A Weyl law and a closing lemma

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The Closing Lemma

Background: the Calabi invariant

A Weyl law and the idea of the proof

The Periodic Floer homology spectral invariants

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Bonus 2: The Seiberg-Witten equations

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Section 1

The Closing Lemma
Some questions

**Question (Smale, Problem 10: “The Closing Lemma”, 1998)**

Let $p$ be a non-wandering point of a diffeomorphism $S : M \to M$ of a compact manifold. Can $S$ be arbitrarily well approximated in $C^r$ by $T : M \to M$, so that $p$ is a periodic point of $T$?

Non-wandering point $p$: $S^k U \cap U \neq \emptyset$ for each neighborhood $U$ of $p$. Pugh: true in $C^1$ topology (1967).

**Question (Franks-Le Calvez, ’00; Xia: Poincaré ’99)**

For a generic $C^r$ area-preserving diffeomorphism of a compact surface, is the union of periodic points dense?

Pugh-Robinson (’80s): true in the $C^1$ topology.
Today’s theorem

Theorem (“Generic density theorem”, CG., Prasad, Zhang)

A generic element of $\text{Diff}(\Sigma, \omega)$ has a dense set of periodic points. More precisely, the set of elements of $\text{Diff}(\Sigma, \omega)$ without dense periodic points forms a meager subset in the $C^\infty$-topology.

Definition of meager: countable union of nowhere dense subsets.

Remarks. Let $\Sigma$ be a closed surface:

- Case $\Sigma = S^2$ previously shown by Asaoka-Irie (2015); more generally for any Hamiltonian diffeomorphism of any $\Sigma$.
- Case $\Sigma = T^2$ proved simultaneously to us by Edtmair-Hutchings using related, but different methods; more generally for any $\Sigma$ when a certain Floer-homological condition holds. We later showed (with Prasad, Pomerleano,
Section 2

Background: the Calabi invariant
Recall. Let \((M^{2n}, \omega)\) be a symplectic manifold. (Example: any surface with area form.)

Any \(H : S^1 \times M^{2n} \to \mathbb{R}\) induces a corresponding (possibly time varying) **Hamiltonian vector field** \(X_{H_t}\) by the rule

\[\omega(X_{H_t}, \cdot) = dH_t(\cdot).\]

Denote its flow by \(\psi^t_H\).
Definition of the Calabi invariant

Let \( \text{Diffeo}_c(D^2, dx \wedge dy) \) denote the set of diffeomorphisms

\[
f : D^2 \rightarrow D^2, \quad f^*(dx \wedge dy) = dx \wedge dy, \quad f = \text{id} \text{ near } \partial D^2.
\]

There is a surjective homomorphism \( \text{Calabi} \)

\[
\text{Cal} : \text{Diffeo}_c(D^2, dx \wedge dy) \rightarrow \mathbb{R},
\]

defined as follows:

- Given \( \varphi \in \text{Diffeo}_c(D^2, dx dy) \), write \( \varphi = \psi^1_H, \quad H = 0 \text{ near } \partial D^2. \)
- Define \( \text{Cal}(\varphi) := \int_{D^2} \int_{S^1} H dt dx dy. \)
- Fact: \( \text{Cal}(\varphi) \) doesn’t depend on choice of \( H! \)
Calabi measures the “average rotation” of the map $\varphi$:

$$\text{Cal}(\varphi) = \int \int \text{Var}_{t=0}^{t=1} \text{Arg}(\varphi_H^t(x) - \varphi_H^t(y)) dx dy.$$
Section 3

A Weyl law and the idea of the proof
Warm-up case: compactly supported disc maps

We’ll first explain the idea in the case of $G := \text{Diffeo}_c(D^2, dx \wedge dy)$. We’ll define a sequence of maps

$$c_d : \text{Diffeo}_c(D^2, dx \wedge dy) \rightarrow \mathbb{R}$$

with the following properties:

- (**Continuity.**) Each $c_d$ is continuous (e.g. in $C^0$ topology).
- (**Spectrality.**) For any $\varphi \in G$, $c_d(\varphi)$ is the “action” of a set of periodic points of $\varphi$.
- (**Weyl Law.**) $\lim_{d \rightarrow \infty} \frac{c_d(\varphi)}{d} = \text{Cal}(\varphi)$. (c.f. “ECH volume property”)

We will now sketch proof of our Theorem in this case, following ideas of Asaoka-Irie, after reviewing more background.
Background: the action

What is the action?

