Homework Day 9 Solutions - ECON 186

Problem 1. Chiang and Wainwright 12.3 #1(d)

1)

d) The bordered Hessian is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} = -2$$

So z is positive definite, which means that z^* is a minimum.

Problem 2. Chiang and Wainwright 12.5 #1(c)

c) Recall that the Lagrangian function is

$$L = (x+2)(y+1) + \lambda(130 - 4x - 6y)$$

Then, the bordered Hessian is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & -4 & -6 \\ -4 & 0 & 1 \\ -6 & 1 & 0 \end{vmatrix} = 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 4 \begin{vmatrix} -4 & 1 \\ -6 & 0 \end{vmatrix} - 6 \begin{vmatrix} -4 & 0 \\ -6 & 1 \end{vmatrix} = 24 + 24 = 48 > 0$$

So U is negative definite and thus $U^* = (16+2)(11+1) = 18(12) = 216$ is a maximum.

Problem 3.

c. To find whether $x^* = y^* = \sqrt{\frac{16}{15}}$ are the maximum input levels for maximizing profits, we need to check the definiteness of the function. Then, if we let the constraint be the function $g(x,y)$ where $x = y$, then the bordered hessian is

$$\begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 10y \\ -1 & 10y & 10x \end{vmatrix} = 0 \begin{vmatrix} 0 & 10x \\ 10y & 10x \end{vmatrix} - \begin{vmatrix} 1 & 10y \\ -1 & 10x \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ -1 & 10y \end{vmatrix}$$

$$= -10x - 10y - 10y = -30x$$

At the optimal value, the bordered hessian is equal to

$$-30\sqrt{\frac{16}{15}} < 0$$

So the bordered hessian is positive definite, which means that this is actually a minimum! But I thought we were trying to find the maximum values! Well, if we plug the constraint into the price function, we can see that

$$f(x, y) = 5x^3 - 16x$$

which means that as $x \to \infty$, profit actually goes to ∞, so the optimal value of each input is ∞!

Problem 4.

The bordered Hessian looks like

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & -2x & -2y & 0 \\ -2 & -2x & -2\mu & 0 & 0 \\ 1 & -2y & 0 & -2\mu & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Problem 5. Chiang and Wainwright 12.6 #1(a, c, f), 6

1)

a)

$$\sqrt{(jx)(jy)} = j = \sqrt{xy}$$

So the function is homogeneous of degree one.

c)

$$(jx)^3 - (jx)(jy) + (jy)^3 = j^3x^3 - j^2xy + j^3y^3$$

Since j cannot be factored out in any degree and leave the function as it was originally, this function is not homogeneous.

f)

$$(jx)^4 - 5(jy)(jw)^3 = j^4\left(x^4 - 5yw^3\right)$$

So the function is homogeneous of degree four.

6)

a)

$$A(jK)^\alpha(jL)^\beta = Aj^\alpha K^\alpha j^\beta L^\beta = Aj^{\alpha+\beta}K^\alpha L^\beta$$

So the Cobb-Douglas production function is homogeneous of degree $\alpha + \beta$. So, if $\alpha + \beta > 1$, this means that if you increase K and L $j - fold$, then output will increase more than $j - fold$, which by definition is increasing returns to scale.

b) Similarly, if $\alpha + \beta < 1$, then if you increase K and L $j - fold$, output will increase by less than $j - fold$, which by definition is decreasing returns to scale.

c) Taking the natural log of both sides of the function, we have

$$ln\ Q = ln\ A + \alpha ln\ K + \beta ln\ L$$

Then,

$$\epsilon_{Q,K} = \frac{\partial(ln\ Q)}{\partial(ln\ K)} = \frac{\frac{\alpha}{K}}{\frac{1}{K}} = \alpha$$

$$\epsilon_{Q,L} = \frac{\partial(ln\ Q)}{\partial(ln\ L)} = \frac{\frac{\beta}{L}}{\frac{1}{L}} = \beta$$

Problem 6.

