SUBLEADING ASYMPTOTICS OF THE ECH SPECTRUM

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ABSTRACT. In previous work [1], the first author and collaborators showed that the leading asymptotics of the embedded contact homology (ECH) spectrum recover the contact volume. Our main theorem here is a new bound on the sub-leading asymptotics.

1. Introduction

1.1. The main theorem. Let $Y$ be a closed, oriented three-manifold. A contact form on $Y$ is a one-form $\lambda$ satisfying

$$\lambda \wedge d\lambda > 0.$$ 

A contact form determines the Reeb vector field, $R$, defined by

$$\lambda(R) = 1, \quad d\lambda(R, \cdot) = 0, \quad \lambda(R) = 1,$$

and the contact structure $\xi := \text{Ker}(\lambda)$. Closed orbits of $R$ are called Reeb orbits.

If $(Y, \lambda)$ is a closed three-manifold equipped with a nondegenerate contact form and $\Gamma \in H_1(Y)$, then the embedded contact homology $ECH(Y, \lambda, \Gamma)$ is defined. This is the homology of a chain complex freely generated over $\mathbb{Z}_2$ by certain sets of Reeb orbits in homology class $\Gamma$, relative to a differential that counts certain $J$-holomorphic curves in $\mathbb{R} \times Y$. (ECH can also be defined over $\mathbb{Z}$, but for the applications in this paper we will not need this.) It is known that the homology only depends on $\xi$, and so we sometimes denote it $ECH(Y, \lambda, \xi)$. Any Reeb orbit $\gamma$ has a symplectic action

$$A(\gamma) = \int_{\gamma} \lambda$$

and this induces a filtration on $ECH(Y, \lambda)$; we can use this filtration to define a number $c_\sigma(Y, \lambda)$ for every nonzero class in ECH, called the spectral invariant associated to $\sigma$; the spectral invariants are $C^0$.

2000 Mathematics Subject Classification. 53D35, 57R57, 57R58.
D. C-G. is partially supported by NSF grant 1711976.
N. S. is partially supported by the DFG funded project CRC/TRR 191.
continuous and so can be extended to degenerate contact forms as well. We will review the definition of ECH and of the spectral invariants in 2.1.

When the class \(c_1(\xi) + 2P.D.(\gamma) \in H^2(Y; \mathbb{Z})\) is torsion, then \(ECH(Y, \xi, \Gamma)\) has a relative \(\mathbb{Z}\) grading, which we can refine to a canonical absolute grading \(gr^Q\) by rationals \([17]\), and which we will review in 2.3. It is known that for large gradings the group is eventually 2-periodic and non-vanishing:

\[
ECH_*(Y, \xi, \Gamma) = ECH_{*+2}(Y, \xi, \Gamma) \neq 0, \quad * \gg 0.
\]

The main theorem of \([11]\) states that in this case, the asymptotics of the spectral invariants recover the contact volume

\[
\text{vol}(Y, \lambda) = \int_Y \lambda \wedge d\lambda.
\]

Specifically:

**Theorem 1.** \([11]\) Thm. 1.3] Let \((Y, \lambda)\) be a closed, connected oriented three-manifold with a contact form, and let \(\Gamma \in H_1(Y)\) be such that \(c_1(\xi) + 2P.D.(\Gamma)\) is torsion. Then if \(\{\sigma_j\}\) is any sequence of nonzero classes in \(ECH(Y, \xi, \Gamma)\) with definite gradings tending to positive infinity,

\[
\lim_{j \to \infty} \frac{c_{\sigma_j}(Y, \lambda)^2}{gr^Q(\sigma_j)} = \text{vol}(Y, \lambda).
\]

The formula \([11]\) has had various implications for dynamics. For example, it was a crucial ingredient in recent work \([10]\) of the first author and collaborators showing that many Reeb vector fields on closed three-manifolds have either two or infinitely many distinct closed orbits, and it was used in \([9]\) to show that every Reeb vector field on a closed three-manifold has at least two distinct closed orbits. It has also been used to prove \(C^\infty\) closing lemmas for Reeb flows on closed three-manifolds and Hamiltonian flows on closed surfaces \([1, 16]\).

By (1.1), we can write

\[
c_{\sigma_j}(Y, \lambda) = \sqrt{\text{vol}(Y, \lambda) \cdot gr^Q(\sigma_j)} + d(\sigma_j),
\]

where \(d(\sigma_j)\) is \(o(gr^Q(\sigma_j)^{1/2})\) as \(gr^Q(\sigma_j)\) tends to positive infinity. It is then natural to ask:

**Question 1.** What can we say about the asymptotics of \(d(\sigma_j)\) as \(gr^Q(\sigma_j)\) tends to positive infinity?

Previously, W. Sun has shown that \(d(\sigma_j) = O(gr^Q(\sigma_j))^{125/252}\) \([26]\) Thm. 2.8]. Here we show:
Theorem 2. Let \((Y, \lambda)\) be a closed, connected oriented three-manifold with a nondegenerate contact form, and let \(\Gamma \in H_1(Y)\) be such that \(c_1(\xi) + 2P.D.(\Gamma)\) is torsion. Let \(\{\sigma_j\}\) be any sequence of nonzero classes in \(ECH(Y, \lambda, \Gamma)\) with definite gradings tending to positive infinity. Define \(d(\sigma_j)\) by (1.2). Then \(d(\sigma_j)\) is \(O(\text{gr}^Q(\sigma_j))^{2/5}\) as \(\text{gr}^Q(\sigma_j) \to +\infty\).

We do not know whether or not the \(O(\text{gr}^Q(\sigma_j))^{2/5}\) asymptotics here are optimal — in other words, we do not know whether there is some contact form on a three-manifold realizing these asymptotics — although our best guess is that such a flow does not exist. We will show, though, in Section 4.3 that there exist contact forms with \(O(1)\) asymptotics for the \(d(\sigma_j)\). In Remark 12, we clarify where the exponent \(2/5\) comes from in our proof, and why the methods in the current paper can not improve on it.

Another topic which we do not address here, except in a very specific example, see Section 4.3, but which is of potential future interest, is whether the asymptotics of the \(d(\sigma_j)\) carry interesting geometric information. In this regard, a similar question in the context of the spectral flow of a one-parameter family of Dirac operators was recently answered in [21]. This is particularly relevant in the context of the argument we give here, as it also involves estimating spectral flow, see Remark 12.

1.2. A dynamical zeta function and a Weyl law. We now mention two corollaries of Theorem 2.

Given \(\Gamma \in H_1(Y)\), define a set of nonnegative real numbers, called the \(ECH\) spectrum for \((Y, \lambda, \Gamma)\)

\[
\Sigma_{(Y, \lambda, \Gamma)} := \cup_{*} \Sigma_{(Y, \lambda, \Gamma), *}
\]

\[
\Sigma_{(Y, \lambda, \Gamma), *} := \{ c_\sigma(\lambda) \mid 0 \neq \sigma \in ECH_{*}(Y, \xi, \Gamma; \mathbb{Z}_2) \}.
\]

(To emphasize, in the set \(\Sigma_{(Y, \lambda, \Gamma), *}\) we are fixing the grading \(*\).) Then, define the Weyl counting function for \(\Sigma_{(Y, \lambda, \Gamma)}\)

\[
N_{(Y, \lambda, \Gamma)}(R) := \# \{ c \in \Sigma_{(Y, \lambda, \Gamma)} \mid c \leq R \}.
\]

We now have the following:

Corollary 3. If \(c_1(\xi) + 2PD(\Gamma)\) is torsion, then the Weyl counting function (1.3) for the \(ECH\) spectrum satisfies the asymptotics

\[
N(R) = \left\lceil \frac{2d - 1}{\text{vol}(Y, \lambda)} \right\rceil R^2 + O(R^{9/5})
\]

where \(d = \dim ECH_*(Y, \xi, \Gamma; \mathbb{Z}_2) + \dim ECH_{*+1}(Y, \xi, \Gamma; \mathbb{Z}_2), * \gg 0\).
As another corollary, one may obtain information on the corresponding dynamical zeta function. To this end, first note that the ECH zeta function

\[ \zeta_{ECH}(s; Y, \lambda, \Gamma) := \sum_{c \neq 0 \in \Sigma(Y, \lambda, \Gamma)} c^{-s} \]

converges for \( \text{Re}(s) > 2 \) by 1.1 and defines a holomorphic function of \( s \) in this region whenever \( c_1(\xi) + 2\text{P.D.}(\Gamma) \) is torsion, by 1.4.

In view of for example [13, 14], one can ask if \( \zeta_{ECH} \) has a meromorphic continuation to \( \mathbb{C} \), and, if so, whether it contains interesting geometric information. The Weyl law (1.4) then shows:

**Corollary 4.** The zeta function (1.5) continues meromorphically to the region \( \text{Re}(s) > \frac{5}{2} \). The only pole in this region is at \( s = 2 \) which is further simple with residue \( \text{Res}_{s=2} \zeta_{ECH}(s; Y, \lambda, \Gamma) = \left[ \frac{2^d - 1}{\text{vol}(Y, \lambda)} \right] \).

In Section 4.3, we give an example of a contact form for which \( \zeta_{ECH} \) has a meromorphic extension to all of \( \mathbb{C} \) with two poles at \( s = 1, 2 \). The meromorphy and location of the poles of (1.5) would be interesting to figure out in general.

1.3. Idea of the proof and comparison with previous works.

The method of the proof uses previous work by C. Taubes relating embedded contact homology to Monopole Floer homology. By using Taubes’s results, we can estimate spectral invariants associated to nonzero ECH classes by estimating the energy of certain solutions of the deformed three-dimensional Seiberg-Witten equations. This is also the basic idea behind the proofs of Theorem 1 and the result of Sun mentioned above, and it was inspired by a similar idea in Taubes’s proof of the three-dimensional Weinstein conjecture 23.

The essential place where our proof differs from these arguments involves a particular estimate, namely a key “spectral flow” bound for families of Dirac operators that appears in all of these proofs. This estimate bounds the difference between the grading of a Seiberg-Witten solution, and the “Chern-Simons” functional, which we review in 2.2 and is important in all of the works mentioned above. We prove a stronger bound of this kind than any previous bound, see Proposition 5 and the discussion about the eta invariant below, and this is the key point which allows us to prove \( O(\text{gr}^Q(\sigma_j))^{2/5} \) asymptotics. Spectral flow bounds for families of Dirac operators were also considered in [19, 20, 25]. The main difference here is that in those works the bounds were proved on reducible solutions where the connections needed to
define the relevant Dirac operators were explicitly given. Here we must consider irreducible solutions, and so we rely on a priori estimates.

