ECON 186 Class Notes: Differential Equations

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Differential Equations

- Many times and in many different ways, we have solved equations. Consider the equation $\frac{dy}{dx} = 2x$. We solve this equation by integration.

$$\int dy = \int 2x dx \to y = x^2 + c$$

- This is an example of a differential equation, which is any equation which involves a derivative. Many differential equations we can solve simply by integrating, but how about an equation of the form

$$\frac{1}{y}\frac{dy}{dx} = -a$$

- This type of differential equation we cannot simply integrate to find the solution, so we must learn new techniques to solve other types of differential equations.

Differential Equations

- As economists, why do we care about how to solve these types of differential equations?
- The primary use of differential equations in general is to model motion, which is commonly called growth in economics. Specifically, a differential equation expresses the rate of change of the current state as a function of the current state.
- Example: Suppose that GDP grows at some constant rate g over time. Then, if y is the current level of GDP in the economy, we can express the relationship as

$$\frac{dy}{dt} = \dot{y} = gy(t)$$

- We will soon discover that this differential equation has a simple solution:

$$y(t) = ce^{gt}$$

Direction Fields

- Differential equations are useful because there is a differential equation to describe nearly any physical situation. Or, for our purposes, we can use them to model many economic situations as well.
- Given this, we want to be able to describe how some variable changes over time, regardless of whether we are actually able to solve a given differential equation.
- Using arrows to represent the direction of motion, we can create geometric representations of differential equations called direction fields.
- Consider the differential equation

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

Direction Fields

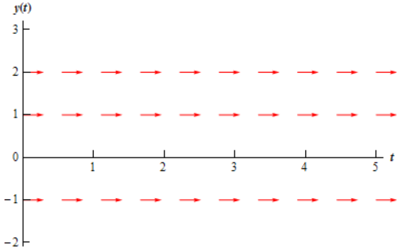
- Although we do not know how to find the solution of this differential equation, we can still take steps to characterize the solution.
- First, we must find where the derivative is zero

$$0 = \left(y^2 - y - 2\right)(1-y)^2 = (y-2)(y+1)(1-y)^2$$

- So, the derivative is zero at three values, $y = -1, 1, 2$.

Direction Fields

- So, since the derivative is zero, this means that the slope of the tangent lines are zero at these values. So, we will start our direction field with this knowledge.



- So, the graph is now divided into four regions. We now want to see how y evolves over time within each region. To do this, plug in values of y within each region and check the sign and magnitude of the derivative.

Direction Fields

- First, we check the region $y < -1$, so we first plug in $y = -2$ to get

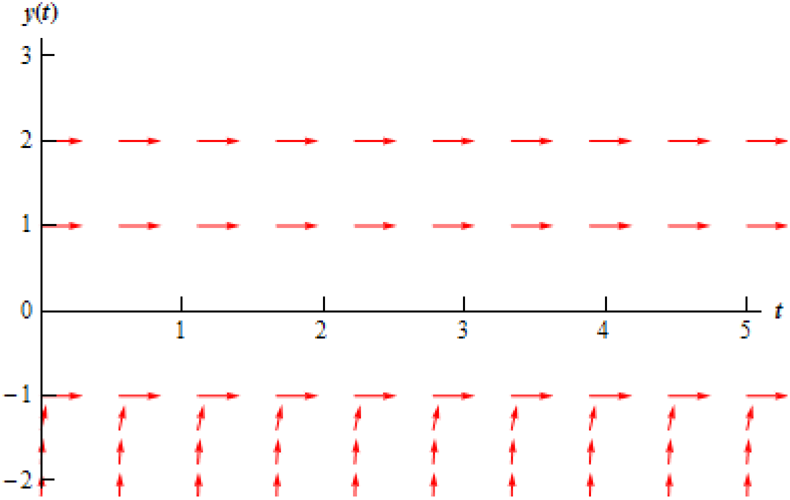
$$y^{'} = (-2-2)(-2+1)(1-(-2))^2 = \left(-4 * -1 * 3^2\right) = 36$$

- However, if we then look even closer to the edge of the region, say, $y = -1.1$, we get

$$y^{'} = (-1.1-2)(-1.1+1)(1-(-1.1))^2 = -3.1 * -.1 * -2.1^2 = 1.3671$$

- So, the slope is very steep and positive as y becomes more negative, and very close to $y = -1$, the slope is still positive but less steep.

