The smooth closing lemma for area-preserving maps of surfaces

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Section 1

The Closing Lemma
Some questions


Let $p$ be a non-wandering point of a diffeomorphism $S : M \rightarrow M$ of a compact manifold. Can $S$ be arbitrarily well approximated in $C^r$ by $T : M \rightarrow M$, so that $p$ is a periodic point of $T$?

Non-wandering point $p$: $S^k U \cap U \neq \emptyset$ for each neighborhood $U$ of $p$. Pugh: true in $C^1$ topology (1967).

Question (Franks-Le Calvez, ’00; Xia: Poincaré ’99)

For a generic $C^r$ area-preserving diffeomorphism of a compact surface, is the union of periodic points dense?

Pugh-Robinson (’80s): true in the $C^1$ topology.
The Closing Lemma

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Today’s theorem

Theorem (CG., Prasad, Zhang)

A generic element of $\text{Diff}(\Sigma, \omega)$ has a dense set of periodic points. More precisely, the set of elements of $\text{Diff}(\Sigma, \omega)$ without dense periodic points forms a meager subset in the $C^\infty$-topology.

Definition of meager: countable union of nowhere dense subsets.

Remarks:

- Case $\Sigma = S^2$ previously shown by Asaoka-Irie (2015); in fact, they prove this for any Hamiltonian diffeomorphism of any closed surface $\Sigma$.
- Case $\Sigma = T^2$ proved simultaneously to us by Edtmair-Hutchings using related, but different methods.
Section 2

Background: the Calabi invariant
A pair \((M^{2n}, \omega)\) with \(\omega\) a differential 2-form is called a **symplectic manifold** if \(d\omega = 0, \omega \wedge \ldots \wedge \omega\) a volume form.

Example: any surface with area form.

Any \(H : S^1 \times M^{2n} \to \mathbb{R}\) induces a corresponding (possibly time varying) **Hamiltonian vector field** \(X_{H_t}\) by the rule

\[
\omega(X_{H_t}, \cdot) = dH_t(\cdot).
\]

Denote its flow by \(\psi^t_H\).
Let $\text{Diffeo}_c(D^2, dx \wedge dy)$ denote the set of diffeomorphisms

$$f : D^2 \to D^2, f^*(dx \wedge dy) = dx \wedge dy, f = id \text{ near } \partial D^2.$$ 

There is a surjective homomorphism $\text{Calabi}$

$$\text{Cal} : \text{Diffeo}_c(D^2, dx \wedge dy) \to \mathbb{R},$$

defined as follows:

- Given $\varphi \in \text{Diffeo}_c(D^2, dx dy)$, write $\varphi = \varphi^1_H$, $H = 0$ near $\partial D^2$.
- Define $\text{Cal}(\varphi) := \int_{D^2} \int_{S^1} H dt dx dy$.
- Fact: $\text{Cal}(\varphi)$ doesn’t depend on choice of $H$!
The Calabi invariant

Calabi measures the “average rotation” of the map $\varphi$:

$$Cal(\varphi) = \int \int Var_{t=0}^{t=1} Arg(\varphi_H^t(x) - \varphi_H^t(y)) \, dx \, dy.$$
Section 3

A Weyl law and the idea of the proof
Warm-up case: compactly supported disc maps

We’ll first explain the idea in the case of 
\( G := \text{Diffeo}_c(D^2, dx \wedge dy) \). We’ll define a sequence of maps

\[
c_d : \text{Diffeo}_c(D^2, dx \wedge dy) \rightarrow \mathbb{R}
\]

with the following properties:

- (Continuity.) Each \( c_d \) is continuous (e.g. in \( C^0 \) topology).
- (Spectrality.) For any \( \varphi \in G \), \( c_d(\varphi) \) is the action of a set of periodic points of \( \varphi \).
- (Weyl Law.) \( \lim_{d \to \infty} \frac{c_d(\varphi)}{d} = \text{Cal}(\varphi) \)

We can now sketch proof of the key fact: given \( U \) open, nonzero \( H \geq 0 \) supported in \( U \), \( \varphi \circ \psi_H^t \) has a periodic point in \( U \) for some \( 0 \leq t \leq 1 \).
Background: the action

What is the action?

Background: On \((S^2, \omega)\), any \(H \in C^\infty(S^1 \times S^2)\) has an associated action functional

\[
A_H(z, u) = \int_0^1 H(t, z(t))dt + \int_{D^2} u^* \omega
\]

defined on capped loops \((z, u)\).

- Critical points of \(H\): capped 1-periodic orbits of \(\varphi^t_H\).
- Critical values of \(H\): called the action spectrum \(Spec(H)\), has Lebesgue measure 0.
- Fact: Each \(c_d(\varphi^1_H) \in Spec_d(H)\) the degree \(d\) action spectrum, also has measure 0.
A similar argument works over an arbitrary closed surface $\Sigma$. Main challenge: in finding a Weyl law, Calabi homomorphism not in general defined. For example, $\text{Diff}(S^2, \omega_{\text{std}})$ is a simple group!

Solution: We prove a “relative” Weyl law recovering a “relative” Calabi invariant.

**Statement of relative Weyl law:** take $\varphi \in \text{Diff}(\Sigma, \omega)$, fix $U \subset \Sigma$ open, $H$ compactly supported in $U$. Then we define $c_d$ analogously to above and show the relative Weyl law:

$$
\lim_{d \to \infty} \frac{c_d(\varphi \circ \psi^1_H) - c_d(\varphi)}{d} = \int_0^1 \int_U H\omega dt.
$$
Section 4

The Periodic Floer homology spectral invariants
Our proof builds on a great story due to Hutchings, Lee, Taubes.

