EHRHART FUNCTIONS AND SYMPLECTIC EMBEDDINGS OF ELLIPSOIDS

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ABSTRACT. McDuff has previously shown that one four-dimensional symplectic ellipsoid can be symplectically embedded into another if and only if a certain combinatorial criteria holds. We reinterpret this combinatorial criteria using the theory of Ehrhart quasipolynomials, and we use this to give purely combinatorial proofs of theorems of McDuff-Schlenk and Frenkel-Müller, concerning the existence of “infinite staircases” in symplectic embedding problems. We then find a third, new, staircase and conjecture that these are the only three staircases for embeddings into rational ellipsoids. Several other applications are also discussed; for example, we give new examples of triangles whose Ehrhart function exhibits a period collapse.

1. Introduction

1.1. The motivating question. Recently, McDuff has proven a striking theorem concerning when one four-dimensional symplectic ellipsoid

\[ E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\} \]

embeds into another. To state McDuff’s theorem, let \( c_k(E(a, b)) \) be the \((k + 1)\)th smallest element in the matrix of numbers

\[(am + bn)_{m, n \in \mathbb{Z}_{\geq 0}}.\]

The numbers \( c_k \) are called ECH capacities, see [13], and should be thought of as measurements of “symplectic size”; they provide quantitative obstructions\(^2\) to symplectic embeddings beyond the classical obstruction that symplectic embeddings must preserve volume. Denote a symplectic embedding\(^3\) by the symbol \( \xrightarrow{s} \). McDuff showed that the

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\(^1\)Here, the symplectic form is given by restricting the standard form \( \omega = \sum_i dx_i dy_i \) on \( \mathbb{R}^4 = \mathbb{C}^2 \).

\(^2\)For more about symplectic capacities, see [4].

\(^3\)A symplectic embedding \( \Psi : (M_1, \omega_1) \to (M_2, \omega_2) \) is an embedding of smooth manifolds such that \( \Psi^* \omega_2 = \omega_1 \).
numbers \( c_k(E(a, b)) \) give sharp obstructions to symplectic embeddings of ellipsoids:

**Theorem 1.1.** [18, Thm. 1.1]

\[
E(a, b) \hookrightarrow E(c, d),
\]

if and only if

\[
c_k(E(a, b)) \leq c_k(E(c, d))
\]

for all \( k \).

At first glance, it is surprising that the simple combinatorial rule in Theorem 1.1 can explain what is known about the complexity of the symplectic embedding problem for ellipsoids. For example, recall the function

\[
(1.1) \quad c(a) := \min \{ \lambda \mid E(1, a) \hookrightarrow B^4(\lambda) \}.
\]

This was computed previously to Theorem 1.1 in [20], and was shown to have a remarkable structure:

**Theorem 1.2.** [20]

- For \( 1 \leq a \leq \tau^4 \), \( c(a) \) is piecewise linear, given by an “infinite staircase” determined by the odd-index Fibonacci numbers.\(^4\)
- For \( a > \left( \frac{17}{6} \right)^2 \), \( c(a) = \sqrt{a} \), i.e. the only obstruction to the embedding problem is the classical volume obstruction.
- For \( \tau^4 < a < \left( \frac{17}{6} \right)^2 \), we have \( c(a) = \sqrt{a} \), except on finitely many compact connected intervals on which it is linear.

Hutchings observed in [14, §2] that it is a subtle arithmetic problem to explain the rich structure in Theorem 1.2 from Theorem 1.1. The aim of the present paper is to explore this number-theoretic problem, particularly with regards to the first bullet point, and illustrate the value of doing this for both combinatorics and symplectic geometry.

1.2. **The main theorem.** Our main idea here is to connect Theorem 1.1 and Theorem 1.2 through the theory of Ehrhart functions.

Let \( \mathcal{P} \) be a convex polytope in \( \mathbb{R}^n \). The *Ehrhart function* of \( \mathcal{P} \) is the counting function

\[
L_\mathcal{P}(t) := \# \left( \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right).
\]

\(^4\)Here, the odd-index Fibonacci numbers refer to the sequence \( g_n := f_{2n+1} \), where \( f_n \) is the sequence of Fibonacci numbers; for a precise statement of the first bullet point, see Theorem 1.7 below.
There is a well-developed theory of the Ehrhart function in the case where $P$ has rational or integral vertices. In this case, if $P$ is $d$-dimensional, then $L_P(t)$ is a polynomial of degree $d$, with coefficients that are periodic. The minimal common period of the coefficients divides the number $D(P)$, which is the smallest $D \in \mathbb{Z}_{>0}$ such that the vertices of $D \cdot P$ are integral; it is called the denominator of $P$.

Two polytopes are called Ehrhart equivalent if they have the same Ehrhart function. It turns out that we can deduce the first bullet point of Theorem 1.2 from Theorem 1.1 — and we can deduce more as well — by finding unexpected pairs of triangles that are Ehrhart equivalent. More precisely, let $\mathcal{T}_{u,v}$ denote the triangle with vertices $(0,0), (0,u), (v,0)$. Define

$$f_{k,l}(p,q) := kp^2 - (k + l + 1)pq + lq^2 + 1$$

and consider positive integer solutions $(p,q)$ to the equation

$$f_{k,l}(p,q) = 0. \tag{1.2}$$

This is a natural equation to consider in terms of the subleading asymptotics of the Ehrhart function, which we will further discuss below, see Remark 1.5; one should regard (1.2) as a family of Diophantine equations, parametrized by $(k,l)$. For very specific pairs $(k,l)$, solutions to (1.2) give Ehrhart equivalences that one would not otherwise expect:

**Theorem 1.3.** Suppose $(k,l) \in \{(1,1), (2,1), (3,2)\}$ and assume that $kp$ and $lq$ are relatively prime. Then $\mathcal{T}_{\frac{kp}{\gcd(kp,lq)}, \frac{lq}{\gcd(kp,lq)}}$ and $\mathcal{T}_{\frac{k}{\gcd(kp,lq)}, \frac{l}{\gcd(kp,lq)}}$ are Ehrhart equivalent if and only if $(k,l,p,q)$ satisfies (1.2).

While Theorem 1.3 implies parts of Theorem 1.2 by invoking Theorem 1.1, it is not implied by Theorem 1.2 in any obvious way.

**Example 1.4.** The case $(k,l) = (1,1)$ is illustrative. In this case, (1.2) becomes

$$p^2 - 3pq + q^2 = -1.$$  

We will show below, see Proposition 4.1, that the solutions to this equation are precisely the consecutive odd-index Fibonacci numbers $(p,q) = (g_n, g_{n+1})$. So, Theorem 1.3 implies that in this case, the triangles $\mathcal{T}_{\frac{g_{n+1}}{g_n}, \frac{g_n}{g_{n+1}}}$ are all Ehrhart equivalent. We will show that this is enough to deduce the Fibonacci staircase part of Theorem 1.2.

**Remark 1.5.** While the numerology in Theorem 1.3 and (1.2) might appear strange, our choices are in fact natural in view of the asymptotics of the counting function $L_P(t)$. As is well-known, the leading
order asymptotics of $L_{\mathcal{P}}(t)$ are given by the volume of $\mathcal{P}$. Thus, because
\[ \text{area}\left(\mathcal{T}_{\frac{k}{p}, \frac{p}{q}}\right) = \text{area}\left(\mathcal{T}_{\frac{1}{k}, \frac{1}{l}}\right) = \frac{1}{2} kl, \]
the Ehrhart function of these triangles agree to leading order. The subleading asymptotics of the Ehrhart function are given by the “affine perimeter” of $\mathcal{P}$, and this is the genesis of (1.2) — agreement of subleading asymptotics is clearly necessary for Ehrhart equivalence, and Theorem 1.3 states that in special cases, this agreement is in fact sufficient.

