ECON 186 Final Solutions

1)

a) Take the total derivative of the function:

$$0 = d\overline{U} = U_M d_M + U_C d_C$$

$$\frac{dM}{dC} = -\frac{U_C}{U_M}$$

b)

$$\frac{dM}{dC} = -\frac{U_C}{U_M} = -\frac{\frac{3}{4c}}{\frac{1}{4m}} = -\frac{\frac{3}{1}}{\frac{1}{8}} = -24$$

2)

a)

Row reduction method:

$$\left[\begin{array}{cc|cc} 6 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 6 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 7 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & 7 & 1 & -6 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} & -\frac{6}{7} \end{array}\right]$$

So the inverse is

$$\begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{6}{7} \end{bmatrix}$$

Cofactor method:

First, find the matrix of minors

$$\begin{bmatrix} -1 & 1 \\ 1 & 6 \end{bmatrix}$$

Next, find the matrix of cofactors

$$\begin{bmatrix} -1 & -1 \\ -1 & 6 \end{bmatrix}$$

Then find the tranpose of the matrix of cofactors, which is the adjugate

$$\begin{bmatrix} -1 & -1 \\ -1 & 6 \end{bmatrix}$$

Find the determinant of P

$$\begin{vmatrix} 6 & 1 \\ 1 & -1 \end{vmatrix} = -6 - 7 = -7$$

Apply the formula

$$P^{-1} = \frac{1}{-7}\begin{bmatrix} -1 & -1 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{6}{7} \end{bmatrix}$$

b)

$$P^{-1}AP = \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{6}{7} \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7}(2+1) & \frac{1}{7}(6-3) \\ \frac{1}{7}(2) - \frac{6}{7}(1) & \frac{1}{7}(6) + \frac{6}{7}(3) \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ -\frac{4}{7} & \frac{24}{7} \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7}(6+1) & \frac{3}{7}(1-1) \\ -\frac{4}{7}(6) + \frac{24}{7}(1) & -\frac{4}{7}(1) - \frac{24}{7} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

3)

a)

$$dU = U_x dx + U_y dy = (10x + 3y)\, dx + (3x + 4y)\, dy$$

b)

$$d^2U = d(dU) = d\left(U_x dx + U_y dy\right) = d\left(U_x dx\right) + d\left(U_y dy\right)$$

$$= U_{xx}dx^2 + U_{yx}dxdy + U_{xy}dydx + U_{yy}dy^2 = U_{xx}dx^2 + 2U_{xy}dxdy + U_{yy}dy^2$$
$$= 10dx^2 + 2\,(3)\,dxdy + 4dy^2 = 10dx^2 + 6dxdy + 4dy^2$$

c)

$$\begin{vmatrix} 10 & 3 \\ 3 & 4 \end{vmatrix} = 40 - 9 = 31$$

d) The first leading principal minor is $|10| > 0$ and the second leading principal minor is $31 > 0$, so d^2U is positive definite.

4)

$$f(x) = (x+1)^{\frac{1}{2}} \qquad f(0) = 1$$
$$f^{'}(x) = \frac{1}{2}\,(x+1)^{-\frac{1}{2}} \qquad f^{'}(0) = \frac{1}{2}$$
$$f^{''}(x) = -\frac{1}{4}\,(x+1)^{-\frac{3}{2}} \qquad f^{''}(0) = -\frac{1}{4}$$
$$f^{'''}(x) = \frac{3}{8}\,(x+1)^{-\frac{5}{2}} \qquad f^{'''}(0) = \frac{3}{8}$$

$$f(x) = f(0) = f^{'}(0)(x-0) + \frac{f^{''}(0)}{2!}(x-0)^2 + \frac{f^{'''}(0)}{3!}(x-0)^3$$

$$\sqrt{x+1} \approx 1 + \frac{x}{2} - \frac{1}{4}\frac{1}{2!}x^2 + \frac{3}{8}\frac{x^3}{3!} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

5)

a)

$$\int_0^{\frac{1}{4}} x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}}\Big|_0^{\frac{1}{4}} = 2\left(\frac{1}{4}\right)^{\frac{1}{2}} = 2\left(\frac{1}{2}\right) = 1$$

b)

$$E(X) = \int_0^{\frac{1}{4}} x\,(x)^{-\frac{1}{2}}\,dx = \int_0^{\frac{1}{4}} x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^{\frac{1}{4}} = \frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{2}{3}\left(\frac{1}{8}\right) = \frac{1}{12}$$

c)

$$E\left(X^2\right) = \int_0^{\frac{1}{4}} x^2 x^{-\frac{1}{2}} dx = \int_0^{\frac{1}{4}} x^{\frac{3}{2}} dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_0^{\frac{1}{4}} = \frac{2}{5}\left(\frac{1}{4}\right)^{\frac{5}{2}} = \frac{2}{5}\left(\frac{1}{32}\right) = \frac{1}{80}$$

