Radiation Pressure Cooling as a Quantum Dynamical Process

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One of the most fundamental problems in optomechanical cooling is how small the thermal phonon number of a mechanical oscillator can be achieved under the radiation pressure of a proper cavity field. Different from previous theoretical predictions, which were based on an optomechanical system’s time-independent steady states, we treat such cooling as a dynamical process of driving the mechanical oscillator from its initial thermal state, due to its thermal equilibrium with the environment, to a stabilized quantum state of higher purity. We find that the stabilized thermal phonon number left in the end actually depends on how fast the cooling process could be. The cooling speed is decided by an effective optomechanical coupling intensity, which constitutes an essential parameter for cooling, in addition to the sideband resolution parameter that has been considered in other theoretical studies. The limiting thermal phonon number that any cooling process cannot surpass exhibits a discontinuous jump across a certain value of the parameter.

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Preparing the approximate pure quantum states of a sizable mechanical oscillator is a feasible way toward macroscopic quantumness. Starting practically from its thermal equilibrium with the environment, such a process is implemented by coupling the oscillator to a cavity field generated by a red-detuned external drive in order to reduce the associated thermal phonon number to a low level, similar to cooling the oscillator to a lower temperature. An important feature we will illustrate is that the cooling result depends on how fast the optomechanical system (OMS) evolves to the finally stable quantum state.

Numerous experiments have realized the cooling to a few and even less than one mechanical quanta [1–13]. Following earlier studies of quantum fluctuations under radiation pressure [14,15], the theoretical description of such optomechanical cooling (see, e.g., [16–23]) was based on a linearization procedure similar to that described in [24], i.e., to decompose the cavity field mode $a$ into the sum of the classical mean value $\alpha$ and its quantum fluctuation $\delta a$. The linearized Hamiltonian gives the cooling action as a beam splitter (BS)-type coupling between the mechanical mode $b$ and the fluctuation $\delta a$ with the coupling intensity $g$ magnified by $\alpha$, which was generally treated as a constant of steady-state value. In an actual cooling process, however, the cavity mean field $\langle \hat{a}(t) \rangle = \alpha(t)$ is built up from zero (when the mechanical oscillator is in thermal equilibrium with its environment) and takes time to evolve to a stable value. Then, the effective coupling strength $g|\alpha|$ used in the previous studies should be more appropriately taken as a variable, because $\alpha(t)$ keeps changing during a cooling process. Because of the impossibility of finding the time-dependent $\alpha(t)$ analytically, it is difficult to study the cooling as a dynamical process if adopting the above-mentioned linearization.

In the present Letter we put forward a quantum dynamical theory for optomechanical cooling. Using this completely quantum approach, which linearizes a weakly coupled OMS’s dynamics without resorting to its classical mean values such as $\alpha(t)$, one can numerically predict the involved physical quantities that evolve with time. The residual occupation of the thermal excitation of a mechanical oscillator, as found in our approach, is connected with the whole cooling process. Compared with previous theoretical predictions, this dynamical approach provides richer information about how good a quantum OMS can be fully cooled down to.

Figure 1(a) illustrates an example optomechanical cooling setup. In a frame with the system modes rotating at the cavity frequency $\omega_c$ and the mechanical frequency $\omega_m$, respectively, the process is governed by the following Hamiltonians [25]:

1. $H_c(t) = i\frac{E}{\hbar}(\hat{a} e^{i\Delta t^2} - \hat{a}^\dagger e^{-i\Delta t^2})$ for an external drive with the intensity $E$ and the detuning $\Delta = \omega_c - \omega_i$ of its frequency $\omega_i$;
2. $H_{om}(t) = -g\hat{a}^\dagger (\hat{b} e^{-i\omega_m t} + \hat{b}^\dagger e^{i\omega_m t})$ for the optomechanical coupling;
3. the stochastic Hamiltonian