Remark 1.6. One could ask whether the very strong condition on $(k, l)$ in Theorem 1.3 can be replaced by something weaker. Without some extra condition on $k$ and $l$, Theorem 1.3 certainly does not hold. For example, for $(k, l) = (3, 1)$ there are many examples of triangles $\mathcal{T}_{\frac{k}{p}, \frac{p}{q}}$ that are not Ehrhart equivalent to $\mathcal{T}_{\frac{1}{k}, \frac{1}{l}}$ even if $(k, l, p, q)$ satisfies (1.2).

Theorem 1.3 is valuable in several ways that we now explain,

1.3. Classifying infinite staircases in symplectic embeddings.

1.3.1. A new staircase. Define the function
\[ c_b(a) := \inf\{\mu : E(1, a) \xrightarrow{\mu} E(\mu, b\mu)\}. \]

By scaling, the function $c_b(a)$ completely determines when one four-dimensional ellipsoid symplectically embeds into another; the case $b = 1$ is precisely the function from (1.1) studied in [20]. In view of [20], it is interesting to try to better understand the function $c_b(a)$. In [20], the function $c(a)$ was computed exactly; in analogy with [23], however, this is probably not feasible for $c_b(a)$. However, it is interesting to look for specific structures for $c_b(a)$, for example analogues of the Fibonacci staircase. Theorem 1.3 is very useful for this.

More precisely, for positive integers $k, l$, define the sequence $r(k, l)_n$ by $r(k, l)_0 = 1, r(k, l)_1 = 1$ and
\[ r(k, l)_{2n+1} = \frac{k+l+1}{k} r(k, l)_{2n} - r(k, l)_{2n-1}, \]
\[ r(k, l)_{2n} = \frac{k+l+1}{l} r(k, l)_{2n-1} - r(k, l)_{2n-2}. \]

For example, the $r(1, 1)_n$ are the odd-index Fibonacci numbers mentioned above, while the $r(2, 1)_n$ are related to the Pell numbers. Set $r(k, l)_{-1} = 1$. In §4, we show that the solutions of (1.2) are precisely the pairs $(p, q) = (r(k, l)_{2n-1}, r(k, l)_{2n})$ for $n \geq 0$. 
Also define the sequences

\[ a(k, l)_n := \begin{cases} 
  kr(k, l)_{n+1}^2 & \text{if } n \text{ is even,} \\
  lr(k, l)_n^2 & \text{if } n \text{ is odd,}
\end{cases} \]

and

\[ b(k, l)_n := \frac{r(k, l)_{n+1}}{r(k, l)_n}. \]

Finally, define the positive real number

\[ \phi(k, l) = \frac{k}{l} \left( \frac{k + l + 1 + \sqrt{(k + l + 1)^2 - 4kl}}{2k} \right)^2. \]

We always have

\[ a(k, l)_0 < b(k, l)_0 < a(k, l)_1 < b(k, l)_1 < \ldots < \phi(k, l). \]

By combining Theorem 1.1 with Theorem 1.3, we can deduce the following “staircase” theorem concerning the function \( c_b(a) \):

**Corollary 1.7.** Suppose \((k, l) \in \{(1, 1), (2, 1), (3, 2)\}\). For \( a \) in the interval \([1, \phi(k, l)]\),

\[ c^+_b(a) = \begin{cases} 
  1 & \text{if } a \in [1, \frac{k}{l}], \\
  \frac{a}{\sqrt{\frac{k}{l} a(k, l)_n}} & \text{if } a \in [a(k, l)_n, b(k, l)_n], \\
  \sqrt{\frac{l}{k} a(k, l)_{n+1}} & \text{if } a \in [b(k, l)_n, a(k, l)_{n+1}].
\end{cases} \]

Thus, for \((k, l) \in \{(1, 1), (2, 1), (3, 2)\}\), the graph of \( c^+_b(a) \) begins with an infinite staircase, see Figure 1. For \((k, l) = (1, 1)\), Theorem 1.7 is a restatement of [20, Thm. 1.1.i] and for \((k, l) = (2, 1)\) it is a restatement of [16, Thm. 1.1.i], although as previously mentioned the proofs of [20, Thm. 1.1.i] and [16, Thm. 1.1.i] do not use Theorem 1.1. For a survey of the methods from [20], see [19]; the methods in [16] are similar.

1.3.2. **Classification.** One of the attractions of Theorem 1.3 is that the fact that Theorem 1.3 is an if and only if result, with three possibilities for \((k, l)\), suggests a possible classification result.

**Conjecture 1.8.** For fixed rational \( b \geq 1 \), the function \( c_b(a) \) has an infinite staircase if and only if \( b \in \{1, 2, 3/2\} \).

In particular, for rational \( b \not\in \{1, 2, 3/2\} \) we conjecture that \( c_b(a) \) has only finitely many nonsmooth points.

A proof of Conjecture 1.8 will be the subject of future work [5].
Remark 1.9. One should compare Conjecture 1.8 with the results in [23], where it is shown that for embeddings into irrational polydiscs, there are infinitely many infinite staircases. Thus, the rationality assumption in Conjecture 1.8 is presumably crucial, and so the picture in our paper is complementary to what is shown in [23] — for us, staircases in the rational case seem to be a very specific phenomenon, connected to combinatorial occurrences of independent interest.

In fact, inspired by Conjecture 1.8, other ongoing work [7] aims to prove the following. For any symplectic four-manifold $X$, we can define the associated embedding function

$$c_X(a) = \inf\{\lambda | E(1,a) \xrightarrow{\delta} \lambda \cdot X\}$$

where by $\lambda \cdot X$, we mean that the symplectic form is scaled by $\lambda$. It is interesting to try to understand what we can learn about $X$ from its embedding function.

Say that $X$ has an infinite staircase if $c_X$ does. In general, it is likely a very hard problem to determine which symplectic manifolds $X$ have infinite staircases. However, when $X$ is symplectic toric, this might be feasible, by building on the embedding techniques of [6]. This is the subject of [7]:

Conjecture 1.10. A closed symplectic toric 4-manifold has an infinite staircase if and only if, up to scaling, there is an integral moment polytope for $X$ with exactly one interior lattice point.
As with Conjecture 1.8, we hope to show that when the moment polytope does not have an integer scaling with exactly one interior lattice point, then the embedding function has only finitely many nonsmooth points.

Conjecture 1.10 is a kind of generalization of Conjecture 1.8, because while ellipsoids $E(a, b)$ are not quite closed symplectic toric-manifolds, they are “convex toric domains” in the sense of [7], and their moment image is the triangle $\Delta_{a, b}$ with vertices $(a, 0), (0, b)$ and $(0, 0)$. Up to scaling, $\Delta_{a, b}$ is an integer polytope with exactly one integer lattice point precisely in the three cases from Conjecture 1.8.

1.4. Period Collapse. As stated above, for a rational polytope $L_P(t)$ is in general a polynomial with periodic coefficients, of period dividing the denominator of $P$. The minimal period of $L_P(t)$ is called the period of $P$, and period collapse refers to any situation where the period of $P$ is less than the denominator of $P$. There has been considerable interest in understanding when exactly period collapse occurs, see for example [12, 17, 24].

The relevance of Theorem 1.3 to period collapse comes from the observation that if $k$ and $l$ are relatively prime then the period of $T_{\frac{k}{l}, \frac{l}{k}}$ is $kl$. In fact, for $(k, l) \in \{(1, 1), (2, 1)\}$ we can explain how to classify all such triangles for which this period collapse occurs. Specifically, we show:

**Theorem 1.11.** Assume that $p$ and $q$ are relatively prime.

(i) If $(k, l) \in \{(1, 1), (2, 1)\}$, then the Ehrhart quasipolynomial of $T_{\frac{p}{q}, \frac{q}{p}}$ has period $kl$ if and only if for some $n \geq 0$, $(p, q) = (r(k, l)_{2n}, r(k, l)_{2n})$ or $(p, q) = (l r(k, l)_{2n}, kr(k, l)_{2n \pm 1})$.