We have chosen to phrase this spectral flow bound in terms of a bound on the “eta” invariants of a family of operators. By the Atiyah-Patodi-Singer index theorem, the bound we need on the spectral flow is equivalent to a bound on the eta invariant, and we make the relationship between these two quantities precise in the appendix.

The paper is organized as follows. In Section 2 we review what we need to know about embedded contact homology, Monopole Floer cohomology and Taubes’s isomorphism. Section 3 reviews the necessary estimates on irreducible solutions to the Seiberg-Witten equations, and proves the key Proposition 5. We then give the proof of Theorem 2 in Section 4 — while our argument in this section is novel, one could instead argue here as in [26], but we give our own argument here since it might be of independent interest, see Remark 14. The end of the paper reviews the sub-leading asymptotics and the dynamical zeta function in the case of ellipsoids, and an appendix rephrases the grading in Seiberg-Witten in terms of the eta invariant rather than in terms of spectral flow.

1.4. Acknowledgments. The first author thanks M. Hutchings, D. McDuff, and W. Sun for very helpful discussions.

2. Floer homologies

We begin by reviewing the facts that we will need about ECH and Monopole Floer homology.

2.1. Embedded contact homology. We first summarize what we will need to know about ECH. For more details and for definitions of the terms that we have not defined here, see [15].

Let \((Y, \lambda)\) be a closed oriented three-manifold with a nondegenerate contact form. Fix a homology class \(\Gamma \in H_1(Y)\). As stated in the introduction, the embedded contact homology \(ECH(Y, \lambda, \Gamma)\) is the homology of a chain complex \(ECC(Y, \lambda, \Gamma)\). To elaborate, the chain complex \(ECC\) is freely generated over \(\mathbb{Z}/2\) by orbit sets \(\alpha = \{(\alpha_j, m_j)\}\) where the \(\alpha_j\)'s are distinct embedded Reeb orbits while each \(m_j \in \mathbb{N}\); we further have the constraints that \(\sum m_j \alpha_j = \Gamma \in H_1(Y)\) and \(m_j = 1\) if \(\alpha_j\) is hyperbolic. To define the chain complex differential \(\partial\), we consider the symplectization \((\mathbb{R}_t \times Y, d(e^t \lambda))\), and choose an almost complex structure \(J\) that is \(\mathbb{R}\)-invariant, rotates the contact hyperplane \(\xi := \ker \lambda\) positively with respect to \(d\lambda\), and satisfies \(J \partial_t = R\). The differential
on \( ECC (Y, \lambda, \Gamma) \) is now defined via
\[
\partial \alpha = \sum_{\beta} \frac{\sharp [M_1(\alpha, \beta) / \mathbb{R}]}{\langle \partial \alpha, \beta \rangle} \beta.
\]

Here \( M_1(\alpha, \beta) \) denotes the moduli space of \( J \)-holomorphic curves \( C \) of ECH index \( I(C) = 1 \) in the symplectization, modulo translation in the \( \mathbb{R} \)-direction, and modulo equivalence as currents, with the set of positive ends given by \( \alpha \) and the set of negative ends given by \( \beta \). If \( J \) is generic, then the differential squares to zero \( \partial^2 = 0 \) and defines the ECH group \( ECH (Y, \lambda; \Gamma) \). We will not review the definition of the ECH index here, see [15] for more details, but the key point is that the condition \( I(C) = 1 \) forces \( C \) to be (mostly) embedded and rigid modulo translation.

As stated in the introduction, the homology \( ECH (Y, \lambda; \Gamma) \) does not depend on the choice of generic \( J \), and only depends on the associated contact structure \( \xi \); we therefore denote it \( ECH (Y, \xi; \Gamma) \). (In fact, the homology only depends on the spin\( ^c \) structure determined by \( \xi \), but we will not need that.) This follows from a canonical isomorphism between ECH and Monopole Floer cohomology [24], which we will soon review. The ECH index \( I \) induces a relative \( \mathbb{Z} / d \mathbb{Z} \) grading on \( ECH (Y, \xi; \Gamma) \), where \( d \) is the divisibility of \( c_1(\xi) + 2P.D. (\Gamma) \in H^2 (Y; \mathbb{Z}) \) mod torsion. In particular, it is relatively \( \mathbb{Z} \)-graded when this second homology class is torsion.

Recall now the action of a Reeb orbit from the introduction. This induces an action on orbit sets \( \alpha = \{ (\alpha_j, m_j) \} \) by
\[
A (\alpha) := \sum_{j=1}^{N} m_j \left( \int_{\alpha_j} \lambda \right).
\]

The differential decreases action, and so we can define \( ECC^L (Y, \lambda, \Gamma) \) to be the homology of the sub-complex generated by orbit sets of action no more than \( L \). The homology of this sub-complex \( ECH^L (Y, \lambda, \Gamma) \) is again independent of \( J \) but now depends on \( \lambda \); there is an inclusion induced map \( ECH^L (Y, \lambda, \Gamma) \to ECH (Y, \xi, \Gamma) \). Using this filtration, we can define the spectral invariant associated to a nonzero class \( \sigma \) in ECH
\[
c_\sigma (Y, \lambda) := \inf \{ L | \sigma \in \text{image} (ECH^L (Y, \lambda, \Gamma) \to ECH (Y, \xi, \Gamma)) \}.
\]

2.2. Monopole Floer cohomology. We now briefly review what we need to know about Monopole Floer cohomology, referring to [17] for additional details and definitions.
Recall that a spin\(^c\) structure on a three-manifold \(Y\) is a pair \((S,c)\) consisting of a rank 2 complex Hermitian vector bundle and a Clifford multiplication endomorphism \(c : T^*Y \otimes \mathbb{C} \to \text{End}(S)\) satisfying 
\[
c(e_1)^2 = -1 \quad \text{and} \quad c(e_1)c(e_2)c(e_3) = 1
\]
for any oriented orthonormal frame \((e_1,e_2,e_3)\) of \(T_yY\). Let \(\mathfrak{su}(S)\) denote the bundle of traceless, skew-adjoint endomorphisms of \(S\) with inner product \(\frac{1}{2}\text{tr}(A^*B)\). Clifford multiplication \(c\) maps \(T^*Y\) isometrically onto \(\mathfrak{su}(S)\). Spin\(^c\) structures exist on any three-manifold, and the set of spin\(^c\) structures is an affine space over \(H^2(Y;\mathbb{Z})\). A spin\(^c\) connection \(A\) on \(S\) is a connection such that \(c\) is parallel. Given two spin\(^c\) connections \(A_1, A_2\) on \(S\), their difference is of the form \(A_1 - A_2 = a \otimes 1^S\) for some \(a \in \Omega^1(Y, i\mathbb{R})\).

If we denote by \(A_1', A_2'\) the induced connections on \(\text{det}(S) = \Lambda^2 S\), we then have \(A_1' - A_2' = 2a\). Hence prescribing a spin\(^c\) connection on \(S\) is the same as prescribing a unitary connection on \(\text{det}(S)\). We let \(\mathcal{A}(Y,s)\) denote the space of all spin\(^c\) connections on \(S\). Given a spin\(^c\) connection \(A\), we denote by \(\nabla^A\) the associated covariant derivative. We then define the spin\(^c\) Dirac operator \(D_A : C^\infty(S) \to C^\infty(S)\) via 
\[
D_A \Psi = c \circ \nabla^A \Psi.
\]

Given a spin\(^c\) structure \(s = (S,c)\) on \(Y\), monopole Floer cohomology assigns three groups denoted by \(\HM(Y,s), \HM(Y,s)\) and \(\HM(Y,s)\). These are defined via infinite dimensional Morse theory on the configuration space \(\mathcal{C}(Y,s) = \mathcal{A}(Y,s) \times C^\infty(S)\) using the Chern-Simons-Dirac functional \(\mathcal{L}\), defined as 
\[
\mathcal{L}(A,\Psi) = -\frac{1}{8} \int_Y (A^t - A_0^t) \wedge (F_{A^t} + F_{A_0^t}) + \frac{1}{2} \int_Y \langle D_A \Psi, \Psi \rangle \ dy \tag{2.1}
\]

using a fixed base spin-c connection \(A_0\) (with \(A_0^t\) flat) and a metric \(g^{TY}\).

The gauge group \(\mathcal{G}(Y) = \text{Map}(Y,S^1)\) acts on the configuration space \(\mathcal{C}(Y,s)\) by \(u.(A,\Psi) = (A - u^{-1} du \otimes I, u\Psi)\). The gauge group action is free on the irreducible part \(\mathcal{C}^\sigma(Y,s) = \{(A,\Psi) \in \mathcal{C}(Y,s) | \Psi \neq 0\} \subset \mathcal{C}(Y,s)\) and not free along the reducibles. The blow up of the configuration space along the reducibles 
\[
\mathcal{C}^\sigma(Y,s) = \{(A,s,\Phi) \mid \|\Phi\|_{L^2} = 1, s \geq 0\}
\]
then has a free \(\mathcal{G}(Y)\) action \(u \cdot (A,s,\Phi) = (A - u^{-1} du \otimes I, s,u\Phi)\).

To define the Monopole Floer cohomology groups one needs to perturb the Chern-Simons-Dirac functional \(\mathcal{L}\). First given a one form \(\mu \in \Omega^1(Y; i\mathbb{R})\), one defines the functional \(e_\mu(A) := \int_Y \mu \wedge F_{A^t}\) whose gradient is calculated to be \(s d\mu\). To achieve non-degeneracy and transversality of configurations one uses the perturbed Chern-Simons-Dirac
where $\mu$ is a finite linear combination of eigenvectors of $*d$ with non-zero eigenvalue. Next let

$$\mathcal{T} = \{ A \in \mathcal{A}(Y, \mathfrak{s}) \mid F_{A^t} = 0 \} / \mathcal{G}(Y)$$

be the space of $A^t$ flat spin-c connections up to gauge equivalence. We choose a Morse function $f : \mathcal{T} \to \mathbb{R}$ to define the functional $f : C^\sigma(Y, s) \to \mathbb{R}$, $f(A_0 + a, s, \Psi) := f([A_0 + a^{h}])$, where $a^{h}$ denotes the harmonic part of $a \in \Omega^1(Y, \mathbb{i}\mathbb{R})$. The gradient may be calculated

$$\nabla f^\sigma_A = \left( (\nabla f)_{p(A)}, 0, 0 \right).$$

The Monopole Floer cohomology groups are now defined using solutions $(A, s, \Phi) \in C^\sigma(Y, \mathfrak{s})$ to the three-dimensional Seiberg-Witten equations

$$\begin{align*}
\frac{1}{2} * F_{A^t} + s^2 c^{-1} (\Phi \Phi^*)_0 + (\nabla f)_{p(A)} + s d\mu &= 0 \\
\Lambda(A, s, \Phi) &= 0
\end{align*}$$

(2.3)

$$D_A \Phi = \Lambda(A, s, \Phi) \Phi = 0$$

where $\Lambda(A, s, \Phi) = (D_A \Phi, \Phi)_{L^2}$ and $(\Phi \Phi^*)_0 := \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2$ defines a traceless, Hermitian endormophism of $S$. We denote by $\mathfrak{C}$ the set of solutions to the above equations.