$$H_{st}(t) = i\frac{\sqrt{2\kappa}}{2} \{ \hat{a}^\dagger \hat{a} \hat{\xi}_c(t) - \hat{a} \hat{\xi}_c^\dagger(t) \} + i\sqrt{2\gamma_m} \{ \hat{b}^\dagger \hat{\xi}_m(t) - \hat{b} \hat{\xi}_m^\dagger(t) \}$$

accounting for the cavity (mechanical) damping at the rate $\kappa$ ($\gamma_m$), with the correlations of the quantum noise operators satisfying [24]

$$\langle \hat{\xi}_c^\dagger(t) \hat{\xi}_c^\dagger(t') \rangle = 0, \quad \langle \hat{\xi}_m^\dagger(t) \hat{\xi}_m^\dagger(t') \rangle = n_{th} \delta(t-t'), \quad \langle \hat{\xi}_c^\dagger(t) \hat{\xi}_m^\dagger(t') \rangle = \langle \hat{\xi}_m^\dagger(t) \hat{\xi}_c^\dagger(t') \rangle = 0$$

where a zero thermal occupation for the cavity reservoir is assumed and $n_{th}$ is the thermal occupation of the mechanical reservoir. Then we take an interaction picture with respect to...


\[ H_c(t) \text{ is the transformed Hamiltonian } H^{m}(t) = U^\dagger_0(t)(H_{om} + H_{sc})U_0(t), \text{ where } U_0(t) = Te^{-\int_0^t dt H_c(t)}, \text{ gives the dynamical equations} \]

\[
\dot{a} = -\kappa a + gE(t)e^{-i\omega_m t}b + gE(t)e^{i\omega_m t}b^\dagger \\
+ ikf(t)E + \sqrt{2\gamma_c} \xi_c(t),
\]

\[
\dot{b} = -\gamma_m b - gE^\dagger(t)e^{i\omega_m t}a + gEf(t)e^{i\omega_m t}a^\dagger \\
+ ig[f(t)E + \sqrt{2\gamma_m} \psi_m(t)],
\]

with \( f(t) = (e^{i\Delta t} - 1)/\Delta \). A cubic term in \( H^{m}(t) \) is neglected in deriving these linear dynamical equations, since we are dealing with a weakly coupled OMS [25].

We first look at the mechanical oscillator's thermal equilibrium with the environment. In this initially prepared state with no external drive (\( E = 0 \)), the mechanical mode takes the exact form \( \dot{b}(t) = e^{-i\omega_m t}b + \sqrt{2\gamma_m} \int_0^t dt e^{-\gamma_m(t-\tau) \xi_m(\tau)} \) from Eq. (3). The second noise drive term maintains the invariant phonon number \( \langle \hat{b}^\dagger \hat{b} \rangle = n_{th} \) under the thermal equilibrium, while the contribution from the first term lowers with time. The noise actions are thus essential to a quantum OMS.

In a general situation it is clearer to use the equation

\[
\frac{d}{dt} \tilde{c}(t) = \tilde{M}(t) \tilde{c}(t) + \tilde{\lambda}(t) + \tilde{\eta}(t)
\]

about the complete set \( \tilde{c}(t) = (\hat{a}(t), \hat{a}^\dagger(t), \hat{b}(t), \hat{b}^\dagger(t))^T \) of the system modes, where \( \tilde{\lambda} = (\lambda_c, \xi_c, \gamma_m, \lambda_m)^T \) with \( \lambda_c(t) = ikf(t)E + \sqrt{2\gamma_c} \xi_c(t) \) and \( \lambda_m(t) = ig[f(t)E + \sqrt{2\gamma_m} \psi_m(t)] \) with \( \xi_c(t) = \sqrt{2\gamma_c} \xi_c(t) \) and \( \psi_m(t) = \sqrt{2\gamma_m} \psi_m(t) \), and the detailed matrix \( \tilde{M}(t) \) can be found in [25]. The general solution to Eq. (4) reads

\[
\tilde{c}(t) = Te^{\int_0^t dt \tilde{M}(t)} \tilde{c}(0) \\
+ \int_0^t dt Te^{\int_0^t dt \tilde{M}(t)} [\tilde{\lambda}(t) + \tilde{\eta}(t)].
\]

