ECON 186 Class Notes: Derivatives and Differentials

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Partial Differentiation

- Consider a function $y = f(x_1, x_2, ..., x_n)$ where the $x_i's$ are all independent, so each can vary without affecting the others.
- Suppose that only x_1 changes, then we will have the difference quotient

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, ..., x_n) - f(x_1, x_2, ..., x_n)}{\Delta x_1}$$

- Then, the partial derivative with respect to the *ith* argument of $f(x_1, x_2, ..., x_n)$ is

$$f_i \equiv \frac{\partial y}{\partial x_i} \equiv \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}$$

- In order to take the partial derivative with respect to x_i, we must treat all other $x's$ as constant, and then the process is exactly the same.

Examples of Partial Differentiation

- Example 1: Let $y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$. Find the partial derivatives with respect to x_1 and x_2.
  - $\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$ and $\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$
- Example 2: Let $y = f(u, v) = (u+4)(3u+2v)$. Find the partial derivatives with respect to u and v.
  - $f_u = (u+4)(3) + (3u+2v)(1) = 3u + 12 + 3u + 2v = 6u + 2v + 12$ and $f_v = (u+4)(2) + (3u+2v)(0) = 2(u+4)$
- Example 3: Let $y = f(u, v) = \frac{3u-2v}{u^2+3v}$. Find the partial derivatives with respect to u and v.
  - $f_u = \frac{(u^2+3v)(3)-(3u-2v)(2u)}{(u^2+3v)^2} = \frac{3u^2+9v-6u^2+4uv}{(u^2+3v)^2} = \frac{-3u^2+4uv+9v}{(u^2+3v)^2}$ and $f_v = \frac{(u^2+3v)(-2)-(3u-2v)(3)}{(u^2+3v)^2} = \frac{-2u^2-6v-9u+6v}{(u^2+3v)^2} = \frac{-2u^2-9u}{(u^2+3v)^2} = \frac{-u(2u+9)}{(u^2+3v)^2}$

Gradient Vector

- The gradient vector is the collection of all partial derivatives of a function f.
- Notation: To denote a gradient, we use the word "grad" or more commonly, an upside down capital Delta, ∇.
- Example 1: If we take the previous example where $y = f(u,v) = \frac{3u-2v}{u^2+3v}$, then we showed the gradient vector would be

$$\nabla y = grad\ f(u,v) = (f_u, f_v) = \left(\frac{-3u^2+4uv+9v}{(u^2+3v)^2}, \frac{-u(2u+9)}{(u^2+3v)^2} \right)$$

- Example 2: Suppose we have a utility function which depends only on consumption and leisure, where $u(c,l) = \frac{c^\sigma}{\sigma} + \alpha \frac{l^\gamma}{\gamma}$
  - $u_c = c^{\sigma-1}$ and $u_l = \alpha(l^{\gamma-1})$. So the gradient vector is $\nabla u(c,l) = (u_c, u_l) = \left(c^{\sigma-1}, \alpha(l^{\gamma-1})\right)$

Jacobian Determinants

- The Jacobian matrix is a matrix of partial derivatives of a series of differentiable functions. Specifically, each row of the Jacobian matrix is the gradient vector of a function.
- Suppose we have n differentiable functions in n variables:

$$\begin{gathered} y_1 = f^1(x_1, x_2, ..., x_n) \\ y_2 = f^2(x_1, x_2, ..., x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = f^n(x_1, x_2, ..., x_n) \end{gathered}$$

- The Jacobian determinant, often referred to as "The Jacobian" is the determinant of the Jacobian matrix. The Jacobian determinant for the above system of equations is

$$|J| \equiv \left| \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} \right| \equiv \begin{vmatrix} \partial y_1/\partial x_1 & \ldots & \partial y_1/\partial x_n \\ \vdots & & \vdots \\ \partial y_n/\partial x_1 & \ldots & \partial y_n/\partial x_n \end{vmatrix} \equiv \begin{vmatrix} f_1^1 & \ldots & f_n^1 \\ \vdots & & \vdots \\ f_1^n & \ldots & f_n^n \end{vmatrix}$$

Jacobian Determinants

- Example: Let $y_1 = 2x_1 + 3x_2$ and $y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2$
  - $\frac{\partial y_1}{\partial x_1} = 2$, $\frac{\partial y_1}{\partial x_2} = 3$, $\frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2$, $\frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$
  - $|J| \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 8x_1 + 12x_2 & 12x_1 + 18x_2 \end{vmatrix} =$
    $2(12x_1 + 18x_2) - 3(8x_1 + 12x_2) = 24x_1 + 36x_2 - 24x_1 - 36x_2 = 0$
- What does a Jacobian determinant of 0 mean? Recall that when a determinant is equal to 0, there is linear dependence among the equations. Similarly, if a Jacobian determinant is equal to 0, there is functional dependence among the system of equations. Functional dependence could be a linear or nonlinear relationship.
  - In fact, the previous result is simply a special case of the Jacobian criterion of functional dependence (read on page 176 in Chiang and Wainwright).
- In this case, $y_2 = y_1^2$, so y_1 and y_2 are nonlinearly dependent.