$$(E\,(X))^2 = \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

$$Var(X) = E\left(X^2\right) - (E\,(X))^2 = \frac{1}{80} - \frac{1}{144} = \frac{9}{720} - \frac{5}{720} = \frac{4}{720} = \frac{1}{180}$$

d)

$$\psi(t) = E\left(e^{ty}\right) = \int_{-\infty}^{0} e^{ty}e^{3y}dx = \int_{-\infty}^{0} e^{y(3+t)}dy = \lim_{a \to -\infty} \int_a^0 e^{y(3+t)}dy$$

$$= \lim_{a \to -\infty} \frac{e^{y(3+t)}}{3+t}\Big|_a^0 = \lim_{a \to -\infty} \frac{1}{3+t}\left(e^{0(3+t)} - e^{a(3+t)}\right) = \frac{1}{3+t}$$

$$\psi^{'}(t) = -\frac{1}{(3+t)^2}$$

$$\psi^{''}(t) = \frac{2\,(3+t)}{(3+t)^4} = \frac{2}{(3+t)^3}$$

$$\psi^{'}(0) = -\frac{1}{9} = E(Y)$$

$$\psi^{''}(0) = \frac{2}{27} = E\left(Y^2\right)$$

$$Var(Y) = E\left(Y^2\right) - (E\,(Y))^2 = \frac{2}{27} - \frac{1}{81} = \frac{6}{81} - \frac{1}{81} = \frac{5}{81}$$

e) Find $Var\left(3X + 4Y + 1\right)$. Assume X and Y are correlated, and that $Cov(X, Y) = 2$.

$$Var(3X + 4Y + 1) = 9Var(X) + 16Var(Y) + 2\left(3\right)\left(4\right)Cov\left(X, Y\right)$$

$$= 9\left(\frac{1}{180}\right) + 16\left(\frac{5}{81}\right) + 2\left(3\right)\left(4\right)\left(2\right) = \frac{79441}{1620} \approx 49.0377$$

6)

a)

$$L = x_1^{0.4}x_2^{0.5} + \lambda\left(108 - 3x_1 - 4x_2\right)$$

b)

$$\frac{\partial L}{\partial x_1} = 0.4x_1^{-0.6}x_2^{0.5} - 3\lambda = 0 \rightarrow \lambda = \frac{0.4x_2^{0.5}}{3x_1^{0.6}}$$

$$\frac{\partial L}{\partial x_2} = 0.5x_1^{0.4}x_2^{-0.5} - 4\lambda = 0 \rightarrow \lambda = \frac{0.5x_1^{0.4}}{4x_2^{0.5}}$$

$$\frac{\partial L}{\partial \lambda} = 108 - 3x_1 - 4x_2 = 0$$

Setting the $\lambda^{'s}$ equal,

$$\frac{0.4x_2^{0.5}}{3x_1^{0.6}} = \frac{0.5x_1^{0.4}}{4x_2^{0.5}} \rightarrow \frac{16}{10}x_2 = \frac{15}{10}x_1 \rightarrow x_1 = \frac{16}{15}x_2$$

Plugging into the budget constraint:

$$108 - 3\left(\frac{16}{15}x_2\right) - 4x_2 = 0 \rightarrow 108 - \frac{48}{15}x_2 - \frac{60}{42}x_2 = 0 \rightarrow 108 = \frac{108}{15}x_2 \rightarrow x_2^* = 15$$

Plugging back into the marginal rate of substitution

$$x_1^* = 16$$

c)

$$L_{11} = \frac{4}{10}\left(-\frac{6}{10}\right)x_1^{-1.6}x_2^{0.5} = -\frac{6}{25}x_1^{-1.6}x_2^{0.5}$$

$$L_{12} = \frac{4}{10}\frac{1}{2}x_1^{-0.6}x_2^{-0.5} = L_{21} = \frac{1}{5}x_1^{-0.6}x_2^{-0.5}$$

$$L_{22} = \frac{1}{2}\left(-\frac{1}{2}\right)x_1^{0.4}x_2^{-1.5} = -\frac{1}{4}x_1^{0.4}x_2^{-1.5}$$

$$g_1 = 3,\ g_2 = 4$$

Since x_1 and x_2 are always positive, the value of these variables will not change the sign of the bordered hessian, so we can form the bordered hessian with just the coefficients.

$$\begin{vmatrix} 0 & 3 & 4 \\ 3 & -\frac{6}{25} & \frac{1}{5} \\ 4 & \frac{1}{5} & -\frac{1}{4} \end{vmatrix} = -3\begin{vmatrix} 3 & \frac{1}{5} \\ 4 & -\frac{1}{4} \end{vmatrix} + 4\begin{vmatrix} 3 & -\frac{6}{25} \\ 4 & \frac{1}{5} \end{vmatrix}$$

$$= -3\left(-\frac{3}{4} - \frac{4}{5}\right) + 4\left(\frac{3}{5} + \frac{24}{25}\right) > 0$$

Since the bordered Hessian is greater than 0, d^2u is negative definite, which means that $u\left(x_1^*, x_2^*\right)$ is a maximum.