Background: On \((S^2, \omega)\), any \(H \in C^\infty(S^1 \times S^2)\) has an associated action functional

\[
A_H(z, u) = \int_0^1 H(t, z(t)) dt + \int_{D^2} u^* \omega
\]

defined on capped loops \((z, u)\).

- Critical points of \(H\): capped 1-periodic orbits of \(\varphi^t_H\).
- Critical values of \(H\): called the action spectrum \(\text{Spec}(H)\), has Lebesgue measure 0.
Each $c_d(\phi^1_H) \in Spec_d(H)$, the **degree d action spectrum**, which also has measure 0.

Here,

$$Spec_d(H) := \bigcup_{k_1 + \cdots + k_j = d} Spec(H^{k_1}) + \ldots + Spec(H^{k_j}),$$

where $H^k$ denotes the $k$-fold “composition” of $H$ with itself.

Won’t define the composition here; key point: $\phi^1_{H^k} = (\phi^1_H)^k$. Can think of $Spec_d$ as the sum of actions of capped periodic orbits with periods summing to $d$. 
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Sketch of proof of Generic Density Theorem

Key claim: given $U$ open, nonzero $H \geq 0$ supported in $U$, $\varphi \circ \psi_H^t$ has a periodic point in $U$ for some $0 \leq t \leq 1$. Given claim, theorem follows by a Baire Category Theorem argument.

Proof of claim (a la Asaoka-Irie):

- Assume the opposite. Then $\varphi \circ \psi_H^t$ and $\varphi$ have the same set of periodic points.
- Hence, $\text{Spec}_d(\varphi \circ \psi_H^t) = \text{Spec}_d(\varphi)$ for all $d$.
- Hence, by Continuity, $c_d(\varphi \circ \psi_H^t) = c_d(\varphi)$ for all $d$.
- However, $\text{Cal}(\varphi \circ \psi_H^t) > \text{Cal}(\varphi)$. Contradiction.
More general surfaces

A similar argument works over an arbitrary closed surface $\Sigma$. Main challenge: in finding a Weyl law, Calabi homomorphism not in general defined. For example, $\text{Diff}(S^2, \omega_{\text{std}})$ is a simple group!

Solution: We prove a “relative” Weyl law recovering a “relative” Calabi invariant.

**Statement of relative Weyl law:** take $\varphi \in \text{Diff}(\Sigma, \omega)$, fix $U \subset \Sigma$ open, $H$ compactly supported in $U$. Then we define $c_d$ analogously to above and show the relative Weyl law:

$$\lim_{d \to \infty} \frac{c_d(\varphi \circ \psi^1_H) - c_d(\varphi)}{d} = \int_0^1 \int_U H \omega dt.$$ 

In fact, we prove a more general Weyl law although this generality is not needed for the Generic Density Theorem.
Section 4

The Periodic Floer homology spectral invariants
Our proof builds on a great story due to Hutchings, Lee, Taubes.

Let $\varphi \in \text{Diffeo}(\Sigma, \omega)$. Recall the mapping torus

$$Y_\varphi = \Sigma_x \times [0, 1] \sim, \quad (x, 1) \sim (\varphi(x), 0).$$

Has a canonical vector field

$$R := \partial_t,$$

a canonical two-form $\omega_\varphi$ induced by $\omega$, and a canonical plane field $\xi = \text{Ker}(dt)$. 
The definition of PFH

Useful for us to assume **monotonicity equation**:

\[ c_1(\xi) + 2PD(\Gamma) = \lambda[\omega_\varphi] \]

for some \( \Gamma \in H_1(Y_\varphi), \lambda \in \mathbb{R} \). There’s a **degree map**
\[ d : H_1(Y_\varphi) \longrightarrow H_1(S_1) = \mathbb{Z}, \]
and we also assume \( d(\Gamma) \) sufficiently large.