Applications and Uses of the Jacobian Determinant

- Applications of Jacobian Matrices and the Jacobian Determinant
  - The Jacobian determinant is used in the process of changing variables from cartesian to polar coordinates when evaluating double or higher integrals.
  - The Jacobian matrix is the best linear approximation to a differentiable function near a given point. So, we can think of the Jacobian as the derivative of a multivariate function.
  - The inverse of the Jacobian matrix of a function is the Jacobian matrix of the inverse of that function. (We will come back to this shortly).
  - The behavior of a system near a stationary point is related to the eigenvalues of the Jacobian matrix of the system.
  - Jacobians are closely related to Hessians, which play a crucial role in the second order conditions of constrained optimization.

Differentials and Derivatives

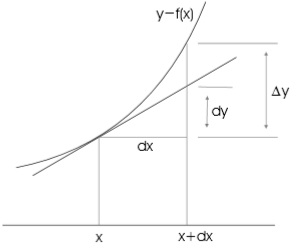
- Recall that we defined the derivative as

$$\frac{dy}{dx} = f^{'}(x)dx$$

- Then, consider dx as an independent variable, and dy an independent variable, and multiply through to get

$$dy = f^{'}(x)dx$$

Differentials and Derivatives



- The tangent line at point x on $y = f(x)$ has slope $f^{'}(x)$. If you move from x to $x + dx$, the tangent line rises by $dy = f^{'}(x)dx$.
- The actual change is $\Delta y = f(x + dx) - f(x)$. So, dy is an approximation to Δy which gets closer as $dx \to 0$.

Differentials and Derivatives

- dx and dy are called the differentials of x and y, respectively.
- Economic application: Recall that the formula for the price elasticity of demand is $\varepsilon_d \equiv \frac{\Delta Q/Q}{\Delta P/P}$
  - However, we have just learned that the differential dQ can serve as an approximation to ΔQ. Then, we can get an approximation elasticity known as the point elasticity of demand, which we can arrange as follows:

$$\varepsilon_d \equiv \frac{dQ/dP}{Q/P}$$

  - The numerator is now the derivative of Q with respect to P, which we call the marginal function. Similarly, the denominator is the average function. In fact, this relationship holds for any y and x of a function $y = f(x)$ so that:

$$\varepsilon_{yx} = \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}}$$

Differentials and Derivatives

- Example: Find the point elasticity of supply ε_s from the supply function $Q = P^2 + 7P$ and determine whether the supply is elastic at $P = 2$.

$$\frac{dQ}{dP} = 2P + 7$$

$$\frac{Q}{P} = P + 7$$

$$\varepsilon_s = \frac{dQ/dP}{Q/P} = \frac{2P+7}{P+7}$$

  - When $P = 2$, $\varepsilon_s = \frac{2(2)+7}{2+7} = \frac{11}{9}$, so the supply is elastic at $P = 2$.

Total Differentials

- The concept of the differential can easily be extended to the case of n independent variables. Suppose we have a function $f(x_1, x_2, ..., x_n)$. The total differential is:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n$$

$$= f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n = \sum_{i=1}^{n} f_i dx_i$$

- How is this different from a derivative? With a derivative, all of the $dx_i's$ are very close to 0, but here they do not have to be, so we can evaluate changes in the $x_i's$ that are not necessarily extremely small.

Total Differential Example

- Example: Let $S = S(Y, i)$ be a saving function where S represents savings, Y is national income, and i is the interest rate. Find the total differential of S.
  - Suppose $S(Y, i) = \frac{1}{2}Y + 3i$. Then
    - ★ $\frac{\partial S}{\partial Y} = \frac{1}{2}$ and $\frac{\partial S}{\partial i} = 3$.
    - ★ $dS = S_Y dY + S_i di = \frac{1}{2}dY + 3di$

Rules of Differentials

- Let k be a constant and u and v be two functions of the variables x_1 and x_2. Then the following rules are valid:
- Rule I: $dk = 0$
- Rule II: $d(cu^n) = cnu^{n-1}du$
- Rule III: $d(u \pm v) = du \pm dv$
- Rule IV: $d(uv) = vdu + udv$
- Rule V: $d(\frac{u}{v}) = \frac{1}{v^2}(vdu - udv)$
- Rule VI: $d(u \pm v \pm w) = du \pm dv \pm dw$
- Rule VII: $d(uvw) = vwdu + uwdv + uvdw$

