ECON 186 Class Notes: Optimization Part 1

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Introduction to Optimization

- Up until this point, we have considered economic equilibrium which is achieved through market forces (such as supply and demand), but the entities involved are not actually vying for any equilibrium.
- For example, consider that a supply curve for a good is made up of many firms, each trying to maximize their own profits (or analogously, minimizing cost). Furthermore, on the demand side, each individual is buying the goods that will maximize their own utility. If we consider all of these choices, a market equilibrium is attained, but we are currently interested in what the optimal choices look like for a specific entity.
- To do this, we must learn how to find optimum and how various entities optimize with and without constraints.

Optimization Terminology

- In optimization problems, the objective function is the equation which is to be optimized.
- The dependent variable is the object of maximization.
- The independent variables, which the economic entity is allowed to choose in order to optimize are called the choice variables.
- Example: Suppose that a firm maximizes profits, which is the difference between total revenue, R, and total cost, C, which both depend on the total quantity Q. Then, the firm seeks to maximize

$$\pi(Q) = R(Q) - C(Q)$$

- This represents the objective function, where π is the object of maximization and Q is the only choice variable.

Relative and Absolute Extremum

- Once again, suppose we have a function $y = f(x)$.
- Recall that the derivative of a function (which we will now refer to as the first derivative) at a point x_0 represents the slope of the tangent line to the curve at x_0.
- Therefore, consider a point where $f'(x_0) = 0$. In this case, the slope of the tangent line at x_0 must be 0, which means $\frac{\Delta y}{\Delta x} = 0$, so when x changes, y does not change.
- This is only possible if x_0 is in fact a relative maximum or minimum of $f(x)$.
  - A function $f(x,y)$ has a relative minimum at the point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some region around (a,b).
  - A function $f(x,y)$ has a relative maximum at the point (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some region around (a,b).
  - On the other hand, a global maximum or minimum is the largest value that the function takes in its entire range.

First-Derivative Test

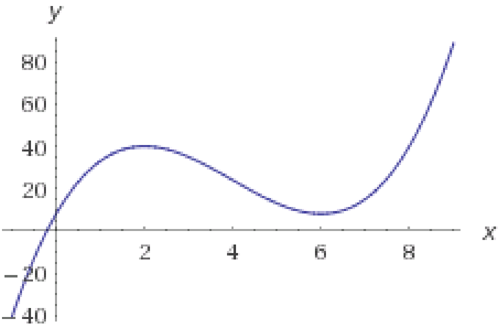
- First-derivative test for relative extremum: If the first derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be
  - a. A relative maximum if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.
  - b. A relative minimum if $f'(x)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right.
  - c. Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and the immediate right of point x_0.
- If $f'(x_0) = 0$, then x_0 is called a critical value and $f(x_0)$ is a stationary value of y.

First-Derivative Test

- $f^{'}(x_0) = 0$ is only a necessary condition for $f(x_0)$ to be a relative extremum since if $f^{'}(x_0) = 0$, $f(x_0)$ could still be an inflection point, which is not a relative extrema.
- So, one method to find relative extrema is to first find the stationary values associated where the condition $f'(x) = 0$ is satisfied, and then to apply the first-derivative test to determine whether the stationary value is a maximum or minimum.

Example of First-Derivative Test

- Find the relative extrema of the function $y = f(x) = x^3 - 12x^2 + 36x + 8$.
- First, we must find the values where $f'(x) = 0$.
  - $f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12) = 3(x-6)(x-2)$
- Setting equal to 0, we can easily see the roots of the polynomial are $x = 2, 6$.
- Then, plug in points just to the left and right of each root to tell if each one is a relative minimum or maximum (or neither).
  - $3(5.99-6)(5.99-2) < 0$ and $3(6.01-6)(6.01-2) > 0$ so $x = 6$ is a relative minimum.
  - $3(1.99-5)(1.99-2) > 0$ and $3(2.01-5)(2.01-2) < 0$ so $x = 2$ is a relative maximum.
- Are $x = 2$ and $x = 6$ also global maximums and minimums, respectively?



