

ECON 186 Class Notes: Derivatives and Differentials

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Rate of Change, Limits, and the Derivative

- Suppose we have the function $y = f(x)$. We want to consider what happens to y when x changes.
- Suppose we want to know what happens to y when x changes from x_0 to x_1 . Let capital Delta represent change, so that $\Delta x = x_1 - x_0$.
- Also, let $f(x_i)$ be used to represent the value of $f(x)$ when $x = x_i$.
 - ▶ Example: Let $f(x) = 5 + x^2$. Then $f(0) = 5 + 0^2 = 5$.
- So, the difference quotient, that is, the average rate of change of y as x changes, can be expressed as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

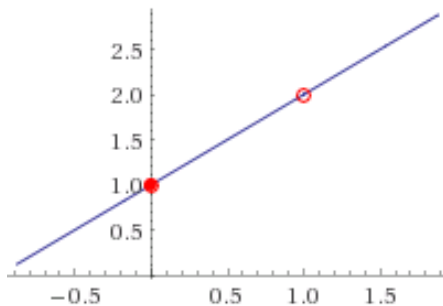
Rate of Change, Limits, and the Derivative

- Informal definition of a limit: Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0 , we say that f approaches the limit L as x approaches x_0 , and we write

$$\lim_{x \rightarrow x_0} f(x) = L$$

- which is read as “the limit of $f(x)$ as x approaches x_0 is L .”
- To evaluate $\lim_{x \rightarrow x_0} f(x)$, simply plug in x_0 for x and evaluate if possible.
- Example 1: Let $f(x) = \frac{x^2-1}{x-1}$. For any $x \neq 1$, $f(x)$ is defined and we may simplify the formula in the following manner.
 - ▶ $f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$
 - ▶ $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$

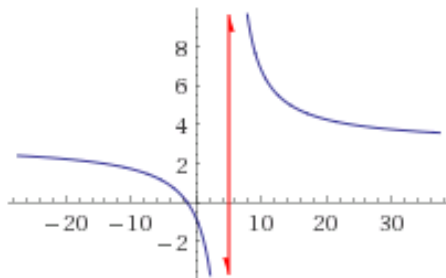
Rate of Change, Limits, and the Derivative



Rate of Change, Limits, and the Derivative

- Example 2: $\lim_{x \rightarrow 2} 4 = 4$
- Example 3: $\lim_{x \rightarrow 2} 5x - 3 = 10 - 3 = 7$
- Example 4: $\lim_{x \rightarrow 5^-} \frac{3x+4}{x-5} = -\infty$ and $\lim_{x \rightarrow 5^+} \frac{3x+4}{x-5} = \infty$
- Example 4 brings to light the important concepts of infinite limits, right and left hand side limits, and vertical asymptotes.
 - ▶ Intuitively, since the function in Example 4 is not defined at 5 and cannot reduce (such as Example 1), a two-sided limit does not exist, and we therefore cannot evaluate such as we did in the previous examples.
 - ▶ To get a sense of why the answer is such, evaluate the function at 4.9999 and 5.0001
 - ★ $f(4.9999) = \frac{3(4.9999)+4}{4.9999-5} = -189997$ and
 - $f(5.0001) = \frac{3(5.0001)+4}{5.0001-5} = 190003$

Rate of Change, Limits, and the Derivative



Rate of Change, Limits, and the Derivative

- If $f(x)$ is defined on an interval (c, b) where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has a right-hand limit L at c . We write

$$\lim_{x \rightarrow c^+} f(x) = L$$

- If $f(x)$ is defined on an interval (a, c) where $a < c$, and approaches arbitrarily close to M as x approaches c from within that interval, then f has a left-hand limit M at c . We write

$$\lim_{x \rightarrow c^-} f(x) = M$$

Rate of Change, Limits, and the Derivative

- The symbol $x \rightarrow c^+$ means we consider only x values greater than c and $x \rightarrow c^-$ means we consider only x values less than c .
- A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$
- A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$

Rate of Change, Limits, and the Derivative

- Example: Let $f(x) = -\frac{8}{x^2-4}$. Find the limit of $f(x)$ as $x \rightarrow \pm\infty$ and $x \rightarrow \pm 2$.

