now, we want to check whether this is significant at the 5% level, which with 30 degrees of freedom gives us $t_c = 2.042$.

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$$t > t_c \rightarrow 2.56 > 2.042$$

So, $\overline{X}$ is significant at the 5% level. That is, we reject the null hypothesis that $\mu = 0$ with 95% confidence.

c)
To find an approximation for the p-value, all we must do is look up the t-statistic and degrees of freedom on the t-table. The p-value is somewhere between 0.01 and 0.02, looks to be about 0.015.

d) A p-value of 0.015 means that if it was true that the true population mean actually is 0, then if we took an infinite number of samples from the population, the effect size we would get would be bigger than the one we found in our one sample roughly 1.5% of the time. So, in conventional significance level terms, we can say that our result is significant at the 5% level, but not the 1% level.

e) 99% confidence interval:

$$\left(\overline{X} - t_c\frac{s}{\sqrt{n}}, \overline{X} + t_c\frac{s}{\sqrt{n}}\right) = \left(2.3 - 2.75\left(\frac{5}{\sqrt{31}}\right), 2.3 + 2.75\left(\frac{5}{\sqrt{31}}\right)\right) = (-0.17, 4.77)$$

Since 0 is in the 99% confidence interval, we cannot say with 99% confidence that we are able to reject the null hypothesis.