y
80
60
40
20
−20
−40
x
2
4
6
8

Second and Higher Derivatives

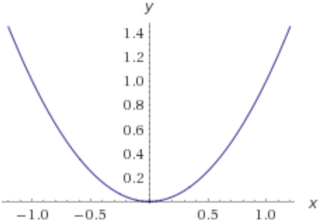
- As noted before, we refer to the derivative of a function as the first derivative. Then, the second derivative is the derivative of the derivative, which we denote either by $f''(x)$ or $\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$.
- If $f''(x)$ exists for all x values in the domain of $f(x)$, the function is twice differentiable.
- Analogously, the *nth* derivative is denoted $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$.
- Example 1: Find the first through fifth derivatives of the function $y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$
  - $f'(x) = 16x^3 - 3x^2 + 34x + 3$, $f''(x) = 48x^2 - 6x + 34$, $f'''(x) = 96x - 6$, $f^{(4)}(x) = 96$, $f^{(5)}(x) = 0$
- Example 2: Find the first four derivatives of the function $y = g(x) = \frac{x}{1+x}$ where $x \neq -1$.
  - $g'(x) = (1+x)^{-2}$, $g''(x) = -2(1+x)^{-3}$, $g'''(x) = 6(1+x)^{-4}$, $g^{(4)}(x) = -24(1+x)^{-5}$

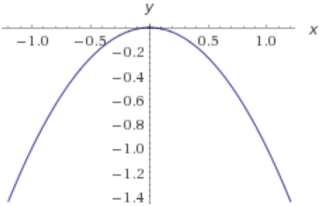
Interpretation of the Second Derivative

- We know that the first derivative represents the rate of change of the original function, $f(x)$. The second derivative then represents the rate of change of the first derivative.
- Suppose that there is an infinitesimal increase in x from a point $x = x_0$.
  - $f'(x_0) > 0$ means that the value of the function tends to increase.
  - $f'(x_0) < 0$ means that the value of the function tends to decrease.
  - $f''(x_0) > 0$ means that the slope of the curve tends to increase.
  - $f''(x_0) < 0$ means that the slope of the curve tends to decrease.

Convexity and Concavity

- If $f''(x) \leq 0$ for all x, then $f(x)$ is called a concave function.
- If $f''(x) < 0$ for all x, then $f(x)$ is called a strictly concave function.
- If $f''(x) \geq 0$ for all x, then $f(x)$ is called a convex function.
- If $f''(x) > 0$ for all x, then $f(x)$ is called a strictly convex function.
- Additionally, a function may be concave or convex on an interval of $f(x)$.


y
1.4
1.2
1.0
0.8
0.6
0.4
0.2
x
−1.0
−0.5
0.5
1.0


y
x
−1.0
−0.5
0.5
1.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4

Example - Attitudes Toward Risk

- Consider a game where for a fixed sum of money paid before hand (the cost of the game), you can throw a die and collect \$10 if an odd number shows or \$20 if an even number shows up. So, since there is an equal probability, the expected value is

$$EV = 0.5(\$10) + 0.5(\$20) = \$15$$

- Therefore, the "fair" cost of the game would be \$15. However, some people, which we call risk-averse, may not want to play the game even if it is "fair." Others, which we call risk-preferring, may want to play the game even if it is less than fair.

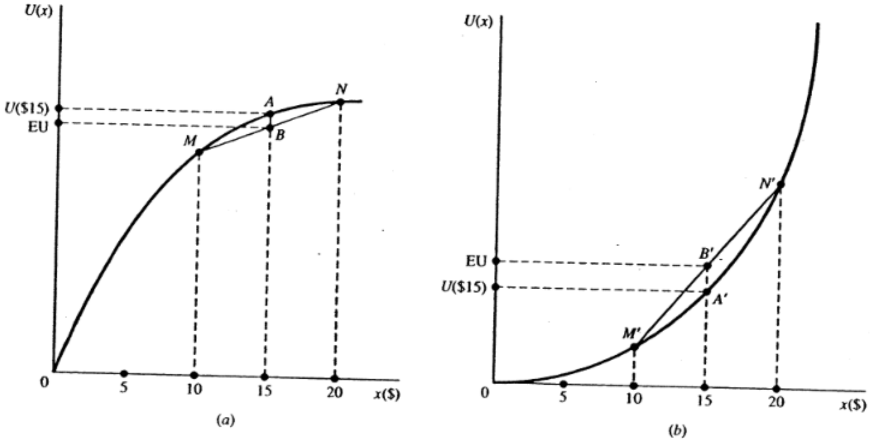
Example - Attitudes Toward Risk

- The difference in risk preference is due to individuals utility function, $U = U(x)$ where x is the payoff, with $U(0) = 0$.
- Then, the expected utility for any individual playing this game is

$$EU = 0.5xU(\$10) + 0.5xU(\$20)$$

- So, whether an individual will play the fair game depends on their utility from each payoff, and therefore the shape of their utility function.