We’ll now define a \( \mathbb{Z}_2 \) vector space \( PFH(\varphi, \Gamma) \), called the **periodic Floer homology**. This is homology of a chain complex \( PFC(\varphi, \Gamma) \), (for nondegenerate \( \varphi \)). Details of \( PFC(\varphi, \Gamma) \):

- Freely generated by sets \( \{(\alpha_i, m_i)\} \), where
- \( \alpha_i \) distinct, embedded closed periodic orbits of \( R \)
- \( m_i \) positive integer; \( (m_i = 1 \text{ if } \alpha_i \text{ is hyperbolic}) \)
- \( \sum m_i[\alpha_i] = \Gamma \).
The differential

- Differential $\partial$ counts $I = 1$ $J$-holomorphic curves in $X := \mathbb{R} \times Y_\varphi$, for generic $J$, where $I$ is the “ECH index”. That is:
  \[
  \langle \partial \alpha, \beta \rangle = \# M^{I=1}_J(\alpha, \beta)
  \]
- $J : TX \longrightarrow TX, J^2 = -1, \mathbb{R}$-invariant (and admissible)
- ECH index beyond scope of talk; basic idea: $I = 1$ forces curves to be mostly embedded,
- Definition of $J$-holomorphic curve:
  $u : (C, j) \longrightarrow (X, J), \quad du \circ j = J \circ du$. 
The differential

Figure: A J-hol curve contributing to $\langle \partial \alpha, \beta \rangle$. 

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A Weyl law and a closing lemma
Example 1: an irrational shift of $T^2$

Write $T^2 = [0, 1]^2 / \sim$.

Let $S : T^2 \to T^2$ be an irrational shift. This has no periodic points at all! So $PFH$ vanishes (other than the empty set). 

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Example 2: an irrational rotation of $S^2$

Let $\varphi$ be an irrational rotation of $S^2$. This has two fixed points $p_+, p_-$. One can check $I(C) \in 2\mathbb{Z}$ for any curve $C$. Conclusion: differential vanishes.

So, degree 1 part generated by $p_+, p_-; \text{ degree 2 part generated by } p_+^2, p_+ p_-, p_-^2 \text{ etc. } \implies \text{ Rank } PFH(S^2, d) = d + 1.$
Lee-Taubes showed that there is a canonical isomorphism

\[ PFH(\varphi, \Gamma) \cong \widehat{HM}_{c-}(Y_\varphi, s_\Gamma), \]

where \( \widehat{HM}_{c-} \) is the (negative monotone) Seiberg-Witten Floer cohomology of \( Y_\varphi \) in the spin-c structure \( s_\Gamma \) corresponding to \( \Gamma \).

This gives a bridge between low-dimensional topology and surface dynamics that is central to our proofs.
Theorem (CG., Prasad, Zhang)

Fix a closed surface $\Sigma$. Then for $C^\infty$-generic $\varphi$, there exists classes $\Gamma_d \in H_1(Y_\varphi)$ with degrees tending to $+\infty$ such that

$$PFH(\Sigma, \varphi, \Gamma_d) \neq 0.$$ 

Compare with our earlier $T^2$ example. Upshot: there is a lot of nonzero homology for defining invariants.

Rough idea of the proof. Assume $(\varphi, \Gamma_d)$ is monotone (recall, this means: $c_1(V) + 2 \text{PD}(\Gamma_d) = \lambda [\omega_\phi]$); holds generically. We use Lee-Taubes to reduce to computation about “reducible” Seiberg-Witten solutions, more coming soon...
Hutchings’ observed that the action can be used to extract invariants $c_\sigma(\varphi)$ from any nonzero (twisted) PFH class $\sigma$.

The numbers $c_\sigma(\varphi)$ are the minimum action required to represent $\sigma$. We call this the **spectral invariant** associated to $\sigma$.

The numbers $c_d(\varphi)$ from before are defined by choosing appropriate nonzero classes with degree $d$. 
Section 5

Impressionistic sketch of the proof of the Weyl law
The Weyl law and Seiberg-Witten theory

The proof of the Weyl law is beyond the scope of the talk. Very rough idea: the Seiberg-Witten equations are equations for a pair $(A, \Psi)$, where $\Psi$ is a section of $s\Gamma$ and $A$ is a spin-c connection.

The configurations with $\Psi = 0$ are called \textbf{reducible} and can be described explicitly. In fact, there is a Floer homology for reducibles computable by classical topology.