with \( \tilde{c} = (\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger)^T \). The time-ordered exponentials appear because we have \( \tilde{M}(t), \tilde{M}(t) \) \( \neq 0 \) for \( t \neq \tau \). In terms of the notation \( \psi_0(t, \tau) \) for the matrix elements \( T e^{\int_0^t dt \tilde{M}(t)} \), the evolving thermal phonon number \( n_m(t) = |\langle \hat{b}^\dagger \hat{b} \rangle(t) - |\langle \hat{b} \rangle(t)\rangle|^2 \) [the subtraction of the contribution from the coherent drive \( \tilde{\lambda}(t) \) will be discussed below] consists of two parts,

\[
n_m^{(s)}(t) = |d_{41}(t, 0)|^2 + |d_{44}(t, 0)|^2 n_{th} \\
+ |d_{43}(t, 0)|^2 (n_{th} + 1),
\]

from taking the expectation value of the first term on the right-hand side of Eq. (5) with respect to the initial cavity vacuum state and mechanical thermal state, and

\[
n_m^{(s)}(t) = 2k \int_0^t dt |d_{41}(t, \tau)|^2 + 2\gamma_m \int_0^t dt |d_{44}(t, \tau)|^2 n_{th} \\
+ 2\gamma_m \int_0^t dt |d_{43}(t, \tau)|^2 (n_{th} + 1),
\]

for averaging the following noise drive term over the reservoir states by means of Eq. (1). A meaningful scenario beyond thermal equilibrium is cooling—the noise contribution in Eq. (7) finally stabilizes to a thermal phonon number \( n_{m,f} \) less than the initial occupation \( n_{th} \), while the contribution \( n_m^{(s)}(t) \) from the evolved system operators gradually tends to zero with time.

Going back to the specific terms in Eqs. (2) and (3), one finds that an increased magnitude of the third term on the right-hand sides (from a squeezing-type coupling) can enhance the phonon numbers in both Eqs. (6) and (7).

The cooling action, on the other hand, manifests as the second term of a BS coupling in these equations. To let the BS action dominate, one could set the detuning \( \Delta \) to be the mechanical frequency \( \omega_m \) so that the factor \( f(t)e^{-i\omega_m t} \) in Eq. (2) will become \( \Gamma_b(t)/\omega_m = (1 - e^{-i\omega_m t})/\omega_m \) containing a nonoscillating term. Meanwhile, the factor \( f(t)e^{i\omega_m t} \) in the squeezing coupling term will be \( \Gamma_s(t)/\omega_m = (e^{i2\omega_m t} - e^{-i\omega_m t})/\omega_m \). A cooling can be performed with a sufficiently large \( \omega_m/k \), because in Eq. (5) it suppresses the integrals of the matrix elements carrying the oscillating factor \( \Gamma_s(t) \).

Our dynamical approach directly gives the picture of how the changes in the system parameters will turn the system from heating to cooling; see the time evolutions of the thermal phonon numbers in Figs. 2(a) and 2(b). In the transitional regimes where the cooling (BS) and heating (squeezing) effect compete with each other, the phonon numbers exhibit oscillations with time, as those in the insets.
of the figures. The corresponding cavity photon numbers in Figs. 2(c) and 2(d) evolve synchronously with the thermal phonon numbers; the phenomenon is discussed further in [25].