Example - Attitudes Toward Risk



Example - Attitudes Toward Risk

- A strictly concave utility function represents a risk-averse individual.
- A strictly convex utility function represents a risk-preferring individual.
- A linear utility function represents a risk neutral individual.

Second-Derivative Test

- Previously, we learned that we can find if a point is a relative maximum or minimum by checking the first derivative just to each side of the point. However, we are also able to use the second derivative, which is usually quicker and easier.
- Second-derivative test for relative extremum: If the value of the first derivative of a function f at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be
  - a. A relative maximum if the second-derivative value at x_0 is $f''(x_0) < 0$.
  - b. A relative minimum if the second-derivative value at x_0 is $f''(x_0) > 0$.
- Example: Find the relative extrema of the function $y = f(x) = 4x^2 - x$.
  - $f'(x) = 8x - 1$, so the only critical value is $x = \frac{1}{8}$.
  - $f''(x) = 8$, so $x = \frac{1}{8}$ is a relative minimum.

Economic Application - Profit Maximization

- Suppose that a firm has the total cost function $C(q)$ and total revenue function $R(q)$. The firm then seeks to maximize profits by choosing the quantity that they are to produce. The objective function of this maximization problem (known as the profit function) is specified as:

$$\pi = \pi(q) = R(q) - C(q)$$

$$\frac{d\pi}{dq} = \pi'(q) = R'(q) - C'(q) = MR - MC = 0 \rightarrow MR = MC$$

- So, profit is either maximized or minimized at the quantity where marginal revenue equals marginal cost. This is known as the first order condition of the optimization problem.

Economic Application - Profit Maximization

- To ensure that this is actually a maximum, we must check the second order condition.

$$\frac{d^2\pi}{dq^2} \equiv \pi^{''}(q) = R^{''}(q) - C^{''}(q) \leq 0 \rightarrow R^{''}(q) < C^{''}(q)$$

- This says that the rate of change of MR is less than the rate of change of MC at the output where MC=MR.

Economic Application - Profit Maximization

- Suppose that a firm has costs given by $C(q) = 120 + 2q^2$ and revenue given by $R(q) = 100q$. What is their profit maximizing quantity?
- $\pi(q) = R(q) - C(q) = 100q - 120 - 2q^2$

$$\frac{d\pi}{dq} = 100 - 4q = 0 \rightarrow q^* = 25$$

- $R^{'}(q) = 100,\ R^{''}(q) = 0,\ C^{'}(q) = 4q,\ C^{''}(q) = 4$
- $R^{''}(q) \leq C^{''}(q) \rightarrow 0 \leq 4$, so $q^* = 25$ is in fact a maximum.

Optimization with more than one Choice Variable

- Suppose we have a function $z = f(x, y)$. The first order condition is then
$$f_x = f_y = 0$$
- In order to describe the second order conditions of an optimization problem with two or more variables, we must understand cross partials and Young's Theorem.
- Just like we can take the partial derivative with respect to the same argument multiple times, we may also look at cross-partials:
$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$
- Young's Theorem: If both f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Second Order Conditions

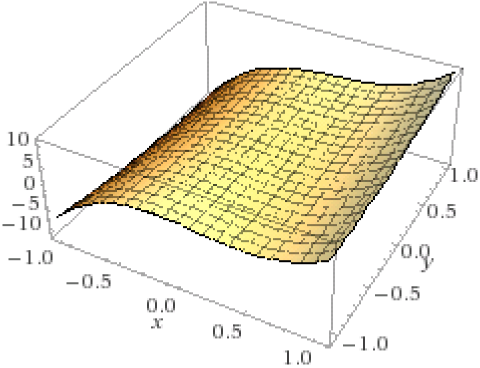
- To determine the necessary and sufficient second order conditions, we must find the second order total differential $z = f(x, y)$.
  - $d^2z = d[dz] = d[f_x dx + f_y dy] = d[f_x dx] + d[f_y dy] = f_{xx}dx^2 + f_{yx}dxdy + f_{xy}dydx + f_{yy}dy^2 = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$
- ▪ Necessary conditions:
  - $d^2z \leq 0$ gives a maximum and $d^2z \geq 0$ gives a minimum.
- Sufficient conditions:
  - $d^2z < 0$ which occurs if and only if $f_{xx} < 0$ and $f_{yy} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$ gives a maximum.
  - $d^2z > 0$ which occurs if and only if $f_{xx} > 0$ and $f_{yy} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$ gives a minimum.