(ii) The Ehrhart quasipolynomial of $T_{\frac{p}{q}, \frac{q}{p}}$ has period 6 if $(p, q) = (r(3, 2)_{2n \pm 1}, r(3, 2)_{2n})$ or $(p, q) = (2r(3, 2)_{2n}, 3r(2, 1)_{2n \pm 1})$.

For example, Theorem 1.11 gives examples of triangles with arbitrarily high denominator and period 1, compare [12, Ex. 2.1].

1.5. Irrational Ehrhart theory. For irrational polytopes, very little is known concerning the counting function $L_P(t)$; it is generally expected that this Ehrhart function is neither an honest polynomial, nor a polynomial with periodic coefficients. However, by applying Theorem 1.3 to well-chosen rational triangles, we can produce irrational triangles with Ehrhart function a polynomial. The following is illustrative:
Example 1.12. Continuing along the lines of Example 1.4, let \((k, l, p, q) = (1, 1, g_n, g_{n+1})\). Then, by Theorem 1.3, the triangles \(T_{g_{n+1}, g_n, g_{n+1}}\) and \(T_{1, 1, g_n, g_{n+1}}\) are Ehrhart equivalent. In particular, the Ehrhart function of \(T_{g_{n+1}, g_n, g_{n+1}}\) is a polynomial, regardless of \(n\). Now, the rational numbers \(g_{n+1}/g_n\) converge to the irrational number \(\tau^2 = \frac{3 + \sqrt{5}}{2}\), where \(\tau\) denotes the Golden Mean. Using this observation, it is not hard to show that the Ehrhart function of the irrational triangle \(T_{\frac{\tau^2}{\tau^2}}\) is a polynomial.

The observation in Example 1.12 was further explored in [10], where some basic examples of “irrational Ehrhart theory” were worked out in detail.

1.6. Subleading asymptotics. ECH capacities \(c_k\) are defined for any symplectic 4-manifold \((X, \omega)\). When every \(c_k(X)\) is finite, and \(X\) is a Liouville domain — that is, a compact symplectic manifold with oriented boundary, such that \(\omega|_{\partial X} = d\lambda\) for a contact form \(\lambda\) — then the ECH capacities satisfy an asymptotic formula. Namely, it is shown in [9] that, under these conditions,

\[
\lim_{k \to \infty} \frac{c_k(X, \omega)^2}{k} = 4 \text{vol}(X, \omega),
\]

where \(\text{vol}(X, \omega) = \frac{1}{2} \int_X \omega \wedge \omega\). In the setting of (1.1), (1.6) implies that ECH capacities recover the volume obstruction \(c(a) \geq \sqrt{a}\).

As mentioned in Remark 1.5, much of the work in this article turns out to be connected to the subleading asymptotics of the ECH capacities \(c_k\). More specifically, in view of (1.6), we can define

\[
e_k := 2\sqrt{\text{vol}(X, \omega)}\sqrt{k} - c_k.
\]

We can now ask for precise asymptotics of the \(e_k\). When \(X = E(a, b)\), then the asymptotics of the \(e_k\) can be computed exactly; when \(a/b\) is irrational, one has \(\lim_{k \to \infty} e_k = \frac{a+b}{2}\), see [11, Prop. 15]. This is relevant to Theorem 1.3, because (1.2) exactly comes from consideration of the subleading asymptotics: (1.2) is the condition obtained by setting the subleading asymptotics of the domain and target to be equal, see Remark 1.5.

The importance of the subleading asymptotics to Theorem 1.2 suggests that it might be valuable to study the subleading asymptotics more generally. The subleading asymptotics of ECH capacities were
explored in much more detail in [11], where it was proved that for a general Liouville domain with all ECH capacities finite, the $c_k$ are $O(k^{2/5})$. Connections with this article and with further symplectic embedding problems were also discussed, see especially [11, §4.4]

1.7. Other aspects of Theorem 1.2. While our primary interest here is the Fibonacci staircase from Theorem 1.2, we mention how to determine parts of the rest of Theorem 1.2 purely combinatorially from Theorem 1.1, and we also briefly touch on some of the significance of these relationships for symplectic geometry.

- The question of deducing the third bullet point of Theorem 1.2 from Theorem 1.1 was partially explored in [8, Lem. 3.2.3], see [8, Rmk. 3.2.5.ii]. More precisely, we show here that from Theorem 1.1, one can deduce that
  $$c(a) = \frac{a + 1}{3}, \quad \tau^4 \leq a \leq 7.$$  
  The combinatorial ideas used to prove this have symplectic significance: indeed, they were a key component in the proof of [8, Prop. 3.2.1], which in turn is important for the work in [8] on higher-dimensional analogues of the function in (1.1).

  In view of [8, Lem. 3.2.3], to obtain all of the full third bullet point using Theorem 1.1, one would only have to show the result for $7 < a < \left(\frac{17}{6}\right)^2$. This is a meaningful simplification to the problem, because one should be able to prove that in this range, in fact only finitely many of the $c_k(E(1,a))$ can even give an obstruction beyond the volume, in analogy with the arguments in [20, §5], particularly [20, Lem. 5.1.2].

- Concerning the second bullet point of Theorem 1.2, in the appendix we use Theorem 1.1 to show that if $a \geq 9$, then $c(a) = \sqrt{a}$. Improving this to hold for $a \geq \left(\frac{17}{6}\right)^2 = 8 \frac{1}{36}$ just by using Theorem 1.2 is an interesting combinatorial problem.

1.8. Outline of the paper. The paper is organized as follows. In §2, we collect some preliminary combinatorial facts that we will need to prove the key Theorem 1.3; some of these facts are well-known, and some are new. Using these preliminaries it is easy to give a quick proof of the “only if” direction of Theorem 1.3, which we also give in §2. We then give the proof of Theorem 1.3 in §3. Next, we find all the solutions to (1.2) in §4. We then use this classification to prove our symplectic staircase theorem, which we do in §5. The applications to period collapse are proven in §6. The appendix addresses the second bullet point of Theorem 1.2.
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2. Preliminaries

We begin by developing the combinatorial machinery that will be used in the proof of our main result, Theorem 1.3. Using this machinery, it is easy to prove the “only if” direction of Theorem 1.3, and so we give a proof of this direction as well.

2.1. Ehrhart quasipolynomials and Fourier-Dedekind sums. Given a triangle of the form

\[ T = \left\{ (x, y) \in \mathbb{R}^2 : x \geq \frac{a}{d}, y \geq \frac{b}{d}, ex + fy \leq r \right\}, \]

one can use generating functions to compute the Ehrhart quasipolynomial of \( T \). This is explained in [1, §2]. We will only need to consider the special case where \( a = b = 0, e = kp^2, f = lq^2, \) and \( r = pq \), for \( p, q, k, l \) positive integers with \( kp^2 \) and \( lq^2 \) relatively prime. In this case, [1, Thm. 2.10] gives:

\[
L_T(t) = \frac{1}{2kl}t^2 + \frac{1}{2} \left( \frac{q}{kp} + \frac{p}{lq} + \frac{1}{klpq} \right) t \\
+ \frac{1}{4} \left( 1 + \frac{k}{kp^2} + \frac{1}{lq^2} \right) + \frac{1}{12} \left( \frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{klpq^2} \right) \\
+ s_{-tpq}(lq^2, 1; kp^2) + s_{-tpq}(kp^2, 1; lq^2),
\]

where \( s_n \) denotes the Fourier-Dedekind sum

\[
s_n(a_1, a_2; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1 - \xi_b^{a_1 k})(1 - \xi_b^{a_2 k})}.
\]

Here and in the following sections, \( \xi_b \) denotes the primitive \( b^{th} \) root of unity \( \xi_b = e^{\frac{2\pi i}{b}} \).