Direction Fields



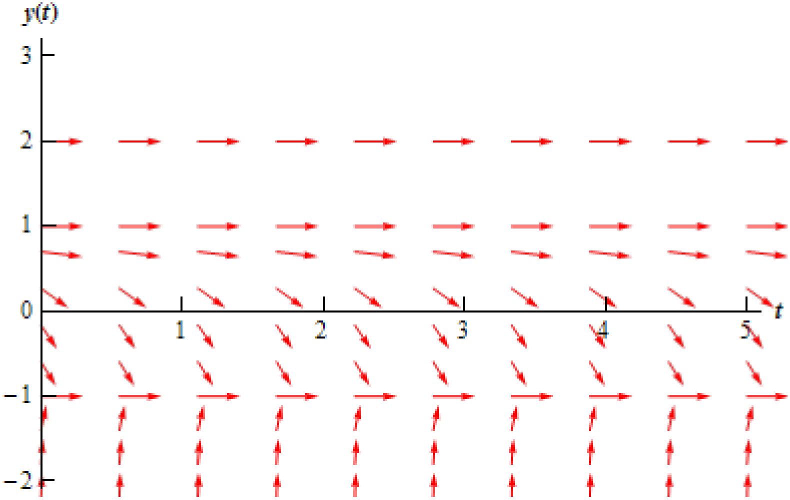
Direction Fields

- Next, we check the middle region $-1 < y < 1$. We will check two points very near the boundary. Let's check $y = -0.9$ and $y = 0.9$.

$$y' = (-0.9 - 2)(-0.9 + 1)(1 - (-0.9))^2 = -1.0469$$

$$y' = (0.9 - 2)(0.9 + 1)(1 - 0.9)^2 = -0.0209$$

Direction Fields



Direction Fields

- Next, check the region $1 < y < 2$. Check $y = 1.1$ and $y = 1.9$.

$$y^{'} = (1.1 - 2)(1.1 + 1)(1 - (1.1))^2 = -0.0189$$

$$y^{'} = (1.9 - 2)(1.9 + 1)(1 - (1.9))^2 = -0.2349$$

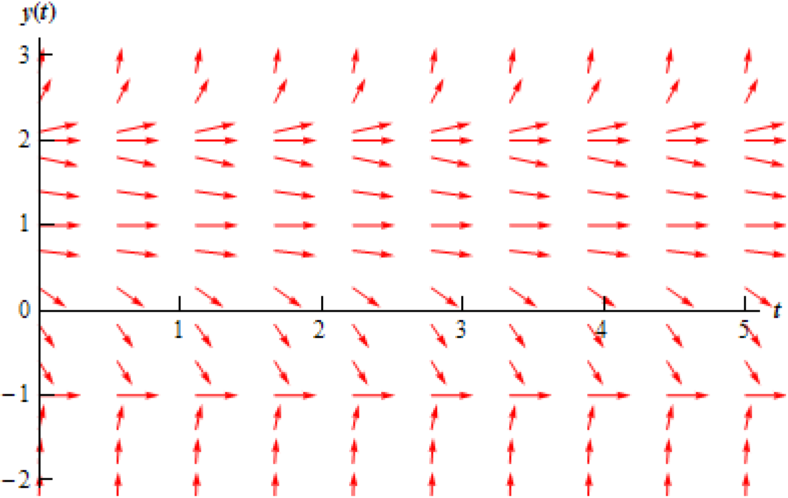
Direction Fields

- Finally, we want to check the region $y > 2$. Let's check $y = 2.1$ and $y = 3$.