New understanding of optomechanical cooling can be obtained from this dynamical picture, as it brings about another important parameter for cooling in addition to the widely concerned sideband resolution \( \omega_m/\kappa \). A relevant phenomenon shown in Fig. 3(a) is that a higher sideband resolution beyond a certain value will be actually worse for cooling if the drive intensity \( E \) is fixed. Our predicted thermal phonon number \( n_m \) comes from the noise contributions as the integrals in Eq. (7), and is therefore determined by the whole process from \( t = 0 \) to the approximate end time \( t = t_s \), when \( n_m(t_s) \) begins to be stable and \( n_m(t_s) \to 0 \) (the associated dynamical behaviors at \( t_s \) asymptotically approach those at \( t = \infty \)). A slower process will accumulate a higher noise contribution \( n_m \), which can be the time average of \( n_m(t) \) \( (t > t_s) \) for the finally oscillating ones in Fig. 2(b). At the BS coupling resonance \( \Delta = \omega_m \), the coefficient of the BS (squeezing) coupling term in Eqs. (2) and (3) takes the form \( |J_T(0)| \). The parameter defined as \( J = (g/\omega_m)(E/\kappa) \) decides how fast the cooling of a specific OMS will be; see the illustration in Fig. 2(a). Were there no quantum noises, an uncoupled system with \( J = 0 \) would take the longest time scale \( 1/\gamma_m (\gamma_m \ll \kappa) \) to reach its stability; cf. the solution \( \hat{b}(t) = e^{-\gamma_m t} \hat{b} \) to Eq. (3) when \( E, \hat{x}(t) = 0 \). With the drive intensity \( E \) fixed as in Fig. 3(a), a lower sideband resolution \( \omega_m/\kappa \) corresponding to a higher \( J \) makes the cooling faster. It is also shown in Fig. 2(b) that, once the parameter \( J \) is fixed, the OMSs with different sideband resolution \( \omega_m/\kappa \) will evolve to the stabilized phase almost together.

Apart from speeding up the cooling of an OMS, increasing \( J \) will make the coexisting squeezing (heating) effect stronger. These two tendencies strike a balance somewhere in the parameter space, so that the best cooling under a fixed drive intensity \( E \) takes place at an optimum \( \omega_m/\kappa \) [see Fig. 3(b)]. The latter tendency will dominate when the parameter \( \omega_m/\kappa \) is continuously lowered, which will increase \( n_m(t) \) significantly on the left of the optimum \( \omega_m/\kappa \). On the other hand, the suppression of the factor \( \Gamma_s(t) \) by faster oscillations due to larger \( \omega_m/\kappa \) will diminish the squeezing effect, leading to the tendency of the stabilized \( n_m \) in Fig. 2(b). Among the three different terms in Eq. (7), the first one independent of \( n_m \) monotonically decreases with increased \( \omega_m/\kappa \), as illustrated in Figs. 3(c) and 3(d). The experimental investigation of the effect related to this contribution was recently reported in [12]. Such pure cavity noise contribution decided by the squeezing coupling intensity \( J\Gamma_s(\kappa t) \) increases significantly with the drive intensity \( E \), which distinguishes our results fundamentally from the previous predictions; see the comparisons in Fig. 3(d) and Fig. S4(b) in [25].

Another important issue is how to reach the best cooling from the initial thermal equilibrium of an OMS. To answer the question, we first examine the limit of \( \omega_m/\kappa \to \infty \). In this limit, the stabilized thermal phonon number given by \( \Delta = \omega_m \) can be found analytically from Eqs. (2) and (3) as
where $\lambda_\pm = \frac{1}{2}[-1 - \Gamma_m \pm \sqrt{(1 - \Gamma_m)^2 - 4J^2}]$ and $\eta_\pm = -1 + \Gamma_m \pm \sqrt{(1 - \Gamma_m)^2 - 4J^2}$ with $\Gamma_m = \gamma_m / \kappa$. It is solely from the contribution of the second term in Eq. (7), because the squeezing action with its intensity $\mathcal{J}_s(\kappa t)$ has been completely averaged out by an infinite $\omega_m / \kappa$. Analogous to a first-order phase transition, this limiting value exhibits a discontinuous jump across the point $J = 1/2(1 - \Gamma_m)$ (in Fig. 4 it is around $J = 0.5$), which separates the weak-coupling regime from the strong-coupling one that has been experimentally observed [8,10,33]. In the strong-coupling regime, the limiting value becomes the constant $n_{m,f} = \Gamma_m n_{\text{th}}$. Inside the weak-coupling regime the stabilized phonon numbers for different $\omega_m / \kappa$ can be close to the limiting value and drop quickly with increased $J$ (see the inset in Fig. 4). With the parameters of the experimental setup in [9] (equivalent to $J \approx 0.16$), for example, an ideal cooling process would not reach a phonon number lower than the limit of $n_{m,f} \approx 0.26$.

