Contact homology exercises

(1) \textit{(There is no generic $f$ such that all multiply covered curves are transverse)} Let $\gamma$ be a simple Reeb orbit, and consider $C = \mathbb{R} \times \gamma$, the trivial cylinder over $\gamma$.

(a) Show that $\text{ind}(C) = 0$.
(b) Find branched covers (with branch points!) $\tilde{C}$ of $C$ such that $\text{ind}(\tilde{C}) = 0$.
(c) Why does this imply that $\tilde{C}$ can not be cut out transversely?

(2) (a) Let $Y = S^3$. Show that for any orbit $\gamma$

$$CZ(\gamma^d) \geq dCZ(\gamma) - d + 1.$$ 

(b) Now let $Y$ be any dynamically convex three-manifold such that $\pi_1(Y)$ contains no torsion. Show that any $J$-holomorphic plane asymptotic to an orbit $\beta$ must have $\beta$ simple.

(3) \textit{(Hard, but worth doing!)} Assume $(Y, \lambda)$ is dynamically convex, and let $B = (u_1, \ldots, u_n)$ be a building of broken curves with one positive end and one negative end. Show:

(a) $\text{ind}(B) \geq 1$
(b) If $\text{ind}(B) = 1$, then $B$ has one level.
(c) If $\text{ind}(B) = 2$, then either i) $B$ has one level, or; ii) $B$ has two levels, both cylinders; or, iii) $B$ has two levels, $B = (u_1, u_2)$, with $u_1$ an index 0 branched cover of a trivial cylinder and $u_2$ a plane union a trivial cylinder.

(4) Recall the \textit{adjunction formula}. This says that if $C$ is a somewhere injective $J$-holomorphic curve in a closed symplectic 4-manifold $X$, then

$$\langle c_1(TX), C \rangle = \chi(C) + [C] \cdot [C] - 2\delta(C),$$

where $\delta \geq 0$ is a nonnegative count of singularities of $C$, where nodal singularities count with weight 1. Prove this formula, when $C$ is in addition immersed, with only nodal singularities.

(5) Make sure you understand the computations from the 11/15 lecture:

(a) Show that $w_\tau(\zeta_1 \cup \zeta_2) = w_\tau(\zeta_1) + w_\tau(\zeta_2) + 2d \cdot \text{wind}_\tau(\zeta_2)$.
(b) Understand why $w_\tau(\zeta_2) = 0$.  

1
(c) Finish the proof that the “bad breaking” can not occur, by using the relative adjunction formula.

(d) Finish the proof that $d^2 = 0$.

(6) Show that $\lambda_n := \cos(nz)dx + \sin(nz)dy$ is a contact form on $T^3$.

(7) Show that the Reeb vector field associated to $\lambda_n$ is $\cos(nz)\partial_x + \sin(nz)\partial_y$.

(8) Show that for each $(a, b, 0)$ in $H_1(T^3)$ such that $(a, b, 0) \neq 0$, there are exactly $n S^1$ families of Reeb orbits in class $(a, b, 0)$. Show that these are the only Reeb orbits.

(9) Show that each orbit in class $(a, b, 0)$ of any $\lambda_n$ has action $2\pi \sqrt{a^2 + b^2}$.

(10) If $C$ is a $J$-holomorphic cylinder from $\alpha$ to $\beta$, show that $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$, with equality if and only if $\alpha = \beta$, and $c$ is an $\mathbb{R}$-invariant cylinder.

(11) Prove the ECH index inequality

$$\text{ind}(C) \leq I(C) - 2\delta(C)$$

when $C$ is a somewhere injective curve in a closed four-manifold. In fact, show that it is an equality.

(12) Show that $\mathbb{R}$-invariant cylinders have $I = 0$.

(13) (Hard but fun!) Assume $J$ is generic, let $C$ be a $J$-holomorphic current (not necessarily somewhere injective!) in $\mathbb{R} \times Y$, and assume that $I(C) = 1$. Show that

$$C = C_0 \sqcup C_1,$$

where $C_1$ is embedded with $I(C_1) = \text{ind}(C_1) = 1$, and $C_0$ is a union of covers of $\mathbb{R}$-invariant cylinders. (Hint: Use the $\mathbb{R}$-translation, plus the index inequality in the somewhere injective case.)

(14) Show that the ECH index is additive over breakings.

(15) Assume $J$ is generic, and let $C$ be any $J$-holomorphic current in $\mathbb{R} \times Y$. Show that $I(C) \geq 0$, with equality if and only if $C$ is a union of $\mathbb{R}$-invariant cylinders.

(16) Show, similarly to a previous exercise, that if $C$ is a $J$-holomorphic current from an orbit set $\alpha$ to an orbit set $\beta$, then $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$.

(17) Show that if $\lambda$ is nondegenerate, then there are only finitely many orbits $\gamma$ less than any fixed action.

(18) (Hard!) Finish the proof that the differential $d$ on ECH is well-defined, by analyzing possible breakings of the $I = 1$ moduli space.