We define a “Seiberg-Witten” spectral invariant, compute it for the reducibles, and show that it does not change much when compared with PFH via the Lee-Taubes isomorphism.

The nonvanishing also comes from computing the reducibles.
Our most general Weyl law is more general than the version stated earlier, which was about \( c_d(\varphi \circ \psi^t_H) - c_d(\varphi) \):

- The general version allows for any nonzero (twisted) PFH class, not just the fixed ones defining \( c_d \).
- The general version allows one to compare two arbitrary sequences of (twisted) PFH classes with degrees tending to infinity, rather than a single sequence. This introduces a new term involving the ECH index beyond the scope of this talk.
- The general version has no requirement on the support of \( H \).
- The general version holds over any coefficients.
Section 6

Bonus: Twisted PFH and the statement of the Weyl law
Twisted PFH

To get quantitative information, Hutchings’ observed one can work with a “twisted” version of PFH; homology of a complex \( \widetilde{PFC}(\varphi, \Theta) \).

Details of \( \widetilde{PFC}(\varphi, \Theta) \):

- Choose a (trivialized) reference cycle \( \Theta \) with \([\Theta] = \Gamma \) in \( H_1 \).
- Generator of \( \widetilde{PFC}(\varphi, d) \) a pair \((\alpha, Z)\), \( Z \in H_2(\alpha, \Theta) \)
- \( \partial \) counts \( l = 1 \) curves \( C \) from \((\alpha, Z)\) to \((\beta, Z')\):
  - this means: \( C \) a curve from \( \alpha \) to \( \beta \), with \( Z = [C] + [Z'] \).

Then \( \widetilde{PFH} \) has an action defined by \( A(\alpha, Z) = \int_Z \omega_\varphi \) and we can use this to define spectral invariants.

\( \widetilde{PFH} \) also has a grading \( l \) induced by the ECH index.
Statement of the Weyl law

Fix any Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \Sigma_g)$ and let $\phi' = \phi \circ \psi_H^1$. Consider sequences

$$\sigma_m \in \widetilde{PFH}(\phi, \Gamma_m, \Theta_m), \quad \sigma'_m \in \widetilde{PFH}(\phi', \Gamma_m, \Theta_m),$$

where the $\Gamma_m$ have degrees $d_m$ tending to infinity.

Then:

$$\lim_m \frac{c_{\sigma'_m}(\phi', \Gamma_m, \Theta_m) - c_{\sigma_m}(\phi, \Gamma_m, \Theta_m) + \int_{\Theta_m} H dt}{d_m} = \frac{l(\sigma'_m) - l(\sigma_m)}{2d_m(d_m + 1 - g)}$$

$$= \int_{M_\phi} H\omega_\phi \wedge dt.$$
Section 7

Bonus 2: The Seiberg-Witten equations
These are the equations:

\[ F_A = r(\star\langle cl(\cdot)\psi, \psi \rangle - i\omega_\varphi) + \ldots, \quad D_A\psi = 0, \text{ for } (A, \psi). \]

Reducibles are when \( \psi = 0 \). Here, \( r \) is a real number, and the Lee-Taubes isomorphism requires \( r \) very large.

Very rough idea: when \( \psi = 0 \), can understand solutions explicitly; however, for cohomological reasons this fixes \( r_0 \) not particularly large, and one has to understand what happens as \( r \) changes. We prove lots of estimates about this (the main reason the paper is long...).
Section 8

Bonus 3: Comparison with proof of the ECH volume conjecture
ECH volume conjecture $\approx 2012$. Why is PFH case tricky? Some reasons:

- That proof also uses reducibles, but Lee-Taubes equations as written have no reducible solutions!
- The energy $\int \lambda \wedge F_A$ plays a key role, but the analogue $\int dt \wedge F_A$ is not too interesting here (gives the degree).
- Nonvanishing was already known for ECH; nothing known.
- Can PFH even recover anything interesting?? Contribution from Hutchings (conjecture, rotation case); CG-Humiliere-Seyfaddini (twist maps)
- Have to work relative to a base connection in the PFH case, introduces many complications.
- The Lee-Taubes isomorphism is not “quantitative”. Need to add quantitative structure on top of it (many estimates, etc.).