First, set up the Lagrangian function

$$L = -(x_1 - 4)^2 - (x_2 - 4)^2 + \lambda_1 (4 - x_1 - x_2) + \lambda_2 (9 - x_1 - 3x_2)$$

The Kuhn-Tucker conditions are

$$L_{\lambda_1} : 4 - x_1 - x_2 = 0 \qquad \lambda_1 \geq 0$$

$$L_{\lambda_2} : 9 - x_1 - 3x_2 = 0$$

$$L_{x_1} : -2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$L_{x_2} : -2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

First, consider the cases for x_1 and x_2:

Case 1: $x_1 = 0, x_2 = 0$
In this case, $C = -(0-4)^2 - (0-4)^2 = -32$

Case 2: $x_1 = 0, x_2 > 0$
$x_2 \in [-\infty, 3]$, so the largest value C can take on is $C = -(0-4)^2 - (3-4)^2 = -16-1 = -17$

Case 3:
$x_1 \in [-\infty, 4]$, so the largest value C can take on is $C = -(4-4)^2 - (0-4)^2 = -16$

However, we can easily pick any two numbers that satisfy the constraints, such as $x_1 = 2, x_2 = 2$, where $C = -(2-4)^2 - (2-4)^2 = -8$, so none of these 3 cases can give a maximum. So it must be the case that $x_1 > 0, x_2 > 0$. So, let's now look at the first cases for λ_1 and λ_2.

Case 1: $\lambda_1 > 0, \lambda_2 > 0$
By complementary slackness, $x_1 + x_2 - 4 = 0$ and $x_1 + 3x_2 - 9 = 0$. From the first constraint, $x_1 = 4 - x_2$. Plugging in, $4 - x_2 + 3x_2 - 9 = 2x_2 - 5 = 0 \rightarrow x_2^* = \frac{5}{2}$. Then, $x_1^* = 4 - \frac{5}{2} = \frac{3}{2}$. Plugging into the FOC for L_{x_1}, $-2\left(\frac{3}{2} - 4\right) - \lambda_1 - \lambda_2 = 5 - \lambda_1 - \lambda_2 = 0 \rightarrow \lambda_1 = 5 - \lambda_2$. Plugging into the FOC for L_{x_2}, $-2\left(\frac{5}{2} - 4\right) - (5 - \lambda_2) - 3\lambda_2 = 3 - 5 + \lambda_2 - 3\lambda_2 = 0 \rightarrow \lambda_2 = -1$, which violates the constraint that λ_1 is nonnegative, so this cannot be a solution.

Case 2: $\lambda_1 > 0, \lambda_2 = 0$
By complementary slackness, $x_1 + x_2 - 4 = 0$. Plugging in $\lambda_2 = 0$ into the FOC's for x_1 and x_2, we get $-2(x_1 - 4) - \lambda_1 = 0 \rightarrow \lambda_1 = -2(x_1 - 4)$ and $-2(x_2 - 4) - \lambda_1 = 0 \rightarrow \lambda_1 = -2(x_2 - 4)$. Then, $-2(x_1 - 4) = -2(x_2 - 4) \rightarrow x_1 = x_2$. Plugging into the constraint, $x_1 + x_1 = 4 \rightarrow x_1^* = x_2^* = 2$. All the conditions are satisfied so this is a solution.

Case 3: $\lambda_1 = 0, \lambda_2 > 0$
By complementary slackness, $x_1 + 3x_2 - 9 = 0$. Substituting $\lambda_1 = 0$ into the FOCs for x_1 and x_2 gives $-2(x_1 - 4) - \lambda_2 = 0 \rightarrow \lambda_2 = -2(x_1 - 4)$ and $-2(x_2 - 4) - 3\lambda_2 = 0 \rightarrow \lambda_2 = -\frac{2}{3}(x_2 - 4)$. So, $-2(x_1 - 4) = -\frac{2}{3}(x_2 - 4) \rightarrow x_1 - 4 = \frac{1}{3}(x_2 - 4) \rightarrow x_1 = \frac{1}{3}x_2 + \frac{8}{3}$. Plugging into the constraint, $\frac{1}{3}x_2 + \frac{8}{3} + 3x_2 - 9 = \frac{10}{3}x_2 - \frac{19}{3} = 0 \rightarrow x_2 = \frac{19}{10}$. Plugging back in to the marginal rate of substitution between x_1 and x_2, $x_1^* = \frac{1}{3}\left(\frac{19}{10}\right) + \frac{8}{3} = \frac{19}{30} + \frac{80}{30} = \frac{99}{30} = \frac{33}{10}$. However, this violates the constraint $x_1 + x_2 \leq 4$, so this cannot be a solution.

Case 4: $\lambda_1 = 0, \lambda_2 = 0$
The FOC for L_{x_1} gives $-2(x_1 - 4) = 0 \rightarrow x_1^* = 4$ and the FOC for L_{x_2} gives $-2(x_2 - 4) = 0 \rightarrow x_2^* = 4$ which violates $x_1 + x_2 \leq 4$.

So the only values that maximize C are $x_1^* = x_2^* = 2$. So the maximum value that can be obtained is $C = -8$.