We can now give the following proof, which was already alluded to in Remark 1.5.
Proof of the “only if” direction of Theorem 1.3. Since Theorem 1.3 is an equivalence of quasipolynomials, we must have equality in the coefficients of each power of $t$. Equating the linear terms given by (2.1) yields the Diophantine equation (1.2), and thus gives the “only if” direction of Theorem 1.3.

2.2. Convolutions. In view of (2.1), the hard part of Theorem 1.3 is evaluating the expression

\[
 s_{-tpq}(kp^2, 1; lq^2) + s_{-tpq}(lq^2, 1; kp^2).
\]

So for $(k, l, p, q)$ satisfying (1.2), consider the sum

\[
 s_{-tpq}(kp^2, 1; lq^2) = \frac{1}{lq^2} \sum_{j=1}^{lq^2-1} \frac{\xi_{lq^2}^{-tpq} j}{(1 - \xi_{lq^2}^{j kp^2})(1 - \xi_{lq^2}^{j})},
\]

Writing $j = ilq + u$ for $0 \leq i < q$, $0 \leq u < lq$ gives

\[
\frac{1}{lq^2} \sum_{i=1}^{q-1} \frac{\xi_{lq^2}^{-tpq} i}{(1 - \xi_{lq^2}^{kp^2 ilq})(1 - \xi_{lq^2}^{ilq})} + \frac{1}{lq^2} \sum_{u=1}^{lq^2-1} \frac{\xi_{lq^2}^{-tpq} u}{(1 - \xi_{lq^2}^{ukp^2 ilq})(1 - \xi_{lq^2}^{u ilq})}.
\]

The sum

\[
 s_{-tpq}(lq^2, 1; kp^2) = \frac{1}{kp^2} \sum_{j=1}^{kp^2-1} \frac{\xi_{kp^2}^{-tpq} j}{(1 - \xi_{kp^2}^{j lq})(1 - \xi_{kp^2}^{j})}
\]

can be similarly rewritten to obtain

\[
\frac{1}{kp^2} \sum_{i=1}^{p-1} \frac{1}{(1 - \xi_{p}^{-i})(1 - \xi_{p}^{i})} + \frac{1}{kp^2} \sum_{u=1}^{kp^2-1} \frac{1}{(1 - \xi_{kp^2}^{ulq})(1 - \xi_{kp^2}^{u ilq})}.
\]

The inner sums of (2.3) and (2.4) are convolutions that can be evaluated explicitly using the Fourier transform.

2.3. Fourier transform. To evaluate the convolutions in (2.3) and (2.4), we will apply the following general lemma:

Lemma 2.1. Let $a_1$, $a_2$, $b$ and $c$ be integers such that $b$ divides neither $a_1$ nor $a_2$. Then

\[
\frac{1}{c} \sum_{k=0}^{c-1} \frac{1}{(1 - \xi_{bc}^{a_1 + kb})(1 - \xi_{bc}^{a_2 - kb})} = \frac{\gamma}{(1 - \xi_{bc}^{a_1 c})(1 - \xi_{bc}^{a_2 c})},
\]
where

\[
\gamma = \begin{cases} 
\frac{1 - \xi^{-a_1+a_2}}{1 - \xi^{-a_1+a_2}} & \text{if } bc \not| a_1 + a_2, \\
\frac{c}{1 - \xi^{-a_1+a_2}} & \text{if } bc | a_1 + a_2.
\end{cases}
\]

We will be most interested in the lemma when in addition \( b \) divides \( a_1 + a_2 \) but \( bc \) does not divide \( a_1 + a_2 \). In this case, Lemma 2.1 implies that the sum in (2.5) is equal to 0.

**Proof.** Our proof is given in three steps.

**Step 1.** This step summarizes the inputs from finite Fourier analysis.

If \( f \) is a function with period \( b \), recall that its Fourier transform is the function

\[
\hat{f}(n) = \frac{1}{b} \sum_{k=0}^{b-1} f(k) \xi_b^{-kn}.
\]

The convolution of two periodic functions \( f, g \) with period \( b \) is given by

\[
(f \ast g)(n) = \sum_{m=0}^{b-1} f(n - m) g(m).
\]

A version of the convolution theorem [1, Thm. 7.10] for the Fourier transform says that

\[
(2.6) \quad (f \ast g)(n) = b \sum_{k=0}^{b-1} \hat{f}(k) \hat{g}(k) \xi_b^{kn}.
\]

**Step 2.** We can compute the Fourier transform of the family of functions that are relevant to the proof of Lemma 2.1 explicitly:

**Lemma 2.2.** Fix positive integers \( a, b \) and \( c \) such that \( b \) does not divide \( a \). Let \( f_a \) be the periodic function of period \( c \) given by

\[
f_a(n) := \frac{1}{1 - \xi^{a+bn}}.
\]

Then for integers \( 0 \leq n \leq c - 1, \)

\[
\hat{f}_a(n) = \frac{\xi^{an}}{1 - \xi^{ac}}.
\]
Proof. For notational simplicity, throughout this proof let $\xi := \xi_{bc}$. For $n \geq 0$, we have

$$\hat{f}_a(n) = \frac{1}{c} \sum_{k=0}^{c-1} \frac{\xi^{-knb}}{1 - \xi a + kb}$$

$$= \frac{\xi^{an}}{c} \sum_{k=0}^{c-1} \frac{\xi^{-(a+kb)n}}{1 - \xi a + kb}$$

$$= \frac{\xi^{an}}{c} \sum_{k=0}^{c-1} \left( \frac{1}{1 - \xi a + kb} - \frac{1 - \xi^{-(a+kb)n}}{1 - \xi a + kb} \right)$$

$$= \frac{\xi^{an}}{c} \sum_{k=0}^{c-1} \left( \frac{1}{1 - \xi a + kb} + \sum_{m=1}^{n} \xi^{-(a+kb)m} \right).$$

We can break the last line up into two sums and interchange the order of summation in the last sum to get

$$\frac{\xi^{an}}{c} \sum_{k=0}^{c-1} \frac{1}{1 - \xi a + kb} + \frac{1}{c} \sum_{m=1}^{n} \left( \xi^{a(n-m)} \sum_{k=0}^{c-1} \xi^{-km} \right). \tag{2.7}$$

The innermost sum on the right hand side of (2.7) is 0 if $c$ does not divide $m$. Since $m \leq n$, when $0 \leq n \leq c - 1$ the sum always vanishes and

$$\hat{f}_a(n) = \frac{\xi^{an}}{c} \sum_{k=0}^{c-1} \frac{1}{1 - \xi a + kb}. \tag{2.8}$$

To simplify the summation, let $z_k = \frac{1}{1 - \xi a + kb}$ and note that the $z_k$ are the roots of the degree-$c$ polynomial $(z - 1)^c = \xi^{ac} z^c$. Hence

$$(z - 1)^c - \xi^{ac} z^c = (1 - \xi^{ac}) \prod_{k=0}^{c-1} (z - z_k). \tag{2.9}$$

Equating the coefficient of $z^{c-1}$ on each side of (2.9) gives

$$\sum_{k=0}^{c-1} \frac{1}{1 - \xi a + kb} = \frac{c}{1 - \xi^{ac}}, \tag{2.10}$$

and Lemma 2.2 now follows by combining (2.8) and (2.10). \qed
Step 3. The sum in (2.5) is \((f_{a_1} * f_{a_2})(0)\), so by (2.6) and Lemma 2.2,
\[
(f_{a_1} * f_{a_2})(0) = c \sum_{k=0}^{c-1} \left( \frac{\xi_{a_1}^k}{1 - \xi_{a_1}} \right) \left( \frac{\xi_{a_2}^k}{1 - \xi_{a_2}} \right) = \frac{c}{(1 - \xi_{a_1})(1 - \xi_{a_2})} \sum_{k=0}^{c-1} \xi^{(a_1 + a_2)_k}.
\]
The sum in the last line evaluates to \(c\) if \(bc|a_1 + a_2\), and otherwise we have
\[
\sum_{k=0}^{c-1} \xi^{(a_1 + a_2)_k} = \frac{1 - \xi^{(a_1 + a_2)c}}{1 - \xi_{a_1 + a_2}}.
\]
This completes the proof. \(\square\)