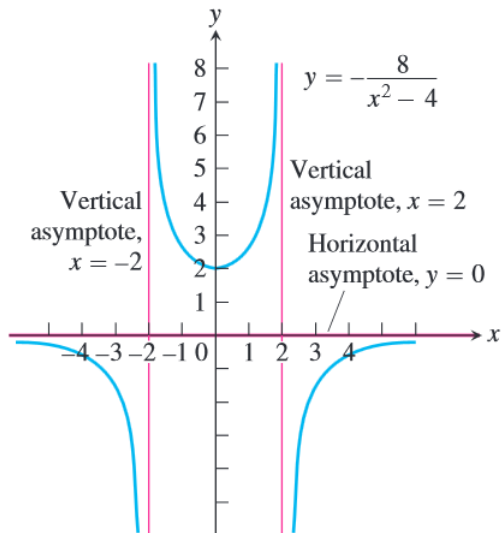
- $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow -\infty} f(x) = 0$, so there is a horizontal asymptote at $y = 0$.

▶ Hint: If the highest degree of x in the denominator is larger than the numerator, the limit will always be 0 as $x \rightarrow \pm\infty$.

- $\lim_{x \rightarrow 2^+} -\frac{8}{x^2-4} = -\infty$ and $\lim_{x \rightarrow 2^-} -\frac{8}{x^2-4} = \infty$

- $\lim_{x \rightarrow -2^+} -\frac{8}{x^2-4} = \infty$ and $\lim_{x \rightarrow -2^-} -\frac{8}{x^2-4} = -\infty$

Rate of Change, Limits, and the Derivative



Limit Theorems

- Theorem I: If $q = av + b$, then $\lim_{v \rightarrow N} q = aN + b$
- Theorem II: If $q = g(v) = b$, then $\lim_{v \rightarrow N} q = b$
- Theorem III: If $q = v^k$, then $\lim_{v \rightarrow N} q = N^k$.
- Suppose $\lim_{v \rightarrow N} q_1 = L_1$ and $\lim_{v \rightarrow N} q_2 = L_2$.
- Theorem IV: $\lim_{v \rightarrow N} (q_1 \pm q_2) = L_1 \pm L_2$
- Theorem V: $\lim_{v \rightarrow N} (q_1 q_2) = L_1 L_2$
- Theorem VI: $\lim_{v \rightarrow N} \frac{q_1}{q_2} = \frac{L_1}{L_2}$ where $L_2 \neq 0$

Rate of Change, Limits, and the Derivative

- Recall that the average rate of change of y for a function $y = f(x)$ is

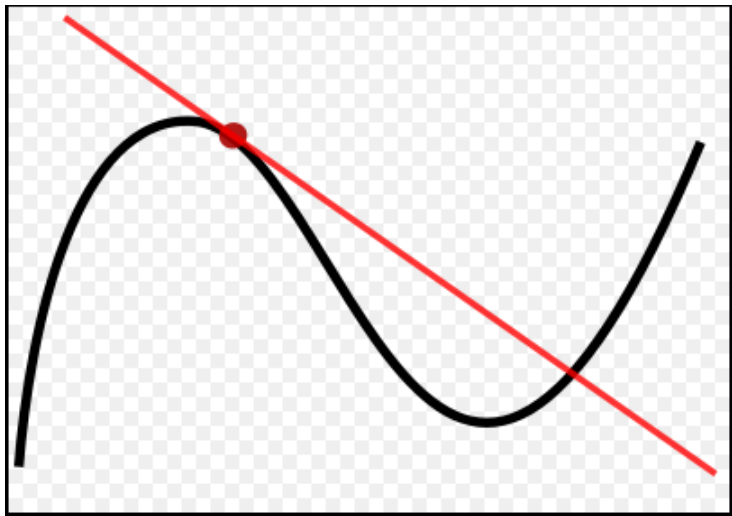
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- Example: Let $y = f(x) = 3x^2 - 4$, then $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x} = \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} = 6x_0 + 3\Delta x$
- We are most interested in the case where Δx is very small, that is,

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- This quantity is called the derivative, and represents an instantaneous rate of change.
- Geometrically, the derivative at x_0 is the slope of the tangent line to x_0 , where the tangent line is in fact the limit of an infinite amount of secant lines.

Rate of Change, Limits, and the Derivative



Derivative Notation

- Since the derivative takes its name because it is in fact a derived function, we denote the derivative of $f(x)$ as $f'(x)$ or simply f' .
- The other common notation is $\frac{dy}{dx}$, which is very similar to the difference quotient formula $\frac{\Delta y}{\Delta x}$ and indicates that the derivative is measuring rate of change.
- Therefore, we can formally define the derivative as follows:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