FOC and SOC Example

- Find the extreme values of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. Are they maximums or minimums?
  - $f_x = 24x^2 + 2y - 6x$, $f_y = 2x + 2y$, $f_{xx} = 48x - 6$, $f_{yy} = 2$, $f_{xy} = f_{yx} = 2$
  - First order condition: $24x^2 + 2y - 6x = 0$ and $2x + 2y = 0 \to x = -y$.
  - Substituting into the first equation, we get $24x^2 - 2x - 6x = 0 \to 24x = 8 \to x_1^* = \frac{1}{3}$. So, $y_1^* = -\frac{1}{3}$.
  - From either equation, we can see that another solution is $x_2^* = y_2^* = 0$.

FOC and SOC Example

- When $x_2^* = y_2^* = 0$, $f_{xx} = -6$, $f_{yy} = 2$, $f_{xy} = 2$, $f_{xy}^2 = 4$, $f_{xx}f_{yy} = -12 < 4$, so $(x_2^*, y_2^*) = (0,0)$ fails the second order condition and is a saddle point since f_{xx} and f_{yy} have different signs.
- When $x_1^* = \frac{1}{3}$ and $y_1^* = -\frac{1}{3}$, $f_{xx} = 10$, $f_{yy} = 2$, $f_{xx}f_{yy} = 20$, $f_{xy}^2 = 4$, so $f_{xx}f_{yy} > f_{xy}^2$. The point $(x_1^*, y_1^*) = \left(\frac{1}{3}, -\frac{1}{3}\right)$ satisfies the sufficient condition for a relative minimum.
- Plugging in, we get $z = 8\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)\left(-\frac{1}{3}\right) - 3\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 1 = \frac{23}{27}$. So, the only relative minimum of this function is $(x^*, y^*, z^*) = \left(\frac{1}{3}, -\frac{1}{3}, \frac{23}{27}\right)$


10
5
0
−5
−10
−1.0
−0.5
0.0
x
0.5
1.0
−1.0
−0.5
0.0
y
0.5
1.0

Quadratic Forms

- Since determining second order conditions is of vital importance, we establish criteria for determining the sign of quadratic forms for arbitrary dx and dy.
- For convenience, let's rename each of the derivatives.
  - $u \equiv dx$, $v \equiv dy$, $a \equiv f_{xx}$, $b \equiv f_{yy}$, $h \equiv f_{xy} \equiv f_{yx}$
  - So, the second order total differential is $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \rightarrow q = au^2 + 2huv + bv^2$
  - In this form, we are treating u and v as variables, and a, h, and b as constants. We want to know what restrictions must be placed upon a, b, and h while u and v are allowed to take any values, to ensure a definite sign for q?

Quadratic Forms

- A quadratic form q is said to be
  - Positive definite if q is invariably positive (> 0).
  - Positive semidefinite if q is invariably nonnegative(≥ 0).
  - Negative semidefinite if q is invariably nonpositive(≤ 0).
  - Negative definite if q is invariably negative(< 0).
    - ⋆ regardless of the values of u and v (the variables in the quadratic form). Otherwise, q is indefinite.

Quadratic Forms

- Notice also that we can rewrite q as a square,

$$\begin{matrix} q = a(u^2) + h(uv) \\ +h(vu) + b(v^2) \end{matrix} = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

- The determinant of the 2×2 coefficient matrix $\begin{vmatrix} a & h \\ h & b \end{vmatrix} = |D|$, is known as the discriminant of the quadratic form q.
- q is $\left\{ \begin{matrix} \textit{positive definite} \\ \textit{negative definite} \end{matrix} \right\}$ iff $\left\{ \begin{matrix} |a| > 0 \\ |a| < 0 \end{matrix} \right\}$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$
  - where $|a| = a$ is the first leading principal minor of $|D|$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is the second leading principal minor of $|D|$.

Quadratic Forms

- Translating back into our previous notation, we now see where the previous second order condition came from:
- - d^2z is $\left\{ \begin{array}{c} \textit{positive definite} \\ \textit{negative definite} \end{array} \right\}$ iff $\left\{ \begin{array}{c} f_{xx} > 0 \\ f_{xx} < 0 \end{array} \right\}$ and

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 > 0 \rightarrow f_{xx}f_{yy} > f_{xy}^2$$