2.4. Reciprocity. Certain expressions with Fourier-Dedekind sums can be evaluated directly. The proof of Theorem 1.3 uses the following result from [1, Thm. 8.8]:

Lemma 2.3. (Rademacher Reciprocity) Let \(n = 1, 2, \ldots, (a+b+c)-1\). Then
\[
s_n(a, b; c) + s_n(c, a; b) + s_n(b, c; a) = -\frac{n^2}{2abc} + \frac{n}{2} \left( \frac{1}{ab} + \frac{1}{ca} + \frac{1}{bc} \right) - \frac{1}{12} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right).
\]

There is another reciprocity statement for \(n = 0\):

Lemma 2.4. [1, Cor. 8.7]:
\[
s_0(a, b; c) + s_0(c, a; b) + s_0(b, c; a) = 1 - \frac{1}{12} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right).
\]

3. Proof of the main theorem

We will now apply the machinery from the previous section to prove Theorem 1.3. We showed the “only if” direction in §2.1, so it remains to show the “if” direction. Assume \(k\) and \(l\) both divide \(k + l + 1\), that \(kp^2\) and \(lq^2\) are relatively prime, and that \((k, l, p, q)\) satisfies (1.2). The proof that \(T_{\frac{k}{kp^2}, \frac{p}{q}}(t) = T_{\frac{k}{k+1}}(t)\) for all positive integers \(t\) follows in four steps.

Step 1. Both \(T_{\frac{k}{kp^2}, \frac{p}{q}}\) and \(T_{\frac{k}{k+1}}\) are quadratic quasipolynomials in \(t\).

By (2.1) both have the same coefficient of \(t^2\), and by (1.2) both have the same coefficient of \(t\). It remains to show that they have the same constant term.
Step 2. To evaluate the relevant Fourier-Dedekind sums, the following elementary fact will be useful:

**Lemma 3.1.** $q$ is relatively prime to $\frac{k+l+1}{l}$ and $p$ is relatively prime to $\frac{k+l+1}{k}$.

**Proof.** Since $(k, l) \in \{(1, 1), (2, 1), (3, 2)\}$, we can argue case by case. If $(k, l) = (1, 1)$, then Lemma 3.1 follows by reducing (1.2) mod 3. If $(k, l) = (2, 1)$, then reducing (1.2) mod 8 shows that $p$ and $\frac{k+l+1}{k}$ are relatively prime, and reducing (1.2) mod 4 shows that $q$ and $\frac{k+l+1}{l}$ are relatively prime. Finally, if $(k, l) = (3, 2)$ then reducing (1.2) mod 3 shows that $q$ and $\frac{k+l+1}{l}$ are relatively prime, and reducing (1.2) mod 2 shows that $p$ and $\frac{k+l+1}{k}$ are relatively prime. \qed

Step 3. We now begin the computation of the Fourier-Dedekind sums. By (1.2), (2.3), and (2.4) we have

$$s_{-tpq}(kp^2, 1; lq^2) + s_{-tpq}(lq^2, 1; kp^2) =$$

$$\frac{1}{lq^2} \sum_{i=1}^{q-1} \frac{1}{(1 - \xi^{-i})(1 - \xi^1_q)} + \frac{1}{lq^2} \sum_{u=1}^{lq-1} \left( \frac{q-1}{(1 - \xi_{lq^2}^{u} - ilq)(1 - \xi_{u+ilq}^l)} \right) +$$

$$\frac{1}{kp^2} \sum_{i=1}^{p-1} \frac{1}{(1 - \xi^{-i}_p)(1 - \xi^1_p)} + \frac{1}{kp^2} \sum_{u=1}^{kp-1} \left( \frac{p-1}{(1 - \xi_{kp^2}^{u} - ikp)(1 - \xi_{u+ikp}^k)} \right).$$

By (1.2) we always have $q|u(kp^2+1)$, and by Lemma 3.1, $lq^2|u(kp^2+1)$ if and only if $q|u$. So applying Lemma 2.1 with $b = lq, c = q, a_1 = u$ and $a_2 = ukp^2$ gives

$$\sum_{i=0}^{q-1} \frac{1}{(1 - \xi_{lq^2}^{u} - ilq)(1 - \xi_{u+ilq}^l)} = \begin{cases} 0 & \text{if } q \nmid u \\ \frac{q^2}{(1 - \xi_{lq}^u)(1 - \xi_q^1)} & \text{if } q | u \end{cases}$$

An identical argument gives

$$\sum_{i=0}^{p-1} \frac{1}{(1 - \xi_{kp^2}^{u} - ikp)(1 - \xi_{u+ikp}^k)} = \begin{cases} 0 & \text{if } p \nmid u \\ \frac{p^2}{(1 - \xi_{kp}^u)(1 - \xi_k^1)} & \text{if } p | u \end{cases}$$

Now $kp^2 \equiv -1 \pmod{l}$ and $lq^2 \equiv -1 \pmod{k}$ by (1.2), so combining (3.2) and (3.3) gives

$$s_{-tpq}(kp^2, 1; lq^2) + s_{-tpq}(lq^2, 1; kp^2) =$$
(3.4) \[
\frac{1}{lq^2} \sum_{i=1}^{q-1} (1 - \xi_i)(1 - \xi_i^{-1}) + \frac{1}{kp^2} \sum_{i=1}^{p-1} (1 - \xi_i^{-1})(1 - \xi_i) + \\
\frac{1}{l} \sum_{i=1}^{l-1} \frac{\xi_i^{-tq}}{(1 - \xi_i)(1 - \xi_i^{-1})} + \frac{1}{k} \sum_{i=1}^{k-1} \frac{\xi_i^{-tp}}{(1 - \xi_i)(1 - \xi_i^{-1})}.
\]

**Step 4.** By (2.1), we must show

(3.5) \[
\frac{1}{4} \left( 1 + \frac{1}{kp^2} + \frac{1}{lq^2} \right) + \frac{1}{12} \left( \frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{klp^2q^2} \right) \\
+ s_{tpq}(lq^2, 1; kp^2) + s_{tpq}(kp^2, 1; lq^2) \\
= \frac{1}{4} \left( 1 + \frac{1}{k} + \frac{1}{l} \right) + \frac{1}{12} \left( \frac{k}{l} + \frac{l}{k} + \frac{1}{kl} \right) + s_t(l, 1; k) + s_t(k, 1; l).
\]

for all \( t \leq 0 \). The right hand side of (3.5) is periodic in \( t \) with period \( kl \), and by (3.4), the left hand side is as well. For \((k, l) = (3, 2)\), by (3.4) the right hand side is equal at \( t = 1 \) and \( t = 5 \), and is equal at \( t = 2 \) and \( t = 4 \). Thus, when \((k, l) = (1, 1)\) we can assume \( t = 0 \), when \((k, l) = (2, 1)\) we can assume \( t = 0 \) or \( 1 \), and when \((k, l) = (3, 2)\) we can assume \( 0 \leq t \leq 3 \).