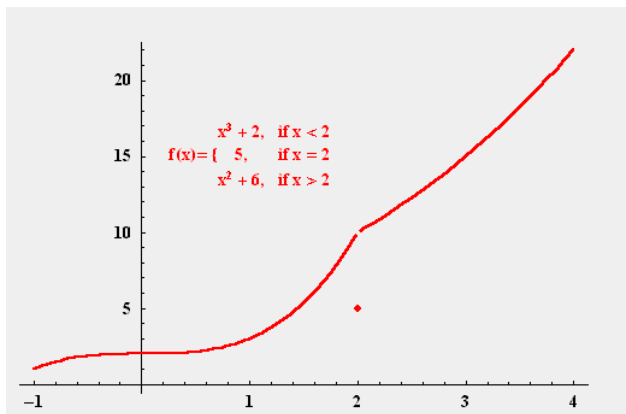
Continuity of a Function

- $f(x)$ is continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.
- If a function is continuous at all points in its domain, then it is called a continuous function.
- Example: Consider the function

$$f(x) = \begin{cases} x^3 + 2 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ x^2 + 6 & \text{if } x > 2 \end{cases}$$

- $\lim_{x \rightarrow 2^-} x^3 + 2 = 10$ and $\lim_{x \rightarrow 2^+} x^2 + 6 = 10$, so $\lim_{x \rightarrow 2} f(x) = 10$
- Therefore, since $\lim_{x \rightarrow 2} f(x) \neq f(2)$, $f(x)$ is discontinuous at $x = 2$.

Continuity and Differentiability of a Function



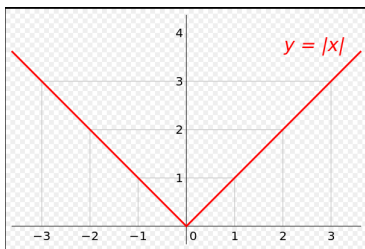
Differentiability of a Function

- A function $f(x)$ is differentiable at x_0 if and only if the following condition is met:

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- That is, a function is differentiable at a point only if the derivative of the function exists at that point.
- A function is differentiable if it is differentiable at each point in its domain.
- Continuity is a necessary, but not sufficient condition for differentiability.

Differentiability of a Function



- A function is differentiable at x_0 if and only if there is a non-vertical tangent line at x_0 . Specifically, since we cannot draw a tangent line to a “kink” or “sharp point” on a graph, the derivative cannot be defined there.
 - ▶ In the above example, there is a kink at $x = 0$, so the function $y = |x|$ is not differentiable at $x = 0$, but is elsewhere.

Rules of Differentiation of One Variable

- Constant-function rule: The derivative of a constant function $y = f(x) = k$ is zero.
 - ▶ Example: Let $f(x) = 5$. Then $f'(x) = 0$.
- Power-function rule: The derivative of a power function $y = f(x) = cx^n$ is cnx^{n-1} .
 - ▶ Example 1: Let $f(x) = 3x^3$. Then $f'(x) = 9x^2$
 - ▶ Example 2: Let $f(x) = \frac{1}{x^3}$. Find $f'(x)$.
 - ★ Easiest way is to rewrite as $f(x) = x^{-3}$. Then $f'(x) = -3x^{-4} = -\frac{3}{x^4}$.

Rules of Differentiation of One Variable

- Sum-difference Rule:

$$\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) = f'(x) \pm g'(x)$$

- ▶ Example: Let $y = 14x^3 = 9x^3 + 5x^3$. Then we will call $f(x) = 9x^3$ and $g(x) = 5x^3$ so that y is the sum of two functions.

$$\frac{dy}{dx} = \frac{d}{dx}(9x^3 + 5x^3) = \frac{d}{dx}9x^3 + \frac{d}{dx}5x^3 = 27x^2 + 15x^2 = 42x^2$$

Rules of Differentiation of One Variable

- Product Rule: The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function.
 - ▶ Mathematically, given two functions $f(x)$ and $g(x)$,
$$\frac{d}{dx} [f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$$
 - ▶ Example 1: Find the derivative of $y = (2x + 3)(3x^2)$. Let $f(x) = 2x + 3$ and $g(x) = 3x^2$.
 - ▶
$$\frac{d}{dx} [(2x + 3)(3x^2)] = (2x + 3)(6x) + (3x^2)(2) = 12x^2 + 18x + 6x^2 = 18x^2 + 18x$$

Marginal Revenue and Average Revenue Example

- Example 2: Suppose we have an arbitrary average revenue function denoted as $f(Q)$. Then the total revenue function will be of the form $R \equiv f(Q)Q$. Find marginal revenue.
- The marginal revenue function will then be
$$\frac{dR}{dQ} = f(Q) \cdot 1 + Q \cdot f'(Q) = f(Q) + Qf'(Q)$$
- Let $f(Q) = 15 - Q$. Then $MR = \frac{dR}{dQ} = 15 - Q + Q(-1) = 15 - 2Q$