We first subdivide the solutions as follows:

$$\mathfrak{C}^o = \{(A, s, \Phi) \in \mathfrak{C} \mid s \neq 0 \} / \mathcal{G}(Y),$$

$$\mathfrak{C}^s = \{(A, 0, \Phi) \in \mathfrak{C} \mid \Lambda(A, 0, \Phi) > 0 \} / \mathcal{G}(Y)$$

$$= \left\{ (A, \Phi) \mid \frac{1}{2} F_{A^t} + d\mu = 0, [A] \text{ is a critical point of } f, \right. \Phi \text{ is a (positive-)normalized eigenvector of } D_A \} / \mathcal{G}(Y)$$

$$\mathfrak{C}^u = \{(A, 0, \Phi) \mid \Lambda(A, 0, \Phi) < 0 \} / \mathcal{G}(Y)$$

$$= \left\{ (A, \Phi) \mid \frac{1}{2} F_{A^t} + d\mu = 0, [A] \text{ is a critical point of } f, \right. \Phi \text{ is a (positive-)normalized eigenvector of } D_A \} / \mathcal{G}(Y).$$

Next, we consider the free $\mathbb{Z}_2$ modules generated by the three sets above

$$C^o = \mathbb{Z}_2[\mathfrak{C}^o], \ C^s = \mathbb{Z}_2[\mathfrak{C}^s], \ C^u = \mathbb{Z}_2[\mathfrak{C}^u].$$

The chain groups for the three versions of Floer cohomology mentioned above are defined by

$$\mathcal{C}^o = C^o \oplus C^s, \ \mathcal{C}^u = C^o \oplus C^u, \ \mathcal{C} = C^s \oplus C^u.$$
These chain groups $\hat{C}, \hat{\hat{C}}, \hat{\hat{C}}$ can be endowed with differentials $\hat{\partial}, \hat{\hat{\partial}}, \hat{\hat{\partial}}$ with square zero; we do not give the precise details here, but the idea is to count Fredholm index one solutions of the four-dimensional equations, see [24] Thm. 22.1.4 for the details. The homologies of these three complexes are by definition the three monopole Floer cohomology groups

$$HM (Y, s), \hat{\hat{H}} (Y, s), \hat{H}M (Y, s).$$

They are independent of the choice of metric and perturbations $\mu, \xi$.

Each of the above Floer groups has a relative $\mathbb{Z}/d\mathbb{Z}$ grading where $d$ is the divisibility of $c_1 (S) \in H^2 (Y; \mathbb{Z})$ mod torsion. This is defined using the extended Hessian

$$\hat{H}_{(A, \Psi)} : C^\infty (Y; iT^* Y \oplus \mathbb{R} \oplus S) \to C^\infty (Y; iT^* Y \oplus \mathbb{R} \oplus S); (A, \Psi) \in \mathcal{C} (Y, s)$$

$$(2.4)$$

$$\begin{bmatrix} a \\ f \\ \psi \end{bmatrix} = \begin{bmatrix} *d a + 2c^{-1} (\psi \Psi)_0 - df \\ -d^* a + i \text{Re} (\psi, \Psi) \\ c (a) \Psi + D_A \psi + f \Psi \end{bmatrix} \begin{bmatrix} a \\ f \\ \psi \end{bmatrix}$$

The relative grading between two irreducible generators $a_i = (A_i, s_i, \Phi_i), (s_i \neq 0), i = 1, 2,$ is now defined via $\text{gr} (a_1, a_2) = \text{sf} \{ \hat{H}_{(A_t, \Psi_t)} \}_{0 \leq t \leq 1}$ (mod $d$) for some path of configurations $(A_t, \Psi_t)$ starting at $(A_2, s_2 \Phi_2)$ and ending at $(A_1, s_1 \Phi_1)$, where $\text{sf}$ denotes the spectral flow.

In the case when the spin-c structure is torsion, the monopole Floer groups are further equipped with an absolute $\mathbb{Q}$-grading, refining this relative grading. As we will review in the appendix, this is given via

$$(2.5)$$

$$\text{gr}^Q [a] = \begin{cases} 2k + \eta (D_A) + \frac{1}{3} \eta_Y - \frac{1}{4 \pi^2} CS (A); & a = (A, 0, \Phi^0_k) \in \mathcal{C}^s, \\ \eta \left( \hat{H}_{(A, s \Phi^k)} \right) - \frac{2}{3} \eta_Y - \frac{1}{4 \pi^2} CS (A); & a = (A, s, \Phi) \in \mathcal{C}^c, \quad s \neq 0. \end{cases}$$

where $\Phi^k$ above denotes the $k$th positive eigenvector of $D_A$ (see Section [A]), and $\eta_Y$ and $\eta_{D_A}$ denote the eta invariant of the corresponding operator, which we will review in Section [3].

2.3. ECH=HM. We now state the isomorphism between the ECH and HM, proved in [24]. Given a contact manifold $(Y^3, \lambda)$ with $d\lambda$-compatible almost complex structure $J$ as before, we define a metric $g^{TV}$ via $g^{TV} |_\xi = d\lambda (., J), |R| = 1$ and $R$ and $\xi$ are orthogonal. This metric is adapted to the contact form in the sense $*d\lambda = 2\lambda, |\lambda| = 1.$
Decompose \( \xi \otimes \mathbb{C} = K_{\xi^{0,0}} \oplus K^{-1}_{\xi^{0,1}} \) into the \( i, -i \) eigenspaces of \( J \). The contact structure now determines the canonical spin-c structure \( s^\xi \) via \( S^\xi = \mathbb{C} \oplus K^{-1} \) with Clifford multiplication \( c^\xi \) given by

\[
c^\xi(R) = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix},
\]

\[
c^\xi(v) = \begin{bmatrix} v^{0,1} \wedge -i v^{1,0} \\ -i v^{0,1} \end{bmatrix}, \quad v \in \xi.
\]

Furthermore, there is a unique spin-c connection \( A_c \) on \( S^\xi \) with the property that \( D A_c \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = 0 \) and we call the induced connection \( A_t \) on \( K^{-1} = \det(S^\xi) \) the canonical connection. Tensor product with an auxiliary Hermitian line bundle \( E \) via \( S^E = S^\xi \otimes E \) and \( c^E = c^\xi \otimes 1 \) gives all other spin-c structures \( s^E \). Furthermore all spin-c connections on \( S^E \) arise as \( A = A_c \otimes 1 + 1 \otimes \nabla^A \) for some unitary connection \( \nabla^A \) on \( E \). The ECH/HM isomorphism is then

\[
(\hat{H}M)^{-\ast}(Y, s^E) = ECH^\ast(Y, \xi; \text{P.D.} c_1(E)).
\]

This reverses the relative gradings. We now state the main ideas involved in the isomorphism (we restrict attention to the case when \( c_1(\det s^E) \) is torsion relevant here). To this end, let \( \sigma \in \hat{H}M(Y, s^E) \). We use the perturbed Chern-Simons-Dirac functional (2.2) and its gradient flow (2.3) with \( \mu = ir\lambda, \ r \in [0, \infty) \), in defining monopole Floer homology. Giving a family of (isomorphic) monopole Floer groups \( \hat{H}M^r(Y, s^E) \), the class \( \sigma \) is hence representable by a formal sum of solutions to (2.3) corresponding to \( \mu = ir\lambda \). Denote by \( \hat{C}^r \) the \( \mu = ir\lambda \) version of the complex \( \hat{C} \) and note that its reducible generators are all of the form \( a = (A, 0, \Phi_k) \) where \( A = A_1 + ir\lambda, \ A_1^t \) flat and \( \Phi_k \) the \( k \)th positive eigenvector of \( D_A \). An important estimate \( \eta(D_{A_1 + r\lambda}) = O \left( r^{31/16} \right) \) now gives that the grading of this generator \( \text{gr}^Q[a] = 2k - \eta(D_{A_1 + r\lambda}) + \frac{1}{4} \eta_Y + r^2 \int \lambda \wedge d\lambda > \text{gr}^Q[\sigma] \) by (2.5) for \( r \gg 0 \). Hence for \( r \gg 0 \) the class \( \sigma \) is represented by a formal sum of irreducible solutions to (2.3) with \( \mu = ir\lambda \), and by a min-max argument, we may choose a family \( (A_r, \Psi_r) := (A_r, s_r, \Phi_r) \) satisfying

\[
\text{gr}^Q[\sigma] = \text{gr}^Q[(A_r, \Psi_r)] = \eta(\hat{H}_{(A_r, \Psi_r)}) - \frac{3}{4} \eta_Y - \frac{1}{4 \pi^2} CS(A_r).
\]

Following a priori estimates on solutions to the Seiberg-Witten equations, one then proves another important estimate \( \eta(\hat{H}_{(A_r, \Psi_r)}) = \)
\( O \left( r^{\frac{21}{16}} \right) \) uniformly in the class \( \sigma \). This gives \( CS(A_r) = O \left( r^{\frac{21}{16}} \right) \) which in turn by a differential relation (see Section 4) leads to \( e_\lambda(A_r) = O(1) \). The final step in the proof shows that for any sequence of solutions \((A_r, \Psi_r)\) to Seiberg-Witten equations with \( e_\lambda(A_r) \) bounded, the \( E \)-component \( \Psi^+_r \) of the spinor \( \Psi_r = \begin{bmatrix} \psi^+_r \\ \psi^-_r \end{bmatrix} \in C^\infty \left( Y; E \oplus K^{-1}E \right) \) satisfies the weak convergence \((\Psi^+_r)^{-1}(0) \rightharpoonup \{(\alpha_j, m_j)\} \) to some ECH orbit set. This last orbit set is what corresponds to the image of \( \sigma \in \hat{HM}(Y, s^E) \) in ECH under the isomorphism (2.6). Furthermore, crucially for our purposes, one has

\[
(2.7) \quad c_\sigma(\lambda) = \lim_{r \to \infty} \frac{e_\lambda(A_r)}{2\pi},
\]

see [11, Prop. 2.6].