When \( t = 0 \), we can apply Lemma 2.4 to both sides of (3.5) to get the desired equality. For other \( t \), we can apply Rademacher reciprocity to evaluate \( s_{tpq}(kp^2, 1; lq^2) + s_{tpq}(lq^2, 1; kp^2) \) as long as \( 0 < tpq < kp^2 + lq^2 \). This holds for all \( p, q \) when \( 0 < t < 2\sqrt{kl} \), and we can always assume \( t \) lies in this range by the previous paragraph. For these \( t \) Lemma 2.3 gives

(3.6) \[
s_{tpq}(kp^2, 1; lq^2) + s_{tpq}(lq^2, 1; kp^2) \\
= -\frac{1}{12} \left( \frac{3}{kp^2} + \frac{3}{lq^2} + 3 + \frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{klp^2q^2} \right) \\
- \frac{t^2}{2kl} + \frac{t}{2} \left( \frac{1}{klpq} + \frac{q}{kp} + \frac{p}{lq} \right).
\]

Since (3.6) also holds for \((p, q) = (1, 1)\), and because

(3.7) \[
\frac{pq}{2} \left( \frac{1}{klp^2q^2} + \frac{1}{kp^2} + \frac{1}{lq^2} \right) = \frac{1}{2} \left( \frac{1}{kl} + \frac{1}{k} + \frac{1}{l} \right)
\]
by (1.2), Theorem 1.3 follows in this case as well by (2.1).

**4. Solving the Diophantine equation**

In this section we prove the following:
Proposition 4.1. Suppose \((k,l) \in \{(1,1), (2,1), (3,2)\}\). Then \((p,q)\) is a solution to (1.2) if and only if \((p,q) = (r(k,l)_{2n+1}, r(k,l)_{2n})\) for some \(n\).

Remark 4.2. To prove Theorem 1.3, we only need the “if” direction of Proposition 4.1. When \((k,l) \in \{(1,1), (2,1)\}\), the “only if” direction will be used in the proof of Theorem 1.11. We include the \((k,l) = (3,2)\) case here for completeness in view of Theorem 1.3.

Proof. Fix \((k,l) \in \{(1,1), (2,1), (3,2)\}\), and consider the pair of congruence relations
\[
ka^2 \equiv -1 \pmod{lb}, \quad lb^2 \equiv -1 \pmod{ka}.
\]
Since \(k\) and \(l\) both divide \(k + l + 1\), if \((p,q)\) satisfies (1.2) then \((a,b) = (p,q)\) is a solution to (4.1). We will show that the converse holds, so it suffices to classify the solutions of (4.1), and we will then show this is precisely the set of \((r(k,l)_{2n+1}, r(k,l)_{2n})\).

We first solve (4.1). The key observation is that if \((p,q)\) is a solution of (4.1), then
\[
p' := \frac{lq^2 + 1}{kp}, \quad q' := \frac{kp^2 + 1}{lq}
\]
are integers and \((p',q')\) and \((p,q')\) are also solutions to (4.1). Motivated by this, we define the involutions
\[
\sigma : (p,q) \mapsto \left(\frac{lq^2 + 1}{kp}, q\right), \quad \tau : (p,q) \mapsto (p, \frac{kp^2 + 1}{lq}).
\]
We claim that if \((p,q) \neq (1,1)\) then either \(\sigma\) or \(\tau\) decreases a coordinate. Suppose that \(p \leq p'\) and \(q \leq q'\). Then \(|kp^2 - lq^2| \leq 1\). If \(kp^2 = lq^2\) then \((k,l,p,q) = (1,1,1,1)\), if \(kp^2 = lq^2 + 1\) then \(lq|2\) so \((l,q) \in \{(1,1), (1,2), (2,1)\}\), and if \(kp^2 = lq^2 - 1\) then \((k,p) \in \{(1,1), (1,2), (2,1)\}\). By examining each of these cases separately we see that if \(k \geq l, p' \geq p, q' \geq q\), then
\[
(k,l,p,q) \in \{(1,1,1,1), (2,1,1,1), (3,2,1,1), (5,1,1,2)\}.
\]
In particular, if we assume in addition that \((k,l) \in \{(1,1), (2,1), (3,2)\}\), then \((p,q) = (1,1)\).

Now define the sequence \(s(k,l)_n\) by \(s(k,l)_0 = s(k,l)_1 = 1\),
\[
s_{2n+1} = \frac{ls(k,l)_{2n}^2 + 1}{ks(k,l)_{2n-1}}, \quad s(k,l)_{2n} = \frac{kls(k,l)_{2n-1} + 1}{ls(k,l)_{2n-2}}.
\]
If \((p,q)\) satisfies (4.1) then \((p,q) = (s(k,l)_{2n+1}, s(k,l)_{2n})\) for some \(n\). This follows by induction after applying either \(\sigma\) or \(\tau\). Another induction using (4.2) shows that \((s(k,l)_{2n+1}, s(k,l)_{2n})\) satisfies (1.2). Thus, the solutions of (1.2) and (4.1) are the same.
To show the \( r_n = s_n \) for all \( n \), we induct using (1.4) and (1.5) to get
\[
kr_{2n+1}^2 - (k + l + 1)r_{2n+1}r_{2n} + br_{2n}^2 = -1,
\]
(4.3)
\[
kr_{2n-1}^2 - (k + l + 1)r_{2n-1}r_{2n} + br_{2n}^2 = -1.
\]
(4.4)
We can then apply a final induction using the recurrence relations (4.2), (1.5) and (1.4).

\[\Box\]

5. Infinite staircases in symplectic embedding problems

We now apply our key Theorem 1.3, and the classification result from the previous section, to deduce the “symplectic staircase theorem” Theorem 1.7. In this section, we give the details for all of this.

There are several basic properties of \( c(a, \frac{k}{l}) \) that significantly simplify the proof of Theorem 1.7:

**Lemma 5.1.** Fix \( k \) and \( l \). Then the function \( c(a, \frac{k}{l}) \) satisfies:

(i) (Continuity) \( c(a, \frac{k}{l}) \) is a continuous function of \( a \).

(ii) (Monotonicity) \( c(a, \frac{k}{l}) \) is a monotonically nondecreasing function of \( a \).

(iii) (Subscaling) \( c(\lambda a, \frac{k}{l}) \leq \lambda c(a, \frac{k}{l}) \) when \( \lambda > 1 \).

**Proof.** To prove statement (i), note that for any \( \epsilon > 0 \), if \( a_i \) is close enough to \( a \) then \( E(1,a_i) \subset (1+\epsilon)E(1,a) \) and \( E(1,a) \subset (1+\epsilon)E(1,a_i) \).

Statement (i) then follows from the observation that if \( E(1,x) \hookrightarrow E(c,c \frac{k}{l}) \) then \( (1+\epsilon)E(1,x) \hookrightarrow (1+\epsilon)E(c,c \frac{k}{l}) \). Statement (ii) follows from the fact that if \( x \leq y \) then \( E(1,x) \subset E(1,y) \). Statement (iii) follows because we have
\[
E(1,a) \subset \sqrt{\lambda}E(1,a)
\]
for any \( \lambda > 1 \), and we know that
\[
\sqrt{\lambda}E(c,c \frac{k}{l}) = E(\lambda c, \lambda c \frac{k}{l}).
\]

\[\Box\]

Our strategy for proving Theorem 1.7 is now to calculate \( c(a(k,l)_n, \frac{k}{l}) \), bound \( c(b(k,l)_n, \frac{k}{l}) \) from below, and apply Lemma 5.1.

5.1. Relevance of Ehrhart functions. We first explain the basic relationship between Ehrhart functions and symplectic embeddings of ellipsoids, via Theorem 1.1.