Quotient Rule

- Quotient Rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- Example: $\frac{d}{dx} \left(\frac{2x-3}{x+1} \right) = \frac{2(x+1) - (2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$
 - ▶ Alternatively, rewrite as $(2x-3)(x+1)^{-1}$, then apply the product rule.
 - ★ $\frac{d}{dx} (2x-3)(x+1)^{-1} = (2x-3)(-1)(x+1)^{-2}(1) + (x+1)^{-1}(2) = -(2x-3)(x+1)^{-2} + 2(x+1)^{-1} = -\frac{2x-3}{(x+1)^2} + \frac{2(x+1)}{(x+1)^2} = \frac{5}{(x+1)^2}$
 - ★ Clearly, this works, but it is more work. This is why the Quotient Rule is often useful.

Chain Rule

- Chain Rule: Suppose we have a differentiable function $z = f(y)$, and y is a differentiable function of x so that $y = g(x)$. Then,

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(y)g'(x)$$

- Intuitively, since y is a function of x and z is a function of y , it must be the case that when x changes, z changes as a result of a change in y . Thus it is a sort of chain reaction.
- Specifically, when x changes, we have the two difference quotients, $\frac{\Delta y}{\Delta x}$ and $\frac{\Delta z}{\Delta y}$. Then, we have $\frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta x}$.

Chain Rule

- Example 1: Let $z = 3y^2$ and $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5)$$

- Example 2: Let $z = (x^2 + 3x - 2)^{17}$. Find $\frac{dz}{dx}$.

- ▶ We really do not want to multiply out z , so instead let $z = y^{17}$ and $y = x^2 + 3x - 2$.
- ▶ $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 17y^{16}(2x + 3) = 17(x^2 + 3x - 2)^{16}(2x + 3)$

Natural Logarithm

- Suppose we want to find the derivative of the function

$f(x) = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$. This would be very messy simply using the product and quotient rules. One answer is to simplify the function using the natural logarithm.

- Properties of Logarithms

- ▶ Rule I: $\log(uv) = \log(u) + \log(v)$
- ▶ Rule II: $\log\left(\frac{u}{v}\right) = \log(u) - \log(v)$
- ▶ Rule III: $\log(u^a) = a\log(u)$

- Example: Simplify $f(x) = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$ using the natural logarithm.

- $\ln(f(x)) = \ln\left(\frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}\right) = \ln(x^2+1) + \ln(x+3)^{\frac{1}{2}} - \ln(x-1) = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$

Derivatives of the Exponential and Natural Logarithmic Functions

- The derivative of the exponential function is itself. Specifically, $\frac{d}{dx}(e^x) = e^x$. More generally, $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
 - ▶ Example 1: Find the derivative of $y = e^{rt}$.

$$\frac{dy}{dt} = \frac{d}{dt}e^{rt} = re^{rt}$$

- ▶ Example 2: Find the derivative of $f(x) = \frac{1}{x}(x^2 + 2e^{2x})$
- ▶ $f'(x) = \frac{1}{x}(2x + 4e^{2x}) + (x^2 + 2e^{2x})\left(-\frac{1}{x^2}\right) = 2 + \frac{4e^{2x}}{x} - 1 - \frac{2e^{2x}}{x^2} = 1 + \frac{4xe^{2x} - 2e^{2x}}{x^2}$

Derivatives of the Exponential and Natural Logarithmic Functions

- The derivative of the natural logarithm, written as $\log(x)$ or $\ln(x)$, is the derivative of the function over the function itself. Specifically,

$$\frac{d}{dx}(\ln(u)) = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}$$

- Example 1: Find the derivative of the function $y = \ln(at)$.

$$\frac{d}{dt} \ln(at) = \frac{a}{at} = \frac{1}{t}$$

- Example 2: Consider the previous example where $f(x) = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$ and $\ln(f(x)) = \ln(x^2+1) - \frac{1}{2}\ln(x+3) - \ln(x-1)$. Find $f'(x)$.

$$\ln(f(x))' = \frac{f'(x)}{f(x)} = \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}$$

$$\rightarrow f'(x) = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1} \right)$$

The Case of Base b

- Consider the function $y = b^t$ where b is a constant and t is a variable. Your first instinct might be that $\frac{d}{dt} b^t = t b^{t-1}$, however this is not the case.
- The more general version is: $\frac{d}{dt} b^{f(t)} = f'(t) b^{f(t)} \ln b$
- Example: Find the derivative of the function $y = 12^{1-2t}$

$$\frac{dy}{dt} = -2(12)^{1-2t} \ln 12$$