3. Estimating the eta invariant

Let \( D \) be a self-adjoint, elliptic, differential operator acting on sections of a complex Hermitian vector bundle \( E \) over a closed, oriented Riemannian manifold \( Y \). Then the sum

\[
(3.1) \quad \eta(D, s) := \sum_{\lambda \neq 0} \frac{\text{sgn}(\lambda)}{|\lambda|^s}
\]

is a convergent analytic function of a complex variable \( s \), as long as \( \text{Re}(s) \) is sufficiently large; here, the sum is over the nonzero eigenvalues of \( D \). Moreover, the function \( \eta(D, s) \) has an analytic continuation to a meromorphic function on \( \mathbb{C} \) of \( s \), which we also denote by \( \eta(D, s) \), and which is holomorphic near 0. We now define

\[
\eta(D) := \eta(D, 0).
\]

We should think of this as a formal signature of \( D \), which we call the \textit{eta invariant} of Atiyah-Patodi-Singer [2].

We will be primarily concerned with the case where \( D = DA_r \), namely \( D \) is the Dirac operator for a connection \( A_r \) solving (2.3). Another case of interest to us is where \( D \) is the \textit{odd signature operator} on \( C^\infty(Y; iT^*Y \oplus \mathbb{R}) \) sending

\[
(a, f) \mapsto (\ast da - df, d^*a),
\]

in which case we denote the corresponding \( \eta \) invariant by \( \eta_Y \).

Now consider the Seiberg-Witten equations (2.3) corresponding to \( \mu = ir\lambda \), and note that an irreducible solution (after rescaling the
spinor) corresponds to a solution \((A_r, \Psi_r)\) to the Seiberg-Witten equations on \(\mathcal{C}(Y, s)\) given via

\[
c (\ast F_{A'}') + r (\Psi \Psi \ast)_{0} + i r \lambda = 0
\]

\[
D_{A} \Psi = 0.
\]

We can now state the primary result of this section:

**Proposition 5.** Any solution to (3.2) satisfies

\[
\eta \left( \mathcal{H}_{(A_r, \Psi_r)} \right) = O \left( r^{3/2} \right).
\]

The purpose of the rest of the section will be to prove this.

3.1. Known estimates. We first collect some known estimates on solutions to the equations (3.2).

**Lemma 6.** For some constants \(c_q, q = 0, 1, 2, \ldots\), we have

\[
|\nabla^q F_{A'}| \leq c_q \left( 1 + r^{1+q/2} \right).
\]

**Proof.** We first note that we have the estimates:

\[
|\Psi_r^{\pm}| \leq 1 + \frac{c_0}{r}
\]

\[
|\Psi_r^{-}| \leq \frac{c_0}{r} \left( 1 - |\Psi_r^{\pm}|^2 \right) + \frac{1}{r}
\]

\[
|\nabla^{A})^{q} \Psi_r^{\pm}| \leq c_q \left( 1 + r^{q/2} \right)
\]

\[
|\nabla^{A})^{q} \Psi_r^{-}| \leq c_q \left( 1 + r^{(q-1)/2} \right)
\]

The first two of these estimates are proved in [23, Lem. 2.2]. The third and fourth are proved in [23, Lem. 2.3].

The lemma now follows by combining the above estimates with the equation (3.2). □

In Section 4, we will also need:

**Lemma 7.** One has the bound

\[
|CS (A_r)| \leq c_0 r^{2/3} e_{\lambda} (A_r)^{4/3}
\]

where the constant \(c_0\) only depends on the metric contact manifold.

**Proof.** This is proven in [23, eq. 4.9], see also [11, Lem. 2.7]. □

3.2. The \(\eta\) invariant of families of Dirac operators. In this section, we prove the key Proposition 8. The main point that we need is the following fact concerning the \(\eta\) invariant:

**Proposition 8.** Let \(A_r\) be a solution to (3.2). Then \(\eta(D_{A_r})\) is \(O(r^{3/2})\) as \(r \to \infty\).
Before giving the proof, we first explain our strategy. The first point is that we have the following integral formula for the $\eta$ invariant:

$$\eta(D_A, r) = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \text{tr} (D_A e^{-tD_A^2}) dt$$

where the right hand side is a convergent integral. This is proved in [7, S 2], by Mellin transform it is equivalent to the fact that eta function $\eta(D_A, s)$ is holomorphic for $\text{Re}(s) > -2$.

We therefore have to estimate the integral in (3.5). To do this, we will need the following estimates:

**Lemma 9.** There exists a constant $c_0$ independent of $r$ such that for all $r \geq 1, t > 0$:

$$\left|\text{tr} (D_A e^{-tD_A^2})\right| \leq c_0 r^2 e^{c_0 rt},$$

and

$$\left|\text{tr} (e^{-tD_A^2})\right| \leq c_0 t^{-3/2} e^{c_0 rt}.$$

Once we have proved Lemma 9, Proposition 8 will follow from a short calculation, which we will give at the end of this section.

The proof of Lemma 9 will require two auxiliary lemmas, see Lemma 10 and Lemma 11 below, and some facts about the heat equation associated to a Dirac operator that we will now first recall. Let $D$ be a Dirac operator on a vector bundle $V$ over a closed manifold $Y$. The heat equation associated to $D$ is the equation

$$\frac{\partial s}{\partial t} + D^2 s = 0$$

for sections $s$, and nonnegative time $t$; the operator $e^{-tD^2}$ is the solution operator for this equation. The heat equation has an associated heat kernel $H_t(x, y)$ which is a (time-dependent) section of the bundle $V \otimes V$ over $Y \times Y$ whose fiber over a point $(x, y)$ is $V_x \otimes V_y^*$; it is smooth for $t > 0$. For any smooth section $s$ of $V$ and $t > 0$, the heat kernel satisfies

$$e^{-tD^2} s(x) = \int_Y H_t(x, y) s(y) \text{vol}(y).$$

Also,

$$\left[ \frac{\partial}{\partial t} + D_x^2 \right] H_t(x, y) = 0,$$

where $D_x$ denotes the Dirac operator applied in the $x$ variables.

Moreover,

$$\text{tr}(e^{-tD^2}) = \int_Y (H_t(y, y)) \text{vol}(y).$$
Hence, we can prove Lemma 9 by bounding $|H_t|$ along the diagonal. The operator $De^{-tD^2}$ has a kernel $L_t(x, y)$ as well, and the analogous results hold.

A final fact we will need is Duhamel’s principle: this says that the inhomogeneous heat equation

$$\frac{\partial \tilde{s}_t}{\partial t} + D^2 \tilde{s}_t = s_t$$

has a unique solution tending to 0 with $t$, given by

$$(3.9) \quad \tilde{s}_t(x) = \int_0^t (e^{-(t-t')D^2}s_{t'})(x) dt',$$

as long as $s_t$ is a smooth section of $S$, continuous in $t$.

Now let $D$ be $D_{A_r}$, and $V$ the spinor bundle for the spin$^c$ structure $S$, and let $H^r_t$ and $L^r_t$ be defined as above, but with $D = D_{A_r}$. Let $\rho(x, y)$ the Riemannian distance function. Define an auxiliary function

$$h_t(x, y) := (4\pi t)^{-3/2} e^{-\rho(x, y)^2/4t}.$$  

In the case of $Y = \mathbb{R}^3$, with $\rho$ the standard Euclidean distance, the function $h_t(x, y)$ is precisely the ordinary heat kernel. In our case, the kernel $H^r_t(x, y)$ has an asymptotic expansion as $t \to 0$,

$$(3.10) \quad H^r_t(x, y) \sim h_t(x, y) (b^0_0(x, y) + b^0_1(x, y)t + b^0_2(x, y)t^2 + \ldots +),$$

that is studied in detail in [6, Ch. 2]; here, the $b^i_r(x, y)$ are defined on all of $Y \times Y$. The following lemma summarizes what we need to know about the results from [6, Ch. 2]:

**Lemma 10.** There exists for all $i = 0, 1, 2, \ldots$ sections $b^i_r(x, y)$ such that:

- The $b^i_r$ are supported in any neighborhood of the diagonal.
- The asymptotic expansion (3.10) may be formally differentiated to obtain asymptotic expansions for the derivative. In particular, there is an asymptotic expansion

$$(3.11) \quad L^r_t(x, y) \sim h_t(x, y) (\tilde{b}^0_0(x, y) + \tilde{b}^0_1(x, y)t + \tilde{b}^0_2(x, y)t^2 + \ldots +)$$

where

$$\tilde{b}^0_n(x, y) = (D_{A_r} + c(\rho_d\rho/2t))b^0_n(x, y).$$

- For any $n, t > 0$,

$$L^r_t(x, y) - h_t(x, y) \sum_{i=0}^n \tilde{b}^i_r(x, y)t^i$$

is $O(t^{i-1/2})$, in the $C^0$ - norm on the product, as $t \to 0$. 


\begin{equation}
(3.12) \quad \left( \partial_t + D^2_{A_r} \right) \left( L_t^r(x,y) - h_t(x,y) \sum_{i=0}^n \tilde{b}_r^i(x,y) t^i \right) = -D^2_{A_r} \tilde{b}_n^r t^n.
\end{equation}

Proof. The lemma summarizes those parts of the proof of [6, Thm. 2.30] that we will soon need; the arguments in [6, Thm. 2.30] provide the proof. The idea behind the first bullet point is that \( h_t(x,y) \) is on the order of \( t^\infty \) away from the diagonal. The reason for the \( i - \frac{1}{2} \) exponent in the third bullet point is that \( h_t \) has a \( t^{-3/2} \) term. For the fourth bullet point, the point is that the coefficients \( b_r^i \) are constructed so as to satisfy (3.7) when formally differentiating (3.10) and equating powers of \( t \); this gives a recursion which is relevant for our purposes because it implies that when we truncate the expansion at a finite \( n \), the inhomogeneous equation (3.12) is satisfied. \( \square \)

In view of the first bullet point of the above lemma, we only have to understand the coefficients \( b_r^i \) in a neighborhood of the diagonal. To facilitate this, let \( i_r^{TY} \) denote the injectivity radius of the Riemannian metric \( g \), and given \( y \in Y \), let \( B_y \left( \frac{i_r^{TY}}{2} \right) \) denote a geodesic ball of radius \( \frac{i_r^{TY}}{2} \) centered at \( y \), and let \( q \) denote a choice of co-ordinates on this ball. Define \( G^k_y \subset C^\infty \left( B_y \left( \frac{i_r^{TY}}{2} \right) \right) \) to be the subspace of \( (r \)-dependent) functions \( f \) satisfying the estimate \( \partial^\alpha_q f = O \left( r^{k + \frac{\| \alpha \|}{2}} \right) \) as \( r \to \infty \), \( \forall \alpha \in \mathbb{N}_0^3 \), and for each \( j \in \frac{1}{2}\mathbb{N}_0 \), further define the subspace \( W^j_y \subset C^\infty \left( B_y \left( \frac{i_r^{TY}}{2} \right) \right) \) via
\begin{equation}
(3.13) \quad f \in W^j_y \iff f = \sum_{i=1}^N f_i, \quad \text{with each} \quad f_i \in q^\alpha G^k_y, \quad k \leq j + \frac{\| \alpha \|}{2}.
\end{equation}

Finally, given \( y \in Y \), we choose a convenient frame for \( S_y^\xi \) and \( E \) over \( B_y \left( \frac{i_r^{TY}}{2} \right) \), which we will call a synchronous frame; specifically, choose an orthonormal basis for each of \( S_y^\xi, E_y \), and parallel transport along geodesics with \( A_c, A_r \) to obtain local orthonormal trivializations \( \{ s_1, s_2 \}, \{ e \} \). Now, if \( b \) is any section of \( S \otimes S_y \) over \( B_y \left( \frac{i_r^{TY}}{2} \right) \times \{ y \} \) write
\begin{equation}
(3.14) \quad b(\cdot, y) = \sum_{k,l=1}^2 f^y_{bk,l}(\cdot) (s_k \otimes e)(\cdot) (s_l \otimes e)^*(y).
\end{equation}
Lemma 11. There is a constant $c_0$ independent of $r$ such that for any $t > 0$, $r \geq 1$, we have

\begin{equation}
|H_r^t(x,y)| \leq c_0 h_2t(x,y) e^{c_0 rt}.
\end{equation}

Further, for any $y \in Y$, the restriction of the terms $b_r^j$ to $B_y \left( \frac{t_0 r^2}{2} \right) \times \{y\}$ have the property that their corresponding functions $f_{b_r^j,kl}^y$ in (3.14) are all in $W_j^j$.

