ECON 186 Class Notes: Linear Algebra

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Singularity and Rank

- As discussed previously, squareness is a necessary condition for a matrix to be nonsingular (have an inverse). The sufficient condition is that the rows and columns must be linearly independent.
  - Formally, a matrix A is nonsingular if and only if it is square and its rows and columns are linearly independent.
- Example: Suppose we have the equation system $Ax = d$ taking the form $\begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.
  - Clearly, the first row is just 2 times the second, so the rows are linearly dependent. In effect, this reduces the system to just one equation with an infinite number of solutions.
- So, in order for a system of equations to have a unique solution, all rows must be linearly independent, therefore the rows of a matrix must be linearly independent to have an inverse.

Rank and Row Reduction

- The rank of a matrix is the maximum number of linearly independent rows (and columns) in a matrix.
- The rank of an $m \times n$ matrix can be at most m or n, whichever is smaller.
- To determine the rank of a matrix (and to solve systems of equations in general), we use elementary row operations to reduce a matrix into row echelon form, where we can easily tell how many rows are linearly independent.
- The number of linearly independent rows is given by the number of nonzero rows after putting a matrix into row echelon form.

Rank and Row Reduction

- A matrix is in row echelon form if:
  - All nonzero rows are above any rows of all zeroes.
  - The leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
  - All entries in a column below a leading entry are zeroes.

Rank and Row Reduction

- Example of a matrix in row echelon form:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

- The elementary row operations on a matrix are as follows:
  - 1. Interchange of any two rows in the matrix.
  - 2. Multiplication of a row by any scalar $k \neq 0$.
  - 3. Addition of k times any row" to another row.

Rank and Row Reduction

- Example 1: Let

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \overset{2}{\rightarrow} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} \overset{2}{\rightarrow} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix} \overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Interestingly, A reduces to the identity matrix. Since the echelon form has 2 nonzero rows, the rank of the matrix is 2. Is A nonsingular?

Rank and Row Reduction

- Example 2: Find the rank of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} \overset{1}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} \overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix} \overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\overset{3}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank and Row Reduction

- This matrix in echelon form has 3 nonzero rows, so the rank of the matrix is 3. Is the matrix nonsingular?
- By definition, for an $n \times n$ matrix to be nonsingular, it must have n linearly independent rows, and thus have rank n. This means that for a matrix to be nonsingular, the echelon form of the matrix must have no zero rows.

Determinants

- A determinant is a uniquely defined scalar associated with a matrix, which is only defined for square matrices. The determinant of A is denoted as $|A|$.
- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = -2$.
- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$
  $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$
  $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$
- $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ is an example of a subdeterminant, and is the minor of the element a_{11}. It is denoted by $|M_{11}|$.

Evaluating an nth-Order Determinant

- In general, the symbol $|M_{ij}|$ denotes the minor obtained by deleting the ith row and jth column of a determinant.
- A cofactor, denoted by $|C_{ij}|$ is a minor with a prescribed algebraic sign attached to it. If the sum of the subscripts i and j in the minor $|M_{ij}|$ is even, the cofactor takes the same sign as the minor. If it is odd, the cofactor takes the opposite sign. In short,

$$|C_{ij}| \equiv (-1)^{i+j} |M_{ij}|$$

Example of 3x3 Determinant

- Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.
  - $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 1\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$
    $2(5)(9) - 2(6)(8) - 1(4)(9) + 1(6)(7) + 3(4)(8) - 3(5)(7) =$
    $90 - 96 - 36 + 42 + 96 - 105 = -9$
- We can see that $M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$, $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$, $M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$. Since $1+1=2$ and $1+3=4$ are even, $|M_{11}| = |C_{11}|$ and $|M_{13}| = |C_{13}|$. However, since $1+2=3$ is odd, $|M_{12}| = -|C_{12}|$, which is why there is a negative sign before the second cofactor.

3x3 Determinants

- So, we can write a third-order determinant (determinant of a 3×3) matrix as:
- $|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{1j}|C_{1j}|$
- Thus, the determinant of any square matrix B of size $n \times n$ can be expressed as

$$|B| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

Properties of Determinants

- The interchanging of rows and columns does not affect the determinant of a matrix. That is, $|A| = |A'|$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

- The interchanging of any two rows will alter the sign, but not the numerical value of the determinant

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$$

Properties of Determinants

- The multiplication of any one row by a scalar k will change the value of the determinant by a multiple of k.

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc)$$

- The addition of a multiple of any row to another row will leave the value of the determinant unaltered.

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Value of the Determinant and Nonsingularity

- If one row is a multiple of another row, the value of the determinant will be zero.
- Example:

$$\begin{vmatrix} 5a & 10b \\ a & 2b \end{vmatrix} = 10ab - 10ab = 0$$

  - If we attempt to reduce this matrix to its echelon form, we will obtain a row of all 0's, therefore the matrix is singular.
  - So, the rank of this matrix is 1, and the matrix is singular as discussed previously.