For positive real numbers \( a, b \) and \( t \), let
\[
\mathcal{N}(a, b; t) = \# \{ i : c_i(E(a,b)) \leq t \}.
\]
By Theorem 1.1,

\[
E(a, b) \hookrightarrow E(c, d)
\]

if and only if

\[
\mathcal{N}(a, b; t) \geq \mathcal{N}(c, d; t)
\]

for all \( t \). Now assume \( c \) and \( d \) are integers. Then \( c_k(E(c, d)) \) is always an integer, so it suffices to check \( \mathcal{N}(a, b; t) \geq \mathcal{N}(c, d; t) \) for \( t \) a positive integer. Since \( \mathcal{N}(a, b; t) = L_{T_{\frac{a}{b}, \frac{c}{d}}}(t) \) for such \( t \), we have proven the following in view of Theorem 1.1:

**Lemma 5.2.** Let \( c \) and \( d \) be integers. Then \( E(a, b) \hookrightarrow E(c, d) \) if and only if

\[
L_{T_{\frac{a}{b}, \frac{c}{d}}}(t) \geq L_{T_{\frac{c}{d}, \frac{a}{b}}}(t) \quad \forall t \in \mathbb{Z}_{>0}.
\]

Note that by scaling, to determine when one rational ellipsoid symplectically embeds into another, it suffices to consider integral ellipsoids.

5.2. **Calculating** \( c(a(k, l)_n, \frac{k}{l}) \). We first claim that \( c(a(k, l)_n, \frac{k}{l}) \) is always equal to the volume obstruction. To simplify the notation, we now let \( a_n, b_n \), and \( r_n \) denote \( a(k, l)_n, b(k, l)_n \), and \( r(k, l)_n \) for fixed \( (k, l) \in \{(1, 1), (2, 1), (3, 2)\} \).

By definition, \( \frac{a_{2n}}{k} = \frac{r_{2n+1}}{r_{2n}} \) and \( \frac{a_{2n+1}}{l} = \frac{r_{2n+2}}{k r_{2n+1}} \). To show

\[
c(a_{2n}, \frac{k}{l}) = \sqrt{\frac{a_{2n}l}{k}} = \frac{r_{2n+1}}{r_{2n}}, \tag{5.1}
\]

\[
c(a_{2n+1}, \frac{k}{l}) = \sqrt{\frac{a_{2n+1}l}{k}} = \frac{lr_{2n+2}}{kr_{2n+1}}, \tag{5.2}
\]

it suffices by Lemma 5.2 to show that

\[
L_{T_{\frac{a_n}{r_n}, \frac{k}{r_{2n}}}}(t) \geq L_{T_{\frac{c}{d}, \frac{a_n}{r_n}}}(t) \tag{5.3}
\]

when \( (p, q) = (r_{2n \pm 1}, r_{2n}) \).

By induction, (1.4) and (1.5) show that \( r_{2n+1} \) and \( r_{2n} \) are relatively prime. Since \( k|l|k+l+1 \) for \( (k, l) \in \{(1, 1), (2, 1), (3, 2)\} \), induction also shows that \( k \not{|} r_{2n} \), \( l \not{|} r_{2n+1} \). Then (5.3) follows from (4.3), (4.4) and Theorem 1.3.

5.3. **Calculating** \( c(b(k, l)_n, \frac{k}{l}) \). By Lemma 5.1 and (5.3), to prove Theorem 1.7 it remains to show that

\[
c(b_n, \frac{k}{l}) \geq \sqrt{\frac{la_{n+1}}{k}} = \begin{cases} \frac{r_{n+2}}{r_{n+1}} & n \text{ odd,} \\ \frac{l}{r_{n+2}} & n \text{ even.} \end{cases}
\]
We will show that for the index
\[(5.4) \quad f_n := \frac{r_{n+2}r_n + r_{n+2} + r_n - 1}{2},\]
we have
\[(5.5) \quad c_{f_n}(E(1, b_n)) = r_{n+2},\]
\[(5.6) \quad c_{f_n}(E(1, \frac{k}{l})) = \begin{cases} r_{n+1} & n \text{ odd}, \\ \frac{k}{l}r_{n+1} & n \text{ even}. \end{cases}\]

We begin with the proof of (5.5). We have
\[
\max_m \{m : c_m(E(1, b_n)) \leq r_{n+2}\} = -1 + \sum_{i=0}^{r_{n+2}} \left( \left\lfloor \frac{i}{b_n} \right\rfloor + 1 \right)
= r_{n+2} + r_n + \sum_{i=0}^{r_{n+2}-1} \left\lfloor \frac{ir_n}{r_{n+2}} \right\rfloor
= \frac{(r_{n+2} + 1)(r_n + 1)}{2},
\]
where the last line follows from the well-known identity
\[
\sum_{i=0}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}
\]
for \((p, q) = 1\). The fact that \(\gcd(r_{n+2}, r_n) = 1\) follows from an induction using (1.4) and (1.5). Since
\[
\#\{m : c_m(E(1, b_n)) = r_{n+2}\} = 2,
\]
we have that \(c_{f_n}(E(1, b_n)) = c_{f_{n+1}}(E(1, b_n))\), and (5.5) follows.

We next prove (5.6). We have that
\[
(5.7) \quad f_{2n} = \frac{r_{2n+2}r_{2n} + r_{2n+2} + r_{2n} - 1}{2}
= \frac{1}{2}((k + l + 1) \frac{k}{l} r_{2n+1} - r_{2n}(r_{2n} + 1) + r_{2n} - 1)
= \frac{kr_{2n+1}^2 + (k + l + 1)r_{2n+1} - (l - 1)}{2l},
\]
where the second line follows from (1.5) and the last line follows from (4.3). Similarly, by (1.4) and (4.4),
\[
(5.8) \quad f_{2n-1} = \frac{kr_{2n}^2 + (k + l + 1)r_{2n} - (k - 1)}{2k}.
\]
By (2.1),
\[
\max_m \{ \left(1 - \frac{k}{l} \right) \} = L\sigma_{n,k,l}(lr) - 1
\]
\[
= \frac{l^2r_{2n} + (k + l + 1)r_{2n}}{2k} + \frac{1}{4} \left(1 + \frac{1}{k} + \frac{1}{l}\right)
\]
\[
+ \frac{1}{12} \left(1 + \frac{1}{k} + \frac{1}{l} + \frac{1}{kl}\right) + s_{-lr_{2n}}(l,1;k) + s_{-lr_{2n}}(k,1;l) - 1.
\]
For \( n \) even, this is equal to \( f_{2n-1} \) if
\[
\frac{k + 1}{2k} = \frac{1}{4} \left(1 + \frac{1}{k} + \frac{1}{l}\right) + \frac{1}{12} \left(1 + \frac{1}{k} + \frac{1}{l} + \frac{1}{kl}\right) + s_{-lr_{2n}}(l,1;k) + s_{-lr_{2n}}(k,1;l).
\]
Similarly, (5.6) holds for \( n \) odd if
\[
\frac{l + 1}{2l} = \frac{1}{4} \left(1 + \frac{1}{k} + \frac{1}{l}\right) + \frac{1}{12} \left(1 + \frac{1}{k} + \frac{1}{l} + \frac{1}{kl}\right) + s_{-kr_{2n+1}}(l,1;k) + s_{-kr_{2n+1}}(k,1;l).
\]
Induction on (1.4) and (1.5) gives \( 2 \nmid r_{2n} \), \( 3 \nmid r_{2n+1} \). By direct computation, (5.9) and (5.10) hold for each \((k,l) \in \{(1,1),(2,1),(3,2)\} \). This completes the proof of Theorem 1.7.

6. Period collapse

We conclude by proving our Theorem 1.11, about period collapse. Assume throughout that \((k,l) \in \{(1,1),(2,1),(3,2)\} \), and continue the notation of the previous section by letting \( r_n \) denote \( r(k,l)_n \).

We first prove the “if” statements of Theorem 1.11. If \((p,q) = (r_{2n+1},r_{2n}) \) then, as explained in §4, \( kp^2 \) and \( lq^2 \) are relatively prime and \((k,l,p,q) \) satisfies (1.2). Thus, by Theorem 1.3 \( T_{k,l}^{p,q} \) is Ehrhart equivalent to \( T_{k,l}^{q,p} \) and so \( T_{k,l}^{p,q} \) has period \( kl \). Similarly, if \((p,q) = (lr(k,l)_{2n},kr(k,l)_{2n+1}) \) then for
\[
(p',q') := (\frac{p}{l},\frac{q}{k})
\]
kq'^2 and lp'^2 are relatively prime and \((k,l,q',p') \) satisfies (1.2). Hence, by Theorem 1.3, \( T_{k,l}^{p,q} \) is Ehrhart equivalent to \( T_{k,l}^{q,p} \). Thus, \( T_{k,l}^{p,q} \) has period \( kl \), since \( T_{k,l}^{p,q} \) is Ehrhart equivalent to \( T_{k,l}^{q,p} \).