Proof. The first bullet point is proved in [19, Prop. 3.1].

To prove the second bullet point, we use the fact that the terms $b_r^j$ in the heat kernel expression (3.10) are known to satisfy a recursion, as alluded to above, and explained in the proof of [6, Thm. 2.30]. Specifically, fix $y \in Y$, choose geodesic coordinates $q$ around $y$, mapping 0 to $y$, and choose a synchronous frame as in (3.14). Then, in these coordinates, we have

\begin{equation}
b_r^0(x,y) = \sum_{i=1}^{2} g^{-1/4}(x)(s_i \otimes e)(x)(s_i \otimes e)^*(y),
\end{equation}

where $g = \det(g_{jk})$. Moreover, if use these coordinates to identify sections of $S \otimes S_y^*$ with a vector of functions, then we have

\begin{equation}
b_r^j(x,y) = -\frac{1}{g_{1/4}(x)} \int_0^1 \rho^{j-1} g^{1/4}(\rho x) D_A^2 b_{r-1}^j (\rho x, y) d\rho, \quad j \geq 1
\end{equation}

where on the right hand side of this equation, we mean that we are integrating this vector of functions component by component.

Now recall the Bochner-Lichnerowicz-Weitzenbock formula for the Dirac operator:

\begin{equation}
D_{A_r}^2 = \nabla_{A_r}^* \nabla_{A_r} + \frac{\kappa}{4} - \frac{1}{2} c(\ast F_{A_r}),
\end{equation}

where $\kappa$ denotes the scalar curvature; we will want to combine this with (3.17). In coordinates, we have

\begin{equation}
\nabla_{A_r} = (\partial_1 + \Gamma_1, \partial_2 + \Gamma_2, \partial_3 + \Gamma_3),
\end{equation}

where each $\Gamma_i$ is the $i^{th}$ Christoffel symbol for $A_r$. We also have

\begin{equation}
\nabla_{A_r}^* = -\sum_{j,k}^{3} g^{jk}(\partial_k + \Gamma_k) + \sum_{i,j,k}^{3} g^{jk} \Gamma^i_{jk},
\end{equation}

where the $\Gamma^i_{jk}$ are the Christoffel symbols of the Riemannian metric. Since we have $A_r = 1 \otimes A + A_c \otimes 1$, where $A_c$ is the canonical connection.
on $S^5$, we can decompose each Christoffel symbol
\begin{equation}
\Gamma_i = c_i + a_i, 
\end{equation}
where the $c_i$ are Christoffel symbols for $A_c$ and the $a_i$ are Christoffel symbols for $A$.

The $c_i$ are independent of $r$. To understand the $a_j$, first write the defining equations for the curvature
\[ F_{kj} = \partial_k a_j - \partial_j a_k. \]
Now write the coordinate $x = (x^1, x^2, x^3)$, and consider $\sum_{k=1}^3 x^k (\partial_k a_j - \partial_j a_k)$. Reintroducing the radial coordinate $\rho$, we have
\[ \rho \sum_{k=1}^3 x^k \partial_k a_j = \frac{\partial a_j}{\partial \rho}. \]
On the other hand, since the frame $e$ is parallel, we have $\nabla_{x^i} e = 0$, hence $\sum_{k=1}^3 x^k a_k = 0$. Thus, we have
\begin{equation}
\label{eq:a_j}
a_j (x) = \sum_{k=1}^3 \int_0^1 d\rho x^k F_{kj} (\rho x). 
\end{equation}
In particular, it follows from the a priori estimate (3.23) and (3.22) that each
\begin{equation}
\label{eq:a_j_est}
a_j \in W_{y}^{1/2}. 
\end{equation}
Now note that we have $W^{j+k}_{y} \subset W^{\max\{j,k\}}_{y}$, $W^{j}_{y} \cdot W^{k}_{y} \subset W^{j+k}_{y}$, and $\partial_y W^{j}_{y} \subset W^{j+1/2}_{y}$. Hence, by (3.18), (3.19), (3.20), and (3.23), we have that the square of the Dirac operator has the schematic form
\begin{equation}
\label{eq:dirac_sq}
D^2_{A_r} = \sum_{j,k} -g^{jk} \partial_j \partial_k + P_j \partial_j + Q 
\end{equation}
where $P_j \in W^{1/2}_{y}$ and $Q \in W^{1}_{y}$. The Lemma now follows by induction, using (3.16) and (3.17). \hfill \square

We now give the promised:

Proof of 7. The second bullet point follows by combining (3.8) and (3.15).

To prove the first bullet point, our strategy will be to bound the pointwise size of the kernel $L^r(y,y)$ and appeal to the version of (3.8) for $L^r$. 

\begin{proof} [Proof of 7] The second bullet point follows by combining (3.8) and (3.15).

To prove the first bullet point, our strategy will be to bound the pointwise size of the kernel $L^r(y,y)$ and appeal to the version of (3.8) for $L^r$. 
\end{proof}
To do this, consider the asymptotic expansion (3.11). By a theorem of Bismut-Freed ([7, Thm. 2.4]), for any $y \in Y$, $\text{tr} L^t_r(y, y)$ is $O(t^{1/2})$ as $t \to 0$. So we have $\text{tr} L^t_r(y, y) = \text{tr} R^t_r(y, y)$ for the remainder

$$R^t_r(x, y) := L^t_r - D_A_r \left[ h_t \left( b_0 + tb_1 \right) \right].$$

By (3.12), $R^t_r$ satisfies the inhomogeneous heat equation

$$\left( \partial_t + D^2_{A_r} \right) R^t_r(x, y) = h_t t \left\{ -D_{A_r}^3 b_1 + c \left( \frac{\rho d \rho}{2t} \right) D_{A_r}^2 b_1 \right\},$$

and by the third bullet point of Lemma 10, $R^t_r \to 0$ as $t \to 0$. We can then apply Duhamel’s principle (3.9) to write

$$R^t_r(x, y) = \int_0^t e^{-\left( t-s \right) D^2_{A_r}} h_s(x, y) s \left\{ -D_{A_r}^3 b_1 + c \left( \frac{\rho d \rho}{2s} \right) D_{A_r}^2 b_1 \right\} ds.$$  

We can then apply the key property of the heat kernel (3.6) to write

$$R^t_r(x, y) = \int_0^t \int_Y H_{t-s}(x, z) h_s(z, y) s K_s(z, y) \text{vol}(z) ds,$$

and we can apply the second bullet point of Lemma 10 to conclude that

$$|R^t_r(y, y)| \leq c_0 \int_0^t \int_Y e^{c \rho(t-s) h_s(z, y)} h_2(t-s)(z, y) s K_s(z, y) \text{vol}(z) ds.$$

By the first bullet point of Lemma 10, we can assume that $K_s(z, y)$ is supported in $B_{y} \left( \frac{t \rho}{2} \right) \times \{ y \}$. Thus, we just have to bound

$$\int_0^t \int_{B_q \left( \frac{t \rho}{2} \right)} e^{c \rho(t-s) h_s(q, 0)} h_2(t-s)(q, 0) s K_s(q, 0) dq ds,$$

where $q$ are geodesic coordinates centered at $y$. To do this, choose a synchronous frame for the spinor bundle, as we have been doing above. Then, following (3.21), (3.22), in these coordinates the Dirac operator is seen to have the form

$$D_{A_r} = w^{jk} \partial_j + K,$$

for $r$–independent $w^{jk}$ and $K \in W^2_y$, in the geodesic coordinates and orthonormal frame introduced before. Combining this with the second bullet point of Lemma 11 gives that the term $K_s \in W^2_y$. So, (3.26) is
dominated by a finite sum of integrals of the form
\[ \int_0^t ds \int_{B_y \left( \frac{iy^T}{2} \right)} dq se^{c_{\text{cart}}} h_{2(t-s)}(q, 0) h_s(q, 0) q^\alpha r^k, \quad k \leq \frac{5}{2} + \frac{|\alpha|}{2}. \]

On \( B_y \left( \frac{iy^T}{2} \right) \), we have
\[ q^T h_t(q, 0) \leq c_1 t^{1+|\alpha|} h_{2t}(q, 0), \]
for some constant \( c_1 \). Hence, we can bound the above integral by
\[ (3.27) \quad c_1 r^k e^{c_{\text{cart}}} \int_0^t ds \int_{B_y \left( \frac{iy^T}{2} \right)} dq s^{1+|\alpha|/2} h_{2(t-s)}(q, 0) h_{2s}(q, 0) dq. \]

We also have
\[ \int_Y h_{t}(x, y) h_{t'}(y, z) dy \leq c_2 h_{4(t+t')}(x, z) \]
as proved in [19, Sec. A]. So, we can bound (3.27) by
\[ c_3 r^k e^{c_{\text{cart}}} \int_0^t s^{1+|\alpha|/2} h_{st}(0, 0) ds = c_3 r^k e^{c_{\text{cart}}} \int_0^t s^{1+|\alpha|/2} (4t)^{-3/2} ds \leq c_3 r^k t^{1+|\alpha|/2} e^{c_{\text{cart}}}. \]

We know that \( k - \frac{1+|\alpha|}{2} \leq 2 \). So, putting all of the above together, (3.26) is dominated by a finite sum of terms of the form
\[ r^2 r^{\frac{1+|\alpha|}{2}} t^{\frac{1+|\alpha|}{2}} e^{c_{\text{cart}}} \]
which proves the result. \( \square \)