$$y' = (2.1 - 2)(2.1 + 1)(1 - 2.1)^2 = 0.3751$$

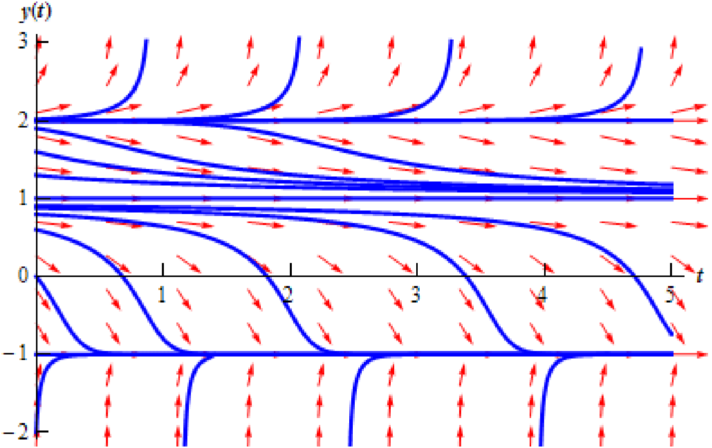
$$y' = (3 - 2)(3 + 1)(1 - 3)^2 = 16$$

Direction Fields



Direction Fields

- Then, we want to draw a set of integral curves, which are the path a variable will take given a certain initial condition. A set of integral curves is displayed in the following



Direction Fields

- Finally, we can characterize the behavior of y based on the initial condition of y as $t \to \infty$.

Value of y(0)	Behavior as $t \to \infty$
$y(0) < 1$	$y \to -1$
$1 \leq y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

Differential Equations

- The order of a differential equation is the order of the highest derivative that appears in the equation.
- A linear differential equation is any differential equation that can be written in the following form:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y^t(t) + a_0(t)y(t) = g(t)$$

- The important thing to note about linear differential equations is that there are no products of the function $y(t)$ and any of its derivatives, and neither $y(t)$ nor any of its derivatives occur to any power other than the first.
- Ordinary differential equations (ode) are equations involving only ordinary derivatives, while partial differential equations (pde) involve partial derivatives.
- A homogeneous differential equation with dependent variable y is a differential equation where each term must contain y or a derivative of y.

Derivation of Solution of First Order Linear Differential Equations

- Thus far we have considered a simple example of GDP growth, but now we want to consider all linear first order differential equations.

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

- Our goal is to find an explicit solution of this differential equation, that is, y by itself on the left hand side.
- Now, suppose that there is some function $\mu(t)$ which we call the integrating factor. Multiply through by $\mu(t)$ to get

$$\frac{dy}{dt}\mu(t) + \mu(t)p(t)y = \mu(t)g(t) \tag{2}$$

- Assume that

$$\mu(t)p(t) = \mu^{'}(t) \tag{3}$$

Derivation of Solution of First Order Linear Differential Equations

- Then, plugging (3) into (2), we get

$$\mu(t)\frac{dy}{dt} + \mu^{'}(t)y = \mu(t)g(t) \tag{4}$$

- Then, notice that the left hand side is in fact the product rule of y and μ:

$$\mu(t)\frac{dy}{dt} + \mu^{'}(t)y = (\mu(t)y(t))^{'}$$

- So, we can write (4) as

$$(\mu(t)y(t))^{'} = \mu(t)g(t) \tag{5}$$

- Then, integrate both sides to get

$$\int (\mu(t)y(t))^{'}dt = \int \mu(t)g(t)dt \rightarrow \mu(t)y(t) + c = \int \mu(t)g(t)dt \tag{6}$$

Derivation of Solution of First Order Linear Differential Equations

- From (6), we can simplify to get

$$\mu(t)y(t) = \int \mu(t)g(t)dt - c$$

$$y(t) = \frac{\int \mu(t)g(t)dt - c}{\mu(t)}$$

- For convenience, since c is an unknown constant, we will change the minus sign to a plus, so that the solution is

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}$$

- However, $\mu(t)$ is simply some function we decided to use in our derivation, but what is it?