The parametric conditions for achieving good cooling manifest more clearly with Fig. 4, in which the limiting value in Eq. (8) gives the boundary that any cooling process cannot surpass. For a setup with the built-in sideband resolution $\omega_m / \kappa$, increasing the drive power can make its cooling better and faster, moving the achieved thermal phonon number along a curve of fixed $\omega_m / \kappa$ to larger $J$, though the enhanced power may change the phonon number less obviously for a system with $\omega_m / \kappa \gg 1$ (in the strong-coupling regime its phonon curve asymptotically approaches the limit line). The higher the parameter $\omega_m / \kappa$ is, the stronger the drive that can be applied to achieve a better cooling; this explains the phonon number tendency to the right of the optimum points in Fig. 3(b). However, the cooling cannot be improved further if $J$ arrives at a turning point (the ending dots of the phonon curves) where the coexisting squeezing effect begins to be significant. After crossing the turning point, the thermal phonon number will increase from the minimum value there, and its evolution will continue to become oscillating and then grow with time as in Fig. 2(a). Such a reference point indicates that the cooling processes in Fig. 2(b) are not the optimum ones—the parameter $J = 1.0$ used there is on the right side of the turning point for $\omega_m / \kappa = 8$.

The evolving Gaussian states of the weakly coupled OMSs, which follow the linear dynamical equations, can be depicted with their Wigner functions. A cooling process starts from a mechanical oscillator’s thermal state with its Wigner function being $W(q_m, p_m) = \left[1/\pi(1 + 2n_{\text{th}})\right] \exp[-1/(1 + 2n_{\text{th}})((q_m - p_m)^2)]$. After turning on a drive, the OMS will evolve under optomechanical coupling to two-mode Gaussian states, whose Wigner functions can be found by numerically calculating the correlation matrix of the two system modes. As illustrated in Fig. 1(b), a perfect “ground-state cooling” to $n_{m,f} = 0$ evolves the mechanical mode’s Wigner function to $W(q_m, p_m) = (1/\pi)e^{-((q_m - q_m^0)^2 - (p_m - p_m^0)^2)}$ of a coherent state, which differs from that of a vacuum state only by the displacements $q_m^0(t), p_m^0(t)$ determined by the coherent drive term $\tilde{A}(t)$ in Eq. (4). Without impairing the purity [34] of the target coherent state, the contribution from the coherent drives should be excluded from the thermal phonon number $n_{m}(t)$ to be reduced from the occupation $n_{\text{th}}$ of an initial thermal state.

In summary, we have developed a quantum dynamical approach to optomechanical cooling. The motivation for the development is to reflect the fact that such cooling is a process for a mechanical oscillator to evolve from its initial thermal state to another state with higher purity, which takes time. Cooling starts after turning on a red-detuned drive that realizes a BS-type coupling between the fast damping cavity mode and the slowly decaying mechanical mode, so that the oscillator’s thermal excitation being converted to cavity photons could be totally eliminated after a time controlled by the parameter $J$. Only with this scenario giving the phonon number in Eq. (6), the mechanical oscillator would be cooled down to a vacuum state modified by its inevitable motion under radiation pressure to a coherent state. The simultaneous quantum noise actions, however, cause the evolution to deviate from going to such a pure quantum state and add to the thermal phonon number in Eq. (7). The thermal occupation left in the end is decided by how soon the coupled cavity and mechanical modes evolve together to a dynamical stability,
after which the system cannot be cooled down further. Like the fully quantum mechanical treatment of OMS in [35] and recent studies of the quantum dynamical features of other physical systems (see, e.g., [36,37]), the properties of the cooling process illustrated here are for a genuine quantum OMS rather than the quantum fluctuations around the trajectories of a classical one; this approach can also be used for blue-detuned drives [38]. This quantum dynamical picture of cooling applies to an OMS truly approaching its macroscopic quantum states.

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