We can finally give:

**Proof of 8.** Define \( E(x) := \text{sign}(x) \text{erfc}(|x|) = \text{sign}(x) \cdot \frac{2}{\sqrt{\pi}} \int_{|x|}^{\infty} e^{-s^2} ds < e^{-x^2} \). This is a rapidly decaying function, so the function \( E(D_{A_r}) \) is defined, and its trace is a convergent sum
\[ \text{tr}(E(D_{A_r})) = \sum_{\lambda} E(\lambda), \]
where \( \lambda \) is an eigenvalue of \( D_{A_r} \). The eta invariant is unchanged under positive rescaling
\[ \eta(D_{A_r}) = \eta \left( \frac{1}{\sqrt{r}} D_{A_r} \right). \]

Now use (3.5) to rewrite the right hand side of the above equation as
\[ \left| \int_0^1 dt \frac{1}{\sqrt{\pi t}} \text{tr} \left[ \frac{1}{\sqrt{r}} D_{A_r} e^{-\frac{t}{r} D_{A_r}^2} \right] + \text{tr} E \left( \frac{1}{\sqrt{r}} D_{A_r} \right) \right|. \]
The absolute value of the first summand in the above expression is bounded from above by a constant multiple of $r^{3/2}$, by the first bullet point in Lemma 9. The absolute value of the second summand in the same expression is bounded from above by $\text{tr} e^{-\frac{1}{2} D^2_{A_r}}$, which by the second bullet point in Lemma 9 is bounded by a constant multiple of $r^{3/2}$ as well.

□

Proof of 5. An application of the Atiyah-Patodi-Singer index theorem as in Section A gives

$$\frac{1}{2} \eta \left( \widehat{\mathcal{H}}_{(A_r, \Psi_r)} \right) = \frac{1}{2} \eta (D_{A_r}) + \frac{1}{2} \eta_Y + \text{sf} \left\{ \mathcal{H}_{(A_r, e\Psi_r)} \right\}_{0 \leq \varepsilon \leq 1}. $$

The spectral flow term above is estimated to be $O \left( r^{3/2} \right)$ as in [23, S 5.4] while $\eta (D_{A_r}) = O \left( r^{3/2} \right)$ by Proposition 8.

□

We also note that the constant in 5 above is only a function of \((Y, \lambda, J)\) and independent of the class $\sigma = [(A_r, \Psi_r)] \in \widehat{HM} (Y, s^E)$ defined by the Seiberg-Witten solution.

Remark 12. The reason that we can not improve upon $\text{gr}^Q$ asymptotics is because we do not know how to strengthen the $O \left( r^{3/2} \right)$ spectral flow estimate on the irreducible solutions of Propositions 5 or 8. A better $O (r)$ estimate does however exist [19, 21] for reducible solutions for which one understands the connection precisely in the limit $r \to \infty$. However, the a priori estimates (3.23) are not strong enough to carry out the same for irreducibles.

4. ASYMPTOTICS OF CAPACITIES

4.1. The main theorem. In this section we now prove our main theorem Theorem 2 on ECH capacities. Let $0 \neq \sigma_j \in ECH (Y, \lambda, \Gamma)$, $j = 0, 1, 2, \ldots$, be a sequence of non-vanishing classes with gradings $\text{gr}^Q \sigma_j = q + 2j$ tending to infinity. As in 2.3, we use the perturbed Chern-Simons-Dirac functional (2.2) and its gradient flow (2.3) with $\mu = ir\lambda$, $r \in [0, \infty)$, in defining monopole Floer homology. Hence for each $r \in [1, \infty)$, the class $\sigma_j$ may be represented by a formal sum of solutions \([a_j]_r\) to (2.3) with $\mu = ir\lambda$. As noted in 2.3, this solution is eventually irreducible and we now estimate $r_1 (j)$ the smallest value of $r$ such that each solution \([a_j]_r\) representing $\sigma_j$ is irreducible. For this note that a reducible solution is of the form $\mathbf{a} = (A, 0, \Phi_k)$ where $A = A_1 + ir\lambda$, $A_1$ flat and $\Phi_k$ the $k$th positive eigenvector of $D_A$. The grading of such a reducible is given by (2.5)

$$\text{gr}^Q [(A, 0, \Phi_k)] = 2k + \eta (D_{A_1 + r\lambda}) + \frac{1}{4} \eta_Y + \frac{r^2}{4\pi^2} \int \lambda \wedge d\lambda.$$
The important estimate \(|\eta_{DA_1+r\lambda}| \leq c_0 r\), (20, Thm. 1.2) now shows
\[ \text{gr}^Q [(A,0,\Phi_k)] > \text{gr}^Q (\sigma_j) = q + 2j \text{ for} \]
\[ r > \tilde{r}_1 (j) := \sup \left\{ r \mid \frac{r^2}{4\pi^2} \text{vol} (Y, \lambda) < c_0 r + \frac{1}{4} \eta_Y + q + 2j \right\}. \]
Hence \( r_1 (j) < \tilde{r}_1 (j) \). Furthermore
\[ (4.1) \quad \tilde{r}_1 (j) = 2\pi \left[ \frac{2j}{\text{vol} (Y, \lambda)} \right]^{1/2} + O (1) \text{ as } j \to \infty \]
by the above definition. A min-max argument then gives \( \forall j \in \mathbb{N} \)
\[ (A,0,\Phi_k) \text{ a piecewise-smooth family of irreducible solutions } [a]_r = (A_r, \Psi_r), \]
\[ r \geq r_1 (j), \text{ of fixed grading } \text{gr}^Q [a] = q + j \text{ such that} \]
\[ v = cs - rE \]
is continuous.

As noted earlier, (2.5), (3.4), Proposition 8, this solution satisfies
\[ (4.2) \quad |CS (A)| \leq c_0 r^{2/3} e_\lambda (A)^{4/3} \]
\[ (4.3) \quad \left| q + 2j - \frac{5}{4} \eta_Y - CS \right| = \left| \frac{1}{2} \eta \left( \hat{H}_{(A,\Psi)} \right) \right| \leq c_0 r^{3/2}, \]
with the constant \( c_0 > 0 \) being independent of the grading \( j \), as well
as the differential equation relation
\[ r \frac{de_\lambda (A)}{dr} = \frac{dCS}{dr}. \]
between the two functionals, away from the discrete set of points where
both derivatives are undefined. Now define \( F (r) = \int_{r_1}^r e_\lambda (A_s) \) \( ds \). This
is a continuous function, and \( v \) is continuous as well, so we may inte-
grate the above equation to conclude that
\[ CS (r) = CS (r_1) - r_1 F' (r_1) + r F' - F \]
\[ = r F' - F, \]
valid for all \( r \). On account of (4.3), \( F \) is then a super/subsolution to
the ODEs
\[ -c_2 r^{3/2} \leq r F' - F - (q + j) \leq c_2 r^{3/2} \]
for \( r \geq r_1 \). This gives
\[ r \left[ \frac{q + j}{r_1} - \frac{q + j}{r} - 2c_2 r^{1/2} + 2c_2 r_1^{1/2} \right] \leq F \leq r \left[ \frac{q + j}{r_1} - \frac{q + j}{r} + 2c_2 r^{1/2} - 2c_2 r_1^{1/2} \right] \]
\[ (4.4) \quad \frac{q + j}{r_1} - 3c_2 r^{1/2} + 2c_2 r_1^{1/2} \leq F' \leq \frac{q + j}{r_1} + 3c_2 r^{1/2} - 2c_2 r_1^{1/2}. \]
Next the estimate (4.2) in terms of $F$ is
\begin{equation}
-c_1 r^{2/3} (F')^{4/3} \leq r F' - F \leq c_1 r^{2/3} (F')^{4/3}.
\end{equation}
We let $\rho_0$ be the smallest positive root of $\frac{1}{3} - [\rho + \rho^2 + \rho^3 + \rho^4] = 0$ and define
\begin{equation*}
\bar{r}_2 (j) = \sup \left\{ r \mid c_1 r^{-2/3} F^{1/3} \geq \rho_0 \right\}
\end{equation*}
which is finite on account of (4.4). Further with $c_3 = 1 + 3 \left( \frac{2 c_1}{3} \right) + 3 \left( \frac{2 c_1}{3} \right)^2$ define
\begin{equation}
\tilde{r}_2 (j) = \sup \left\{ r \mid \frac{q + j}{r_1} + 3 c_2 r^{1/2} - 2 c_2 r_1^{1/2} \geq \left( \frac{3}{4 c_1} \right)^3 r \\
or \frac{q + j}{r_1} - \frac{q + j}{r} + 2 c_2 r^{1/2} - 2 c_2 r_1^{1/2} \geq \left( \frac{1}{9 c_3} \right)^3 r \\
or \frac{q + j}{r_1} - \frac{q + j}{r} + 2 c_2 r^{1/2} - 2 c_2 r_1^{1/2} \geq r \right\}
\end{equation}
and set $r_2 (j) := \max \{ \bar{r}_2 (j), \tilde{r}_2 (j) \}$. We note that $r_2 (j) = O \left( j^{1/2} \right)$.
We now have the following lemma.

Lemma 13. For $r > r_2 (j)$ we have
\begin{equation}
\left( \frac{F}{r} \right)^{1/3} - \frac{2 c_1}{3 r} F^{2/3} \leq (F')^{1/3} \leq \left( \frac{F}{r} \right)^{1/3} + \frac{2 c_1}{3 r} F^{2/3}.
\end{equation}

Proof. By definition,
\begin{equation*}
r > r_2 (j) \geq \bar{r}_2 (j) \\
\implies \rho := c_1 r^{-2/3} F^{1/3} < \rho_0 \\
\implies \rho + \rho^2 + \rho^3 + \rho^4 < \frac{1}{3}
\end{equation*}
as well as
\begin{equation}
r > r_2 (j) \geq \tilde{r}_2 (j) \\
\implies F' \leq c_2 + \frac{q + j}{r_1} + c_2 \ln r - c_2 \ln r_1 < \left( \frac{3}{4 c_1} \right)^3 r
\end{equation}
by (4.4).