Derivation of Solution of First Order Linear Differential Equations

- To determine what $\mu(t)$ is, we must go back to our assumption in equation (3)

$$\mu(t)p(t) = \mu^{'}(t)$$

- Divide both sides by $\mu(t)$:

$$\frac{\mu^{'}(t)}{\mu(t)} = p(t)$$

- Recall that the left hand side is simply the natural logarithm of $\mu(t)$, so we can rewrite as

$$(ln(\mu(t)))^{'} = p(t)$$

Derivation of Solution of First Order Linear Differential Equations

- Then, integrate both sides to get

$$ln(\mu(t)) + k = \int p(t)dt$$

$$ln(\mu(t)) = \int p(t)dt + k$$

$$\mu(t) = e^{\int p(t)dt + k}$$

Derivation of Solution of First Order Linear Differential Equations

- Although this is a correct solution of $\mu(t)$, having the constant k in the exponent is inconvenient, so using the properties of exponents, note that
$$\mu(t) = e^{\int p(t)dt + k} = e^k e^{\int p(t)dt}$$
- Again, k is just some unknown constant, so let's just rename e^k as k so that we can write $\mu(t)$ as
$$\mu(t) = ke^{\int p(t)dt}$$
- Now note however that by combining our solutions for $y(t)$ and $\mu(t)$, we get
$$y(t) = \frac{\int ke^{\int p(t)dt} g(t)dt + c}{ke^{\int p(t)dt}} = \frac{\int e^{\int p(t)dt} g(t)dt + \frac{c}{k}}{e^{\int p(t)dt}}$$
- Then, since c and k are arbitrary constants, so is a ratio of the two, so we will call this ratio c. Then, we can just drop k from the solution of $\mu(t)$ since it will be absorbed into c anyway.

First Order Linear Differential Equations

- So, we can express the solution to a general first order linear differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

- as the following two equations

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}$$

$$\mu(t) = e^{\int p(t)dt} \tag{7}$$

First Order Linear Differential Equations Examples

- Let's return to the GDP growth example given by the differential equation

$$\frac{dy}{dt} = gy$$

- First, put the differential equation in the proper form

$$\frac{dy}{dt} - gy = 0$$

- Apply the formulas

$$\mu(t) = e^{\int -g dt} = e^{-gt}$$

$$y(t) = \frac{\int e^{-gt}(0)dt + c}{e^{-gt}} = ce^{gt}$$

First Order Linear Differential Equations Examples

- Consider the nonhomogeneous differential equation

$$\frac{dv}{dt} = 9.8 - 0.196v$$

- To find the solution, we must first put the differential equation in the proper form

$$\frac{dv}{dt} + 0.196v = 9.8$$

- Apply the formulas

$$\mu(t) = e^{\int 0.196 dt} = e^{0.196t}$$

$$v(t) = \frac{\int e^{0.196t}(9.8)dt + c}{e^{0.196t}} = \frac{\frac{9.8}{0.196}e^{0.196t} + c}{e^{0.196t}} = 50 + ce^{-0.196t}$$

First Order Linear Differential Equations Examples

- The solution to the previous problem is known as a general solution. However, if we specify initial conditions, we can find a particular solution. For first order differential equations, one initial condition will always give a particular solution. This type of problem is known as an initial value problem.
- Consider the previous problem, but now with an initial condition for v.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad\qquad v(0) = 48$$

- We need to find the value of c that satisfies the initial condition.

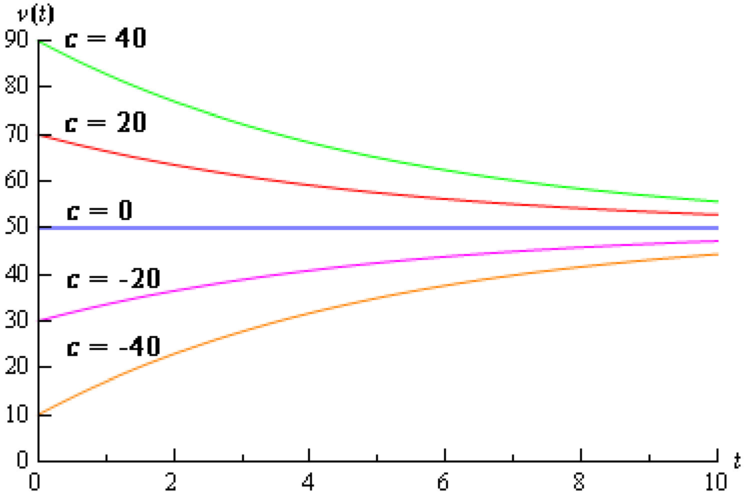
$$v(t) = 50 + ce^{-0.196t} \rightarrow v(0) = 50 + ce^{-0.196(0)}$$

$$= 50 + c \rightarrow 48 = 50 + c \rightarrow c = -2$$

- So, the particular solution to the initial value problem is

$$v = 50 - 2e^{-0.196t}$$

First Order Linear Differential Equations Examples



First Order Linear Differential Equations Examples

- Find the solution to the following initial value problem.