Assume now in addition that \((k,l) \in \{(1,1),(2,1)\} \). We now complete the proof of Theorem 1.3 by proving the “only if” statements. If \( kp^2 \) and \( lq^2 \) are relatively prime and \( T_{k,l}^{p,q} \) has period \( kl \), then by (2.1)
we must have
\[
(6.1) \quad s_{klpq}(kp^2, 1; lq^2) + s_{klpq}(lq^2, 1; kp^2) = s_0(kp^2, 1; lq^2) + s_0(lq^2, 1; kp^2).
\]
We know in addition that \( klpq \leq kp^2 + lq^2 \). Hence, we can apply Lemma 2.3 and Lemma 2.4 to \( (6.1) \) to conclude that \((k, l, p, q)\) satisfies \((1.2)\), so the "only if" direction of Theorem 1.11 follows by Proposition 4.1.

If \( kp^2 \) and \( lq^2 \) are not relatively prime, then we must have \((k, l) = (2, 1)\) and it must also be the case that \( q \) is divisible by 2 and \( p \) is not divisible by 2. Define \( q' := \frac{q}{2} \). We know that \( 2q^2 \) and \( p^2 \) are relatively prime. Moreover, \( \mathcal{T}_{\frac{a}{2p^2}, q'} \) is Ehrhart equivalent to \( \mathcal{T}_{\frac{2p^2}{q'}, \frac{q'}{q}} \). If \( \mathcal{T}_{\frac{2p^2}{q'}, \frac{q'}{q}} \) has period 2 then by \((2.1)\) we must have
\[
(6.2) \quad s_{2pq'}(2p^2, 1; q'^2) + s_{2pq'}(q'^2, 1; 2p^2) = s_0(2p^2, 1; q'^2) + s_0(q'^2, 1; 2p^2).
\]
Since \( 2pq' \leq 2p^2 + q^2 \), we can apply Lemma 2.3 and Lemma 2.4 to \((6.2)\) to conclude that \((2, 1, q', p)\) satisfies \((1.2)\). Theorem 1.11 again then follows by Proposition 4.1.

Appendix A. A combinatorial proof of flexibility for sufficiently stretched ellipsoids

The purpose of this appendix is to explain how Theorem 1.1 implies the following:

**Theorem A.1.** Let \((k, l) \in \{ (1, 1), (2, 1), (3, 2) \} \). Then
\[
(A.1) \quad c_{\frac{k}{l}}(a) = \sqrt{\frac{a}{k}} \quad \text{for all} \quad a \geq \frac{k}{l}(1 + \frac{l+1}{k})^2.
\]

One can view this as a quantitative "flexibility" result. Namely, recall from the introduction that the right hand side of equation \((A.1)\) represents the volume obstruction. Theorem A.1 then says that if an ellipsoid \( E(1, a) \) is sufficiently stretched, then all obstructions to embedding it into an \( E(1, \frac{k}{l}) \) vanish except for the classical volume obstruction; moreover, Theorem A.1 gives a quantitative bound on how stretched the ellipsoid must be.

We include Theorem A.1 to complement Theorem 1.7, in view of Theorem 1.2.

**Remark A.2.** The method in the proof of Theorem A.1 can be adapted to establish equations like \((A.1)\) for other \( k \) and \( l \). As \( k \) and \( l \) vary, all one needs in the proof is a bound like \((A.8)\), which can often be found by direct computation using \((2.1)\).
To place Theorem A.1 in its appropriate context, note that McDuff and Schlenk prove $c(a, 1) = \sqrt{a}$ for all $a \geq \left(\frac{17}{6}\right)^2$, and Frenkel and Müller prove $c(a, 2) = \sqrt{a/2}$ for all $a \geq 1 \left(\frac{15}{4}\right)^2$. Both proofs use methods that differ from ours. In [3, Thm. 1.3], Buse and Hind show that for any $k, l$ with $k \geq l$, (A.1) holds for all

$$a \geq \frac{k}{l} \left(\frac{5}{4} + \frac{4l}{k}\right)^2.$$

When Theorem A.1 applies, it gives sharper bounds than (A.2). In particular, for $(k, l) = (3, 2)$, (A.2) gives the bound $a \geq \frac{2209}{96} \approx 23.01$, while Theorem A.1 gives the bound $a \geq 6$.

**Proof.** If $a_k$ and $b_k$ are two sequences (indexed with $k$ starting at 0), define a new sequence

$$(a \# b)_k := \sup_{k_1 + k_2 = k} a_{k_1} + b_{k_2}.$$

This is the *sequence sum* operation originally defined by Hutchings in [13].

Let $\mathcal{N}(a, b)$ be the sequence whose $k$th term is $c_k(E(a, b))$. For the sequences $\mathcal{N}(a, b)$, the sequence sum operation satisfies the identity

$$\mathcal{N}(a, b) \# \mathcal{N}(a, c) = \mathcal{N}(a, b + c),$$

for $a, b,$ and $c$ any positive integers. This is proven by an elementary argument in [18, Lem. 2.4]. It follows, see [18, §2], that if $a$ is rational then there is a finite sequence of positive rational numbers $(a_1, \ldots, a_n)$ associated to $a$, called a *weight sequence* for $a$, such that

$$(A.3) \quad a_i \leq 1 \quad \forall i,$$

$$(A.4) \quad \sum_{i=1}^{n} a_i^2 = a,$$

and

$$(A.5) \quad \mathcal{N}(1, a) = \mathcal{N}(a_1, a_1) \# \ldots \# \mathcal{N}(a_n, a_n).$$

The weight sequence is closely related to the continued fraction expansion for $a$, see [18, §2].

By (A.5), to prove Theorem A.1, it suffices to show that

$$(A.6) \quad \sum_{i=1}^{n} ld_i a_i \leq d \sqrt{\frac{a_l}{k}}.$$
whenever \((d_1, \ldots, d_n, d)\) are nonnegative integers satisfying

\[(A.7) \quad \sum_{i=1}^{n} \frac{d_i^2 + d_i}{2} + 1 \leq L_{T_i/k,1/l}(d).\]

For \((k, l) \in \{(1, 1), (2, 1), (3, 2)\}\), we know by direct computation that

\[(A.8) \quad L_{T_i/k,1/l}(d) \leq \frac{d^2}{2kl} + (\frac{1}{2k} + \frac{1}{2l} + \frac{1}{2kl})d + 1.\]

If

\[\sum_{i=1}^{n} d_i^2 \leq \frac{d^2}{kl},\]

then \((A.6)\) follows by applying \((A.4)\) and the Cauchy-Schwarz inequality. Thus, if \((d_1, \ldots, d_n, d)\) satisfies \((A.7)\), we can assume that

\[(A.9) \quad \sum_{i=1}^{n} d_i \leq (\frac{1}{k} + \frac{1}{l} + \frac{1}{kl})d.\]

Then by \((A.3)\),

\[(A.10) \quad \sum_{i=1}^{n} ld_i a_i \leq (1 + \frac{l+1}{k})d.\]

Since \((1 + \frac{l+1}{k}) \leq \sqrt{\frac{a}{k}}\) if \(a \geq \frac{k}{l}(1 + \frac{l+1}{k})^2\), Theorem A.1 follows. \(\square\)

References

[5] D. Cristofaro-Gardiner Symplectic embeddings of ellipsoids, preprint available upon request
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