For $y = (F')^{1/3}$ equations (4.5) become the pair of quartic inequalities
\begin{align}
0 &\leq c_1 r^{-1/3} y^4 - y^3 + r^{-1} F' \\
0 &\leq c_1 r^{-1/3} y^4 + y^3 - r^{-1} F'
\end{align}
With \( y_0^\pm = \left( \frac{F}{r} \right)^{1/3} \pm \frac{2c_1}{3r} F^{2/3} \), we calculate

\[
c_1 r^{-1/3} \left( y_0^+ \right)^4 - \left( y_0^+ \right)^3 + r^{-1} F
= - r^{-5/3} c_1 F^{4/3} \left[ 1 - \frac{4}{3} \rho - \frac{64}{27} \rho^2 - \frac{32}{27} \rho^3 - \frac{16}{81} \rho^4 \right] < 0 \quad \text{and}
\]

\[
c_1 r^{-1/3} \left( y_0^- \right)^4 + \left( y_0^- \right)^3 - r^{-1} F
= r^{-5/3} c_1 F^{4/3} \left[ -1 - \frac{4}{3} \rho + \frac{64}{27} \rho^2 - \frac{32}{27} \rho^3 + \frac{16}{81} \rho^4 \right] < 0.
\]

Since the minimum of the quartic (4.9) is attained at \( y_{\text{min}} = (F')^{1/3} = \frac{3}{4c_1} r^{1/3} \), this gives \( (F')^{1/3} = y \leq y_0^+ \) or \( (F')^{1/3} = y \geq y_{\text{min}} = \frac{3}{4c_1} r^{1/3} \). The second possibility being disallowed on account of (4.8), gives the desired upper bound of (4.7). Similarly, the minimum of quartic (4.10) is attained at the negative \( y_{\text{min}} = -\frac{3}{4c_1} r^{1/3} \). Hence \( y_0^- \leq y = (F')^{1/3} \) which is the lower bound in (4.7).

Next we cube (4.7) and use (4.6) to obtain

\[
\frac{F}{r} - c_3 \frac{F^{4/3}}{r^{5/3}} \leq F' \leq \frac{F}{r} + c_3 \frac{F^{4/3}}{r^{5/3}}
\]

for \( r \geq r_2(j) \). This gives

\[
r^{1/3} \left[ \frac{r^{1/3}}{-c_3 + c_0^- r^{1/3}} \right] \leq F^{1/3} \leq r^{1/3} \left[ \frac{r^{1/3}}{c_3 + c_0^+ r^{1/3}} \right], \quad \text{where}
\]

\[
c_0^\pm = \frac{r_3^{1/3}}{F(r_3)^{1/3}} \mp c_3 \left[ \frac{F(r_3)}{r_3^{1/3}} \right]^{1/3}
\]

for \( r \geq r_3(j) \geq r_2(j) \). The last equation with (4.11) gives

\[
\left( \frac{F(r_3)}{r_3^{1/3}} + c_3 \left[ \frac{F(r_3)}{r_3^{1/3}} \right]^{1/3} \right)^3 = \frac{1}{(c_0^+)^3} \leq F'(r) \leq \frac{1}{(c_0^-)^3} = \left( \frac{F(r_3)}{r_3^{1/3}} - c_3 \left[ \frac{F(r_3)}{r_3^{1/3}} \right]^{1/3} \right)^3
\]

and hence

\[
\frac{F(r_3)}{r_3} \left[ 1 - \frac{4c_3}{r_3^{1/3}} \left( \frac{F(r_3)}{r_3^{1/3}} \right)^{1/3} \right] \leq c_\lambda (A_r) \leq \frac{F(r_3)}{r_3} \left[ 1 + \frac{4c_3}{r_3^{1/3}} \left( \frac{F(r_3)}{r_3^{1/3}} \right)^{1/3} \right]
\]
from (4.6) for \( r \gg 0 \). However, (4.4) for \( r = r_3 \) when substituted into the last equation above becomes

\[
\left[ \frac{q + j}{r_1} - \frac{q + j}{r_3} - 2c_2 r_3^{1/2} \right] \left( 1 - \frac{4c_3}{r_3^{1/3}} \left[ \frac{q + j}{r_1} + 2c_2 r_3^{1/2} \right]^{1/3} \right) \leq c_\lambda (A_r) \\
\]

\[
\left[ \frac{q + j}{r_1} + 2c_2 r_3^{1/2} \right] \left( 1 + \frac{4c_3}{r_3^{1/3}} \left[ \frac{q + j}{r_1} + 2c_2 r_3^{1/2} \right]^{1/3} \right) \geq c_\lambda (A_r)
\]

Setting \( r_3 = j^{4/5} \) (satisfying \( r_3 \geq r_2 = O\left(j^{1/2}\right) \) for \( j \gg 0 \)) and using (2.7), (4.1) gives

\[
(4.13) \\
c_{\sigma_j}(\lambda) = j^{1/2} \text{vol}(Y, \lambda)^{1/2} + O\left(j^{2/5}\right),
\]

as \( j \to \infty \), which is our main result Theorem 2.

\textit{Remark 14.} One could replace the arguments in this subsection with the arguments in Sun’s paper [26], if desired — the key reason why we have a stronger bound than Sun is because of our stronger bound on the Chern-Simons functional, and not because of anything we do in this subsection. We have chosen to include our argument here, which we developed independently of the arguments in [26], for completeness, and because it is short and might be of independent interest.

On the other hand, the arguments in [11] are not quite strong enough for Theorem 2, even with the improved bound in Proposition 8.

4.2. \textbf{Proofs of Corollaries.} Here we prove the two corollaries 3 and 4, both following immediately from the capacity formula Theorem 2.

\textbf{Proof of 3.} The \( \mathbb{Z}_2 \) vector space \( ECH(Y, \xi, \Gamma; \mathbb{Z}_2) \) is known to be two-periodic, and nontrivial, in sufficiently high grading, see for example [15]. Thus, for \( * \gg 0 \) sufficiently large there exists a finite set of classes \( \{\sigma_1, \ldots, \sigma_{2^d-1}\} \subset ECH_*(Y, \xi, \Gamma; \mathbb{Z}_2) \cup ECH_{*+1}(Y, \xi, \Gamma; \mathbb{Z}_2) \) such that

\[
\{0, U^j \sigma_1, \ldots, U^j \sigma_{2^d-1}\} = ECH_{*+2j}(Y, \xi, \Gamma; \mathbb{Z}_2) \cup ECH_{*+1+2j}(Y, \xi, \Gamma; \mathbb{Z}_2), \forall j \geq 0.
\]

Thus the ECH spectrum modulo a finite set is given by

\[
\bigcup_{j=0}^{\infty} \left\{ c_{U^j \sigma_1}(\lambda), \ldots, c_{U^j \sigma_{2^d-1}}(\lambda) \right\}.
\]

The corollary now follows as \( c_{U^j \sigma_l}(\lambda) = j^{1/2} \text{vol}(Y, \lambda)^{1/2} + O\left(j^{2/5}\right), 1 \leq l \leq 2^d - 1 \), by Theorem 2. \( \square \)
Proof of 4. As in the previous corollary, the ECH zeta function is given, modulo a finite and holomorphic in \( s \in \mathbb{C} \), sum by

\[
\sum_{j=0}^{\infty} \left[ c_{Uj,\sigma_1}(\lambda)^{-s} + \cdots + c_{Uj,\sigma_{d-1}}(\lambda)^{-s} \right].
\]

With \( \zeta^R(s) \) denoting the Riemann zeta function, we may using Theorem 2 compare

\[
\left| \sum_{j=0}^{\infty} c_{Uj,\sigma_1}(\lambda)^{-s} - \text{vol}(Y,\lambda)^{-\frac{s}{2}} \sum_{j=0}^{\infty} j^{-\frac{s}{2}} = \zeta^R\left(\frac{j}{2}\right) \right| = O\left( \sum_{j=0}^{\infty} \frac{1}{j^{3s/5}} \right)
\]

whence the difference being holomorphic for \( \text{Re}(s) > \frac{5}{3} \). The corollary now follows on knowing \( s = 2 \) to be the only pole of the Riemann zeta function \( \zeta^R\left(\frac{j}{2}\right) \) with residue 1.

4.3. The ellipsoid example. We close by presenting an example with \( O(1) \) asymptotics, and where the corresponding \( \zeta_{ECH} \) function extends meromorphically to all of \( \mathbb{C} \).

Consider the symplectic ellipsoid

\[
E(a, b) := \left\{ \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\} \subset \mathbb{C}^2 = \mathbb{R}^4.
\]

The symplectic form on \( \mathbb{R}^4 \) has a standard primitive

\[
\lambda_{std} = \frac{1}{2} \sum_{i=1}^{2} (x_idy_i - y_idx_i).
\]

This restricts to \( \partial E(a, b) \) as a contact form, and the ECH spectrum of \( (\partial E(a, b), \lambda) \) is known. Specifically, let \( N(a, b) \) be the sequence whose \( j^{th} \) element (indexed starting at \( j = 0 \)) is the \((j+1)^{st}\) smallest element in the matrix

\[
(ma + nb)_{(m,n)\in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}}.
\]

Then, the ECH spectrum \( S_{\partial E(a, b)} \) is precisely the values in \( N(a, b) \). Moreover, the homology

\[
ECH_* (\partial E(a, b))
\]

has a canonical \( \mathbb{Z} \)-grading, such that the empty set of Reeb orbits has grading 0, and it is known to have one generator \( \sigma_j \) in each grading \( 2j \), see [15]. The spectral invariant associated to \( \sigma_j \) is precisely the \( j^{th} \) element in the sequence \( N(a, b) \).
With this understood, we now have:

**Proposition 15.** Let $\sigma_j$ be any sequence of classes in $ECH(\partial E(a,b))$ with grading tending to infinity. Then, the $d(\sigma_j)$ are $O(1)$. In fact, if $a/b$ is irrational, then

$$\lim_{j \to \infty} \frac{d(\sigma_j)}{j} = \frac{a + b}{2}.$$ 

**Proof.** Assume that $a/b$ is irrational. If $t = c_{\sigma_j}$, then by the above description, the grading of $\sigma_j$ is precisely twice the number of terms in $N(a,b)$ that have value less than $t$. With this understood, the example follows from [12, Lem. 2.1].