$$ty^{'} + 2y = t^2 - t + 1 \qquad y(1) = \frac{1}{2}$$

- First, divide through by t to put the differential equation in the proper form.

$$y^{'} + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

- Then, apply the formulas

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2ln|t|} = e^{ln|t|^2} = |t|^2 = t^2$$

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)} = \frac{\int t^2 \left(t - 1 + \frac{1}{t}\right)dt + c}{t^2} = \frac{\int \left(t^3 - t^2 + t\right)dt + c}{t^2}$$

$$= \frac{\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c}{t^2} = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + ct^{-2}$$

First Order Linear Differential Equations Examples

- Applying the initial value, we get

$$y(1) = \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c = \frac{5}{12} + c \to \frac{6}{12} = \frac{5}{12} + c \to c = \frac{1}{12}$$

- So the final solution is

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$$

Separable Differential Equations

- A separable differential equation is any differential equation that can be written in the following form:

$$N(y)\frac{dy}{dx} = M(x)$$

- To solve, first rewrite as follows:

$$N(y)dy = M(x)dx$$

- Then, integrate both sides

$$\int N(y)dy = \int M(x)dx$$

- This will provide an implicit solution that we may sometimes, but not always, solve for an explicit solution.

Separable Differential Equations Example

- Solve the following separable differential equation

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

- First, rewrite as

$$\frac{1}{6y^2}dy = xdx$$

- Then, integrate both sides

$$\int \frac{1}{6y^2}dy = \int xdx \rightarrow -\frac{1}{6y} + c = \frac{x^2}{2} + k \rightarrow -\frac{1}{y} = 3x^2 + c$$

- Plugging in the initial value to find the value of the constant c, we get

$$-\frac{1}{\frac{1}{25}} = 3(1)^2 + c \rightarrow -25 = 3 + c \rightarrow c = -28$$

Separable Differential Equations Example

- So, the implicit solution is

$$-\frac{1}{y} = 3x^2 - 28$$

- We can then easily solve for y to obtain an explicit solution

$$-\frac{1}{y} = 3x^2 - 28 \rightarrow y = -\frac{1}{3x^2 - 28} = \frac{1}{28 - 3x^2}$$

- Note that this solution is only valid if $x \neq \pm\sqrt{\frac{28}{3}}$.
  - Therefore, the range of x can be any of the following intervals: $\left(-\infty, -\sqrt{\frac{28}{3}}\right), \left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right), \left(\sqrt{\frac{28}{3}}, \infty\right)$
    - ★ However, only the interval $\left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$ contains the value of x in the initial condition, so this is known as the interval of validity.

Separable Differential Equations Example

- Solve the initial value problem

$$y' = \frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

- Separate into the proper form and then integrate both sides

$$\int \frac{1}{y^3} dy = \int \frac{x}{\sqrt{1+x^2}} dx$$

- To integrate the right hand side, set $u = 1 + x^2$, so $du = 2xdx$ and $dx = \frac{du}{2x}$. Then, the right hand side becomes

$$\int \frac{x}{u^{\frac{1}{2}}} \frac{du}{2x} = \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = u + c = \sqrt{1+x^2} + c$$

- So, the implicit solution for the differential equation is

$$-\frac{1}{2y^2} + k = \sqrt{1+x^2} + c \rightarrow -\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Separable Differential Equations Example

- Plugging in the initial value

$$-\frac{1}{2(-1)^2} = \sqrt{1+(0)^2} + c \to -\frac{1}{2} = 1 + c \to c = -\frac{3}{2}$$

- Then, we can solve for the explicit solution

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2} \to y^2 = -\frac{1}{2\sqrt{1+x^2}-3} \to y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$