Total Differential Example

- Find the total differential of $y = f(x_1, x_2) = \frac{x_1 + x_2}{2x_1^2}$
- Method 1: Find the partial derivatives and plug into the formula $dy = f_1 dx_1 + f_2 dx_2$
  - $f_1 = \frac{2x_1^2(1) - (x_1+x_2)(4x_1)}{(2x_1^2)^2} = \frac{2x_1^2 - 4x_1^2 - 4x_1x_2}{4x_1^4} = -\frac{x_1+2x_2}{2x_1^3}$ and $f_2 = \frac{1}{2x_1^2}$. So $dy = -\frac{x_1+2x_2}{2x_1^3}dx_1 + \frac{1}{2x_1^2}dx_2$
- Method 2: Use the rules of differentials.

$$dy = \frac{1}{4x_1^4}\left[2x_1^2 d(x_1 + x_2) - (x_1 + x_2)d(2x_1^2)\right]$$

$$= \frac{1}{4x_1^4}\left[2x_1^2(dx_1 + dx_2) - (x_1 + x_2)4x_1 dx_1\right]$$

$$= \frac{1}{4x_1^4}\left[-2x_1(x_1 + 2x_2)dx_1 + 2x_1^2 dx_2\right] = -\frac{x_1 + 2x_2}{2x_1^3}dx_1 + \frac{1}{2x_1^2}dx_2$$

Total Derivatives

- Until now, we have discussed partial derivatives which allow us to measure the instaneous change from one variable while holding all other variables constant.
- However, suppose that we have two functions, $y = f(x, w)$ and $x = g(w)$, or alternatively, we can write these as the composite function $y = f[g(w), w]$.
- Therefore, when w changes, it can affect y directly since y is a function of w, as well as indirectly via a change in x.

Total Derivatives

- We can obtain the indirect effect by using a slight extension to the Chain Rule we learned previously, which would be $f_x \frac{dx}{dw}$ and the direct effect is simply f_w.
  - So the total derivative is

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$$

.

- Alternatively, we can take the total differential of $y = f(x, w)$ to obtain $dy = f_x dx + f_w dw$ and then divide by dw to obtain the same result.
- Note that while $\frac{\partial y}{\partial w}$ is the partial derivative of y with respect to w, $\frac{dy}{dw}$ is the total derivative of y with respect to w.

Total Derivatives

- Example: Find the total derivative $\frac{dz}{dx}$, given $z = f(y, x) = 3y - x^2$ and $y = g(x) = 2x^2 + x + 4$

$$\frac{dz}{dx} = \frac{\partial z}{\partial y}\frac{dy}{dx} + \frac{\partial z}{\partial x} = 3(4x+1) - 2x = 10x + 3$$

- Now suppose we have a function $y = f(x_1, x_2, w)$ where $\begin{cases} x_1 = & g(w) \\ x_2 = & h(w) \end{cases}$
  - In this case, a change in w can actually affect y directly, indirectly through g and then through f, and through h and then f.
  - To find the total derivative, first take the total differential then divide through by w

$$dy = f_{x_1}dx_1 + f_{x_2}dx_2 + f_w dw \rightarrow \frac{dy}{dw} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial y}{\partial w}$$

Total Derivatives

- Let $Q = L^{\alpha} K^{\beta} t^{\gamma}$ be a Cobb-Douglas production function where L represents labor, K represents time, and t represents time. Since K and L can change over time, it is the case that $L = L(t) = 20 + \frac{1}{2}t$ and $K = K(t) = 15 + 2t$. Find the total derivative of this production function.

$$dQ = \frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial t} dt \to \frac{dQ}{dt} = \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial t}$$

$$= \alpha L^{\alpha-1} K^{\beta} t^{\gamma} \frac{1}{2} + \beta K^{\beta-1} L^{\alpha} t^{\gamma}(2) + L^{\alpha} K^{\beta} \gamma t^{\gamma-1}$$

Why do we Care About Total Derivatives

- Why do we care so much about total derivatives in economics other than their apparent mathematical usefulness?
- In economics, everything is related and changes in a given variable not only affect many other variables, but often have a feedback loop so that the variable that is affected also affects the original variable that was changed (referred to as reverse causality in econometrics).
  - For example, consider the interest rate (federal funds rate specifically). Suppose there is an increase in the interest rate, then savings will become more attractive and inevitably increase, capital will flow out of other countries and back to the US (since treasuries now have a higher yield), and the quantity of loans will decrease since loans are more expensive. Although these are actually moving the economy in opposite directions, it has been shown empirically that increasing the interest rate significantly slows down the economy. So, if the economy slows down past the point where the fed desires, they will then have to lower the interest rate, and the cycle works in reverse.