When $a/b$ is rational, a similar argument still works to show $O(1)$ asymptotics. Namely, if $t = c_{\sigma_j}$, then by above, the grading of $\sigma_j$ is precisely twice the number of terms in $N(a,b)$ that have value less than $t$, up to an error no larger than some constant multiple of $\sqrt{j}$. Now apply [5, Thm. 2.10]. □

**Proposition 16.** The $ECH$ $\zeta$ function $\zeta_{ECH}$ for $ECH(\partial E(a,b))$ has a meromorphic continuation to all of $\mathbb{C}$. It has exactly two poles, at $s = 1$ and $s = 2$, with residues

$$\text{Res}_{s=2}\zeta_{ECH}(s; Y, \lambda, \Gamma) = \frac{1}{ab}, \quad \text{Res}_{s=1}\zeta_{ECH}(s; Y, \lambda, \Gamma) = \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right).$$

**Proof.** The $ECH$ zeta function in this example

$$\zeta_{ECH}(s; Y, \lambda, \Gamma) = \sum_{m,n \in \mathbb{N}} (ma + nb)^{-s}$$

(4.14)

$$= \frac{1}{2} \left[ \zeta^B(s,a|a,b) + \zeta^B(s,b|a,b) - a^{-s}\zeta^R(s) - b^{-s}\zeta^R(s) \right]$$

is given in terms of the classical zeta functions of Riemann and Barnes [4, 18]

$$\zeta^B(s,w|a,b) := \sum_{m,n \in \mathbb{N}_0} (w + ma + nb)^{-s}, \quad w \in \mathbb{R}_{>0}.$$ 

Thus $\zeta_{ECH}(s; Y, \lambda, \Gamma)$ (4.14) is known to possess a meromorphic continuation to the entire complex plane in this example. Its only two poles are at $s = 1, 2$ with residues

$$\text{Res}_{s=2}\zeta_{ECH}(s; Y, \lambda, \Gamma) = \frac{1}{ab}$$

$$\text{Res}_{s=1}\zeta_{ECH}(s; Y, \lambda, \Gamma) = \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right)$$
respectively; while its values at the non-positive integers are also known [22]. In particular its value at zero is
\[ \zeta_{ECH}(0; Y, \lambda, \Gamma) = \frac{1}{4} + \frac{1}{12} \left( \frac{b}{a} + \frac{a}{b} \right). \]

\[ \square \]

Appendix A. The $Q$-grading and the $\eta$ invariant

In this appendix we give a formula for the absolute grading $\text{gr}^Q$ on monopole Floer groups of torsion spin-c structures (from [17, S 28.3]) in terms of a relevant eta invariant. First to recall the definition of $\text{gr}^Q(A)$, $A = (A, s, \Phi)$, choose a four manifold $X$ which bounds $Y$ and form the manifold with cylindrical end $Z = X \cup (Y \times [0, \infty))$. Choose a metric $g^{TZ}$ on $Z$ which is of product type $g^{TZ} = g^{TX} + dt^2$ on the cylindrical end. Choose a spin-c structure $(S^{TZ}, c^Z)$ over $Z$ which is of the form
\[ S^{TZ} = S^{TZ}_+ \oplus S^{TZ}_-, \quad S^{TZ}_+ = S^{TY}, \]
\[ c^Z(\alpha) = \begin{bmatrix} 0 & c^Y(\alpha) \\ c^Y(\alpha) & 0 \end{bmatrix}, \quad \alpha \in T^*Y \]
\[ c^Z(dt) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \]
over the cylindrical end and a spin-c connection on $S^{TZ}$ of the form $B = dt \wedge \partial_t + A \oplus A$ on the cylindrical end. One may now form the Fredholm elliptic operator
\[ -d^* + d^+ + D^{+}_B : L^2(Z; \Lambda^{1}_{TZ} \oplus S^{TZ}_+) \rightarrow L^2(Z; \Lambda^{0}_{TZ} \oplus \Lambda^{2+}_{TZ} \oplus S^{TZ}_-) . \]
The absolute grading of $[(A, 0, \Phi^A_0)] \in \zeta^*$, with $A^t$ flat and $\Phi^A_0$ the first positive eigenvector of $D_A$, is given in terms of this operator. The precise formula [17, Defn. 28.3.1] simplifies to
\[ \text{gr}^Q[(A, 0, \Phi^A_0)] = -2 \text{ind} \left( D^{+}_B \right) + \frac{1}{4} \left( c_1(S^+), c_1(S^+) \right) - \frac{1}{4} \sigma(Z), \]
with $\sigma(Z)$ denoting the signature of $Z$, $S^+$ the bundle $S^{TZ}_+$ from above, and ind denoting the complex index, namely the difference in complex dimensions, compare [8, S 3.4].

We now want to rewrite this using the Atiyah-Patodi-Singer index theorem. First, the version that we want for twisted Dirac operators, see [21, eq. 4.3], gives
\[ \text{ind}(D^{+}_B) = \int_X \text{ch}(\xi) \hat{A} - \frac{\eta(D_A)}{2}, \]
where \( \text{ch} \) is the Chern character for the coefficient line bundle \( \xi \), and \( \hat{A} \) is the \( \hat{A} \) polynomial. The line bundle \( \xi \) here satisfies \( \xi^2 = \det(S^+) \).

Hence, the integral of the Chern character term is

\[
\int_X \text{ch}(\xi) = \frac{1}{8} \langle c_1(S^+), c_1(S^+) \rangle.
\]

Meanwhile, the contribution from the \( \hat{A} \) polynomial is

\[
-\frac{1}{24} \int_X p_1 = -\frac{1}{8} \sigma(Z) - \frac{1}{8} \eta_Y,
\]

where \( \eta_Y \) is the eta invariant of the odd signature operator on \( C^\infty(Y; T^*Y \oplus \mathbb{R}) \) sending

\[
(a, f) \mapsto (*da - df, d^*a),
\]

and we have used [2, Thm. 4.14] to compute the integral of \( p_1 \). Hence, we learn that

\[
\text{gr}_Q[(A, 0, \Phi^A)] = \eta(D_A) + \frac{1}{4} \eta_Y.
\]

A reducible generator \([a_k] = [(A, 0, \Phi^A_k)] \in \mathcal{C}^\ast\) however has \( \frac{1}{2} F_{A^k} = -d \mu \) and \( \Phi^A_k \) the \( k \)th eigenvector of \( D_A \). Hence,

\[
\text{gr}_Q[(A, 0, \Phi^A_k)] = 2k + (\eta(D_A) + \frac{1}{4} \eta_Y) - \text{sf}\{D_{A_s}\}_{0 \leq s \leq 1},
\]

where \( D_{A_s} \) is a family of Dirac operators, associated to a family of connections starting at the flat connection and ending at one satisfying \( \frac{1}{2} F_{A'} = -d \mu \). Hence, by interpreting this spectral flow as an index through another application of Atiyah-Patodi-Singer [3, p. 95], and applying [2, eq. 4.3] to compute this index, we get

\[
\text{gr}_Q[(A, 0, \Phi^A_k)] = 2k + \eta(D_A) + \frac{1}{4} \eta_Y + \frac{1}{4} \int_Y \mu \wedge F_{A^k}
\]

(A.1)

\[
= 2k + \eta(D_A) + \frac{1}{4} \eta_Y + \frac{1}{4 \pi^2} \int_Y \mu \wedge F_{A^k}
\]

(A.2)

as the absolute grading of a reducible generator.

The absolute grading of an irreducible generator \([a'] = (A', s, \Phi')\), \( s \neq 0 \), is then given by

\[
\text{gr}_Q[a'] = \text{gr}_Q[a_0] - \text{sf}\{\mathcal{H}_{(A_\varepsilon, \Psi_\varepsilon)}\}_{0 \leq \varepsilon \leq 1}
\]

in terms of spectral flow of the Hessians [2.4] for a path \((A_\varepsilon, \Psi_\varepsilon) \in \mathcal{A}(Y, s) \times C^\infty(S), \varepsilon \in [0, 1]\) starting at \((A, 0)\) and ending at \((A', s\Phi')\).

As above, we can interpret this spectral flow as an index; this time, to
compute the relevant index, we need to apply ([2, Thm. 3.10]), which gives that the above is equal to

\[ \text{gr}^\mathbb{Q} [\mathbf{a}'] = \eta \left( \hat{\mathcal{H}}_{(A,s,\Phi')} \right) - \frac{3}{4} \eta_Y - 2 \int_{Y \times [0,1]} \rho_0. \]

Here \( \rho_0 \) is the usual Atiyah-Singer integrand, namely the local index density defined as the constant term in the small time expansion of the local supertrace \( \text{str} \left( e^{-t \mathcal{D}^2} \right) \) with

\[ \mathcal{D} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \partial_\varepsilon + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{\mathcal{H}}_{(A_\varepsilon, \Psi_\varepsilon)}, \]

where \((A_\varepsilon, \Psi_\varepsilon)\) is a path of configurations with \((A_0, \Psi_0)\) satisfying \( \Psi_0 = 0 \) and \( A_0 \). To compute the index density we choose a path of the form

\[ (A_\varepsilon, \Psi_\varepsilon) = \begin{cases} \left( A + 2 \varepsilon (A' - A), 0 \right); & 0 \leq \varepsilon \leq \frac{1}{2}, \\ (A', (2 \varepsilon - 1) \Psi); & \frac{1}{2} \leq \varepsilon \leq 1. \end{cases} \]

On the interval \([0, \frac{1}{2}]\), the integral of the local density is given by the usual local index theorem: as above, we have

\[ -2 \int_{Y \times [0,\frac{1}{2}]} \rho_0 = -\frac{1}{4 \pi^2} CS \left( A \right). \]

On the other hand, for the calculation on \( Y \times \left[ \frac{1}{2}, 1 \right] \), we have \( \rho_0 = 0 \). To see this, first note \( \mathcal{D}^2 = -\partial_\varepsilon^2 + \hat{\mathcal{H}}^2_{(A_\varepsilon, \Psi_\varepsilon)} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2 M\Psi \) gives

\[ \text{str} \left( e^{-t \mathcal{D}^2} \right) = \text{tr} \left( e^{-t \left[ -\partial_\varepsilon^2 + \hat{\mathcal{H}}^2_{(A_\varepsilon, \Psi_\varepsilon)} - 2 M\Psi \right]} - e^{-t \left[ -\partial_\varepsilon^2 + \hat{\mathcal{H}}^2_{(A_\varepsilon, \Psi_\varepsilon)} + 2 M\Psi \right]} \right). \]

Duhamel’s principle (3.9), as in the proof of the previously mentioned [6, Thm. 2.30], then gives that the coefficients in the small time heat kernel expansion of the difference satisfy a recursion of the form (3.17), but with \( D^2_{A_\varepsilon} \) replaced by \( -4 M\Psi \), so that the coefficients are of the form

\[ \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \]

with respect to the decomposition \( iT^* Y \oplus \mathbb{R} \oplus S \).

Hence we have in summary:

\[ \text{gr}^\mathbb{Q} [\mathbf{a}] = \begin{cases} 2k + \eta \left( D_A \right) + \frac{1}{4} \eta_Y - \frac{1}{4 \pi^2} CS \left( A \right); & \mathbf{a} = \left( A, 0, \Phi^4 \right) \in \mathcal{C}^s, \\ \eta \left( \hat{\mathcal{H}}_{(A,s,\Phi')} \right) - \frac{3}{4} \eta_Y - \frac{1}{4 \pi^2} CS \left( A \right); & \mathbf{a} = \left( A, s, \Phi \right) \in \mathcal{C}^o, s \neq 0. \end{cases} \]
SUBLEADING ASYMPTOTICS OF THE ECH SPECTRUM

References


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