

ECON 186 Class Notes: Taylor and Maclaurin Series

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Power Series

- A power series is an infinite sum of power functions. Specifically, a power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

- A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots + c_n (x - a)^n + \dots$$

- where a and $c_0, c_1, \dots, c_n, \dots$, are constants.
- Theorem: If $\sum c_n (x - a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f :

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \quad a - R < x < a + R$$

- This function f has derivatives of all orders inside the interval of convergence.

Taylor and Maclaurin Series

- We now know that within its interval of convergence, the sum of a power series is a continuous function with derivatives of all orders. However, can a function $f(x)$ which we know has derivatives of all orders on an interval I be expressed as a power series on I ?
- Consider the function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots$$

- We now want to form a power series with the derivatives of $f(x)$, so let's find out some of its derivatives.

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots + na_n(x-a)^{n-1} + \dots$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \dots$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \dots$$

Taylor and Maclaurin Series

- Now, suppose that we evaluate each derivative at the point we are centering at, a . Then, we get

$$f'(a) = a_1$$

$$f''(a) = 1 \cdot 2a_2$$

$$f'''(a) = 1 \cdot 2 \cdot 3a_3$$

- So, we can now write the power series $f(x)$ using the derivatives of f evaluated at a .

Taylor and Maclaurin Series

- Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

- The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

Taylor and Maclaurin Series

- Taylor and Maclaurin series are used to approximate non-polynomial functions as a series of polynomials since polynomials are generally easy to work with and compute.
- A Taylor polynomial is any finite number of initial terms of a Taylor series.
- Note that Chiang and Wainwright define R_n to be the remainder where n means that there are n terms in the Taylor polynomial.

Example of Taylor Series

- Suppose we want to expand the nonpolynomial function

$$f(x) = \frac{1}{1+x}$$

- around the point $a = 1$ with $k = 4$.
- So, we must find the 1st-4th derivatives of $f(x)$ and evaluate them at $a = 1$.

$$f'(x) = -(1+x)^{-2} \qquad f'(1) = -\frac{1}{4}$$

$$f''(x) = 2(1+x)^{-3} \qquad f''(1) = \frac{1}{4}$$

$$f'''(x) = -6(1+x)^{-4} \qquad f'''(1) = -\frac{3}{8}$$

$$f^{(4)}(x) = 24(1+x)^{-5} \qquad f^{(4)}(1) = \frac{3}{4}$$

Example of Taylor Series

- Additionally, $f(1) = \frac{1}{2}$. So, applying the formula we get

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{4(2!)}(x-1)^2 \\ &\quad - \frac{3}{8(3!)}(x-1)^3 + \frac{3}{4(4!)}(x-1)^4 + R_4 \\ &= \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4 + R_4 \end{aligned}$$

- This is a 4th order Taylor polynomial with remainder.
- A first order Taylor polynomial is a linear approximation to a function.

Log-Linearization

- One of the most useful applications of Taylor series is that there are many nonlinear functions which we are not able to solve with analytical solutions, so it is often necessary to find a linear approximation to the function, so that numerical techniques can be used to evaluate the function.
- For example, suppose that we have the following nonlinear function

$$f(x) = \frac{g(x)}{h(x)}$$

- First, to linearize, we take the log of both sides

$$\ln f(x) = \ln g(x) - \ln h(x)$$

Log-Linearization

- Now use the first order Taylor series expansion around the point x^* .

$$\ln f(x) \approx \ln f(x^*) + \frac{f'(x^*)}{f(x^*)} (x - x^*)$$

$$\ln g(x) \approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)} (x - x^*)$$

$$\ln h(x) \approx \ln h(x^*) + \frac{h'(x^*)}{h(x^*)} (x - x^*)$$

Log-Linearization

- Then, putting these all together

$$\begin{aligned} & \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) \\ &= \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) - \ln h(x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*) \end{aligned}$$

- Grouping terms

$$\begin{aligned} & \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) \\ &= \ln g(x^*) - \ln h(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*) \end{aligned}$$

Log-Linearization

- Then, since $\ln f(x^*) = \ln g(x^*) - \ln h(x^*)$, these terms cancel out, leaving us with

$$\frac{f'(x^*)}{f(x^*)}(x - x^*) = \frac{g'(x^*)}{g(x^*)}(x - x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

- Although this is enough to be able to numerically evaluate the function at x^* , we often want to express the Taylor expansion as percent deviations from the point which we are expanding around, since x^* is often the steady-state of the system, and we want to know how much the system is deviating from the steady state.

Log-Linearization

- To do this, first multiply and divide each term by x^* .

$$\frac{x^* f'(x^*)}{f(x^*)} \frac{(x-x^*)}{x^*} = \frac{x^* g'(x^*)}{g(x^*)} \frac{(x-x^*)}{x^*} - \frac{x^* h'(x^*)}{h(x^*)} \frac{(x-x^*)}{x^*}$$

- For convenience, define $\tilde{x} = \frac{(x-x^*)}{x^*}$, which is the percentage deviation of x about x^* . Then, our final result is

$$\frac{x^* f'(x^*)}{f(x^*)} \tilde{x} = \frac{x^* g'(x^*)}{g(x^*)} \tilde{x} - \frac{x^* h'(x^*)}{h(x^*)} \tilde{x}$$

Example Log-Linearization

- Consider the closed economy accounting identity

$$y_t = c_t + i_t$$

- Take logs:

$$\ln y_t = \ln(c_t + i_t)$$

- First order Taylor series expansion:

$$\ln y^* + \frac{1}{y^*} (y_t - y^*) = \ln(c^* + i^*) + \frac{1}{(c^* + i^*)} (c_t - c^*) + \frac{1}{(c^* + i^*)} (i_t - i^*)$$

- Note that $\ln(c^* + i^*) = \ln y^*$, so these terms cancel and we are left with

$$\frac{1}{y^*} (y_t - y^*) = \frac{1}{(c^* + i^*)} (c_t - c^*) + \frac{1}{(c^* + i^*)} (i_t - i^*)$$

Example Log-Linearization

- Now, multiply and divide the terms on the right by c^* and i^* , respectively.

$$\frac{1}{y^*} (y_t - y^*) = \frac{c^*}{(c^* + i^*)} \frac{(c_t - c^*)}{c^*} + \frac{i^*}{(c^* + i^*)} \frac{(i_t - i^*)}{i^*}$$

- Then, define

$$\tilde{y}_t = \frac{(y_t - y^*)}{y^*}, \quad \tilde{c}_t = \frac{(c_t - c^*)}{c^*}, \quad \tilde{i}_t = \frac{(i_t - i^*)}{i^*}$$

- Then, the log-linearized version of the accounting identity is

$$\tilde{y}_t = \frac{c^*}{(c^* + i^*)} \tilde{c}_t + \frac{i^*}{(c^* + i^*)} \tilde{i}_t = \frac{c^*}{y^*} \tilde{c}_t + \frac{i^*}{y^*} \tilde{i}_t$$