Value of the Determinant and Nonsingularity

- Therefore, given a linear equation system $Ax = d$, where A is an $n \times n$ coefficient matrix, the following are equivalent:
  - $|A| \neq 0$
  - A has row independence, that is, there are n linearly independent rows (equivalently columns) in A.
  - A is nonsingular.
  - A^{-1} exists.
  - The rank of A is n.
  - A unique solution $x^* = A^{-1}d$ exists.

Finding the Inverse of a Matrix Method 1

- Let A and B be $m \times n$ and $m \times p$ matrices, respectively. Then the augmented matrix $(A|B)$ is the $m \times (n+p)$ matrix. That is, the matrix whose first n columns are the columns of A, and whose last p columns are the columns of B.
- If A is an invertible (nonsingular) $n \times n$ matrix, then it is possible to transform the matrix $(A|I_n)$ into the matrix $(I_n|A^{-1})$ with elementary row operations.

Example of Finding the Inverse of a Matrix using Elementary Operations

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\rightarrow} \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example of Finding the Inverse of a Matrix using Elementary Operations

$$\xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -2 & -\frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{3}{10} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{3}{10} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{10} & 1 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{10} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{3}{10} & 0 \end{array}\right]$$

- So the inverse of this matrix is $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

Finding the Inverse of a Matrix Method 2

- In general, the formula to find the inverse of a matrix is

$$A^{-1} = \frac{1}{|A|} Adj\, A$$

- where *Adj* represents the adjugate or adjoint of a matrix. This is found by first finding the matrix of cofactors of a matrix and then swapping their positions over the main diagonal (also known as finding the transpose).

Example of Finding Inverse using Cofactors

- Let $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.
- Step 1: Construct a matrix of minors.

$$\begin{bmatrix} 0(1)+2(1) & 2(1)+2(0) & 2(1)-0(0) \\ 0(1)-2(1) & 3(1)-2(0) & 3(1)-0(0) \\ 0(-2)-0(2) & 3(-2)-2(2) & 3(0)-2(0) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

- Step 2: Construct a matrix of cofactors by simply applying the previously defined rule to the matrix of minors.

$$\begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{bmatrix}$$

Example of Finding Inverse using Cofactors

- Step 3: Find the adjugate by taking the transpose of the cofactor matrix.

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

- Step 4: Find the determinant of A. Since we already found the matrix of minors, all we have to do is multiply the top row of A by each element's corresponding minor.

$$\begin{vmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 3(2) - 0(2) + 2(2) = 10$$

.

- Step 5: Apply the formula.

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$$

Formula for the 2x2 case

- A well known and useful formula to find the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{detA}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Derivation using row echelon form:
- $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} a-\frac{cb}{d} & 0 & 1 & \frac{-b}{d} \\ c & d & 0 & 1 \end{array}\right] \to$
$\left[\begin{array}{cc|cc} \frac{ad-bc}{d} & 0 & 1 & \frac{-b}{d} \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} \frac{ad-bc}{d} & 0 & 1 & \frac{-b}{d} \\ 0 & d & \frac{-cd}{ad-bc} & \frac{ad}{ad-bc} \end{array}\right] \to$
$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & d & \frac{-cd}{ad-bc} & \frac{ad}{ad-bc} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$
- So the inverse of A is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Cramer's Rule

- Another sometimes useful way of solving a system of equations is called Cramer's Rule.
- Suppose we have a system of equations $Ax = d$, then Cramer's Rule says that

$$x_j^* = \frac{|A_j|}{|A|}$$

- where $|A_j|$ is the determinant of A with the jth column replaced by d.
- Clearly, it must be the case that $|A| \neq 0$, that is, A has an inverse, in order for Cramer's Rule to be applied.

Example of Cramer's Rule

- Suppose we have the system of equations

$$7x_1 - x_2 - x_3 = 0$$

$$10x_1 - 2x_2 + x_3 = 8$$

$$6x_1 + 3x_2 - 2x_3 = 7$$

Example of Cramer's Rule

Then, $A = \begin{bmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $d = \begin{bmatrix} 0 \\ 8 \\ 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{vmatrix} = 7\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} - \begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} =$$

$$7(4) - 7(3) + 10(-2) - 6 - 10(3) + 6(-2) = 28 - 21 - 20 - 6 - 30 - 12 = -61$$

Example of Cramer's Rule

$$|A_1| = \begin{vmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{vmatrix} = 0\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} - \begin{vmatrix} 8 & -2 \\ 7 & 3 \end{vmatrix} =$$
$$-16 - 7 - 24 - 14 = -61$$

$$|A_2| = \begin{vmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{vmatrix} = 7\begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} + 0\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} - \begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix} =$$
$$-16(7) - 7(7) - 70 + 8(6) = -183$$

$$|A_3| = \begin{vmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{vmatrix} = 7\begin{vmatrix} -2 & 8 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix} - 0\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} =$$
$$-14(7) - 24(7) + 70 - 8(6) = -244$$

$$x_1^* = \frac{|A_1|}{|A|} = \frac{-61}{-61} = 1,\ x_2^* = \frac{|A_2|}{|A|} = \frac{-183}{-61} = 3,\ x_3^* = \frac{|A_3|}{|A|} = \frac{-244}{-61} = 4$$