Implicit Functions

- Suppose we have the function $y = f(x) = 3x^4$. This is called an explicit function because y is explicitly expressed as a function of x.
- However, if we express the function as $y - 3x^4 = 0$, this function is now an implicit function, since the explicit function is only implied. We denote this by $F(x, y) = 0$. Note that capital F is used to prevent confusion with the lowercase f used to denote explicit functions.
- Naturally, this can be extended to the case with more than two arguments, $F(y, x_1, ..., x_m) = 0$.
- Clearly, any explicit function can become an implicit function simply by moving over the left hand side, but implicit functions cannot necessarily be written as explicit functions, which is why we must know how to take the derivatives of implicit functions as well.

Implicit Functions

- Moreover, if we come upon an equation written implicitly, how can we be sure that it actually is defining a function at all?
  - Consider the implicit equation $F(y,x) = x^2 + y^2 - 9 = 0$. Recall that this represents a circle centered at the origin with radius 3. Therefore, each x does not correspond with a unique value of y, so this is by definition a relation, but not a function.
  - However, if we restrict y only to nonnegative values, we can obtain $y = \sqrt{9 - x^2}$ and if we restrict y only to nonpositive values, we can obtain $y = -\sqrt{9 - x^2}$, which are functions. So, even though the implicit relation is not a function on all points of its domain, it is a function on some subset of its domain.

Implicit Functions

- So, how do we know whether an implicit function

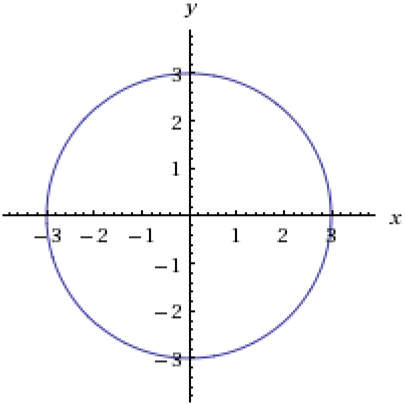
$$F(y, x_1, ..., x_m) = 0 \tag{1}$$

- actually does define a function

$$y = f(x_1, ..., x_m) \tag{2}$$

- The answer lies in the Implicit Function Theorem, which states the following:
  - Given (1), if (a) the function F has continuous partial derivatives $F_y, F_1, ..., F_m$, and if (b) at a point $(y_o, x_{10}, .., x_{m0})$ satisfying the equation (1), F_y is nonzero, then there exists an m-dimensional neighborhood of $(x_{10}, ..., x_{m0})$, N, in which y is an implicitly defined function of the variables $x_1, ..., x_m$, in the form of (2). This implicit function satisfies $y_0 = f(x_{10}, ..., x_{m0})$. It also satisfies the equation (1) for every m-tuple $(x_1, ..., x_m)$ in the neighborhood N. Moreover, the implicit function f has continuous partial derivatives $f_1, ..., f_m$.

Implicit Functions



The Implicit Function Rule

- How do we find the derivative of an implicit function that we cannot solve for explicitly?
- Again, suppose that we have the equation $F(y, x_1, ..., x_m) = 0$. Taking the total differential we get

$$F_y dy + F_1 dx_1 + F_2 dx_2 + ... + F_m dx_m = 0 \tag{3}$$

- Additionally, taking the total differential of the explicit function $y = f(x_1, x_2, .., x_m)$, we get

$$dy = f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n \tag{4}$$

- We can then substitute (4) into (3) and collect terms to obtain

$$(F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 + ... + (F_y f_m + F_m) dx_m = 0$$

The Implicit Function Rule

- Recall that the $x_i^{'}s$ can vary independently, so that the $dx_i^{'}s$ are independent. Therefore, it must be the case that

$$F_y f_i + F_i = 0 \quad \forall i$$

- So, the implicit function rule states that

$$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$

- In the case of two variables, x and y, specified by the equation $F(x, y) = 0$, the implicit function rule states

$$f_y \equiv \frac{dy}{dx} = -\frac{F_x}{F_y}$$

The Implicit Function Rule Examples

- Example 1: Let $F(x, y) = y - 3x^4 = 0$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-12x^3}{1} = 12x^3$$

- Clearly, we could have easily moved y to the other side and solved the explicit function just as we have been. But what if we can't do this?
- Example 2: Let $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$. Find $\frac{\partial y}{\partial x}$.
  - Clearly, this equation is not easy to solve for y, but since F_y is nonzero at least at some points, and F_y, F_x, and F_w are continuous, the derivative is meaningful and can be found using the implicit function rule.

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}$$