7)

a) First, put the differential equation in the standard form

$$y^{'} - \left(\frac{2}{x}\right) y = x$$

So, the integrating factor is

$$\mu(t) = e^{-\int \frac{2}{x} dx} = e^{-2ln|x|} = e^{ln\, x^{-2}} = \frac{1}{x^2}$$

Applying the formula

$$y(x) = \frac{\int \frac{1}{x^2} x dx + c}{\frac{1}{x^2}} = \frac{ln\,|x| + c}{\frac{1}{x^2}} = x^2\left(ln\,|x| + c\right)$$

b) This is a separable differential equation, so first put it in the proper form

$$(2y - 4)\, dy = \left(3x^2 + 4x - 4\right) dx$$

$$\int (2y - 4)\, dy = \int \left(3x^2 + 4x - 4\right) dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

Next, let's apply the initial condition $y(1) = 3$

$$(3)^2 - 4(3) = 1^3 + 2(1)^2 - 4(1) + c \to c = -2$$

So the implicit particular solution to the initial value problem is then

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

To find the explicit solution, first rewrite as

$$y^2 - 4y - \left(x^3 + 2x^2 - 4x - 2\right) = 0$$

Use the quadratic formula

$$y(x) = \frac{4 \pm \sqrt{16 - 4(1)(-\left(x^3 + 2x^2 - 4x - 2\right)}}{2}$$

$$= \frac{4 \pm \sqrt{16 + 4\left(x^3 + 2x^2 - 4x - 2\right)}}{2}$$

$$= 2 \pm \sqrt{4 + x^3 + 2x^2 - 4x - 2}$$

Now, to figure out which one of the signs it should be, we must reapply the initial value

$$3 = y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm 1 = 3, 1$$

So, the " + " sign must be correct for our solution. So the explicit solution is

$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

8)

a)

$$H_0 : \widehat{\beta} = 0$$
$$H_1 : \widehat{\beta} \neq 0$$

b)

$$t = \frac{\sqrt{n}\left(\widehat{\beta} - \beta\right)}{s} = \frac{\sqrt{31}\,(2.3 - 0)}{5} = 2.56$$

Now, we want to check whether this is significant at the 5% level, which with 30 degrees of freedom gives us $t_c = 2.042$.

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$$t > t_c \rightarrow 2.56 > 2.042$$

So, $\widehat{B}$ is significant at the 5% level.

c)
To find an approximation for the p-value, all we must do is look up the t-statistic and degrees of freedom on the t-table. The p-value is somewhere between 0.01 and 0.02, looks to be about 0.015.

d) A p-value of 0.015 means that if it was true that interest rates have no effect on capital flows, if we took an infinite number of samples from the population, the effect size we would get would be bigger than the one we found in our one sample 1.5% of the time. So, in conventional significance level terms, we can say that our result is significant at the 5% level, but not the 1% level.

e) 99% confidence interval:

$$\left(\widehat{\beta} - t_c\frac{s}{\sqrt{n}}, \widehat{\beta} + t_c\frac{s}{\sqrt{n}}\right) = \left(2.3 - 2.75\left(\frac{5}{\sqrt{31}}\right), 2.3 + 2.75\left(\frac{5}{\sqrt{31}}\right)\right) = (-0.17, 4.77)$$

Since 0 is in the 99% confidence interval, we cannot say with 99% confidence that we are able to reject the null hypothesis.

9)

a)

$$Q\left(jK, jL\right) = \left(jK\right)\left(jL\right)^{\frac{1}{2}} = j^{\frac{3}{2}}\left(KL^{\frac{1}{2}}\right)$$

So Q is homogeneous of degree $\frac{3}{2}$, which means that it has increasing returns to scale.

b)

$$Q\left(jK, jL\right) = \left(jK\right)^{\frac{1}{4}}\left(jL\right)^{\frac{1}{4}} = j^{\frac{1}{2}}\left(KL\right)^{\frac{1}{4}}$$

So Q is homogeneous of degree $\frac{1}{2}$, which means that it has decreasing returns to scale.

c)

$$Q\left(jK, jL, jt\right) = \left(jK\right)^{\frac{1}{4}}\left(jK\right)^{\frac{1}{3}} + jt = j^{\frac{7}{12}}K^{\frac{1}{4}}L^{\frac{1}{3}} + jt$$

So in this case, Q is not homogeneous.