Let $\varphi \in \text{Diffeo}(\Sigma, \omega)$. Recall the **mapping torus**

$$Y_\varphi = \Sigma \times [0, 1]_t / \sim, \quad (x, 1) \sim (\varphi(x), 0).$$

Has a canonical vector field

$$R := \partial_t,$$

a canonical two-form $\omega_\varphi$ induced by $\omega$, and a canonical plane field $\xi = \text{Ker}(dt)$. 
The definition of PFH

Useful for us to assume **monotonicity equation:**

\[ c_1(\xi) + 2PD(\Gamma) = \lambda [\omega_\varphi] \]

for some \( \Gamma \in H_1(Y_\varphi), \lambda \in \mathbb{R} \). There’s a **degree map**
\[ d : H_1(Y_\varphi) \to H_1(S_1) = \mathbb{Z}, \]
and we also assume \( d(\Gamma) \) sufficiently large.

The \( \mathbb{Z}_2 \) vector space \( PFH(\varphi, \Gamma) \) is homology of a chain complex \( PFC(\varphi, \Gamma) \), (for nondegenerate \( \varphi \)). Details of \( PFC(\varphi, \Gamma) \):

- Freely generated by sets \( \{ (\alpha_i, m_i) \} \), where
- \( \alpha_i \) distinct, embedded closed periodic orbits of \( R \)
- \( m_i \) positive integer; (\( m_i = 1 \) if \( \alpha_i \) is hyperbolic)
- \( \sum m_i[\alpha_i] = \Gamma \).
The differential

- Differential $\partial$ counts $I = 1$ $J$-holomorphic curves in $X := \mathbb{R} \times Y_\varphi$, for generic $J$, where $I$ is the “ECH index”. That is:
  \[ \langle \partial \alpha, \beta \rangle = \# \mathcal{M}^I_j(\alpha, \beta) \]

- $J : TX \rightarrow TX, J^2 = -1$, $\mathbb{R}$-invariant (and admissible)

- ECH index beyond scope of talk; basic idea: $I = 1$ forces curves to be mostly embedded,

- Definition of $J$-holomorphic curve:
  \[ u : (C, j) \rightarrow (X, J), \quad du \circ j = J \circ du. \]
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The differential

$\alpha$

$\beta$

$\gamma_{\varphi}$

$\mathbb{R}$

Figure: A $J$-hol curve contributing to $\langle \partial \alpha, \beta \rangle$. 
Write $T^2 = [0, 1]^2 / \sim$.

Let $S : T^2 \to T^2$ be an irrational shift. This has no periodic points at all! So $PFH$ vanishes (other than the empty set).
Example 2: an irrational rotation of $S^2$

Let $\varphi$ be an irrational rotation of $S^2$. This has two fixed points $p_+, p_-$. One can check $I(C) \in 2\mathbb{Z}$ for any curve $C$. Conclusion: differential vanishes.

So, degree 1 part generated by $p_+, p_-; \text{ degree 2 part generated by } p_+^2, p_+p_-, p_-^2 \text{ etc. } \implies \text{ Rank } PFH(S^2, d) = d + 1.$
Lee-Taubes showed that there is a canonical isomorphism

$$PFH(\varphi, \Gamma) \cong \widehat{HM}_{c_{-}}(Y_{\varphi}, s_{\Gamma}),$$

where $\widehat{HM}_{c_{-}}$ is the (negative monotone) Seiberg-Witten Floer cohomology of $Y_{\varphi}$ in the spin-c structure $s_{\Gamma}$ corresponding to $\Gamma$.

This gives a bridge between low-dimensional topology and surface dynamics that is central to our proofs.
Application 1: generic non-vanishing of $PFH$

**Theorem (CG., Prasad, Zhang)**

*Fix a closed surface $\Sigma$. Then for $C^\infty$-generic $\varphi$, there exists classes $\Gamma_d \in H_1(Y_{\varphi})$ with degrees tending to $+\infty$ such that*

$$PFH(\Sigma, \varphi, \Gamma_d) \neq 0.$$ 

*Compare with our earlier $T^2$ example. Upshot: there is a lot of nonzero homology for defining invariants.*
Twisted PFH

To get quantitative information, Hutchings’ observed one can work with a “twisted” version of PFH; homology of a complex $\widetilde{PFC}(\varphi, \Theta)$.

Details of $\widetilde{PFC}(\varphi, \Theta)$:
- Choose a (trivialized) reference cycle $\Theta$ with $[\Theta] = \Gamma$ in $H_1$.
- Generator of $\widetilde{PFC}(\varphi, d)$ a pair $(\alpha, Z)$, $Z \in H_2(\alpha, \Theta)$
- $\partial$ counts $I = 1$ curves $C$ from $(\alpha, Z)$ to $(\beta, Z')$:
  - this means: $C$ a curve from $\alpha$ to $\beta$, with $Z = [C] + [Z']$.

Then $\widetilde{PFH}$ has an action defined by $A(\alpha, Z) = \int_Z \omega \varphi$ and for any nonzero $\sigma \in \widetilde{PFH}(\varphi, \Theta)$ we can define $c_\sigma(\varphi)$ to be the minimum action required to represent it. We call this the spectral invariant associated to $\sigma$. 
Section 5

Impressionistic sketch of the Weyl law
The proof of the Weyl law is beyond the scope of the talk. Very rough idea: the Seiberg-Witten equations are equations for a pair \((A, \Psi)\), where \(\Psi\) is a section of \(s_{\Gamma}\) and \(A\) is a spin-c connection.

The configurations with \(\Psi = 0\) are called **reducible** and can be described explicitly. In fact, there is a Floer homology for reducibles computable by classical topology.

We define a “Seiberg-Witten” spectral invariant, compute it for the reducibles, and show that it does not change much when compared with PFH via the Lee-Taubes isomorphism.