TOPICAL REVIEW

Atomic optical bistability in two- and three-level systems: perspectives and prospects

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The atomic optical bistability (AOB) in multi-level systems has advantages over the two-level system as absorption, dispersion, and nonlinear optical properties of one optical field coupled to one atomic transition can be greatly modified by other optical fields coupled to other connected nearby atomic transitions in multi-level atomic systems due to induced atomic coherences. By making use of such changes in linear and nonlinear optical properties around resonance, which are related to the electromagnetically induced transparency (EIT), it is easy to manipulate and control nonlinear optical processes in the multi-level atomic systems. We review earlier two-level AOB theory, as well as experiments and some recent studies that made use of multi-level atomic EIT systems in achieving controls of nonlinear optical processes such as instabilities and stochastic resonance associated with the coupled atom–cavity systems.

Keywords: optical bistability; coherence; stochastic resonance; noise-induced switching

1. Introduction

The phenomenon of atomic optical bistability (AOB) in two-level atoms confined in an optical cavity has been a center of attraction during the 1980s and early 1990s, due to its potential applications in all-optical switches, memories, and optical transistor-like devices [1]. Two different kinds of AOB, i.e. absorptive AOB and dispersive AOB were predicted theoretically and observed experimentally. The mechanisms of absorptive AOB and dispersive AOB were attributed to saturation of atomic transition and intensity-dependent refractive index of the media, respectively [1–3]. Several interesting works were reported in AOB of two-level atomic systems. The mean field model along with quantum statistics was utilized for studying the transmission and fluorescent spectra [4]. The analogy between AOB in two-level systems and first-order phase transition was stipulated using the fact that the diffusion coefficient of the Focker–Plank equation is intensity dependent [5]. The self-pulsing was shown in the two-level AOB analytically in the steady state [6]. The bistable states of the optical cavity output appear only for a definite range of input intensities [7] and the intensity of the upper branch of the bistable curve shows an instability behavior for some parametric conditions [8]. Experimental investigations of single-mode instability induced in the cavity field interacting with a beam of sodium atoms has been carried out.

The detailed comparisons of theoretical and experimental works were reported and behavior of spontaneous output oscillations produced by such instability was pointed out [7,8]. The system of a cold atomic cloud of cesium atoms shows AOB as well as instability in which the Zeeman sublevels of the cesium atoms interact with a circularly-polarized laser beam in an optical cavity. However, there are limitations to the experiments with two-level atoms in general and to this experiment in particular because of the lack of control due to only one laser beam being employed for both optical pumping and saturation, and the optical cavity was not locked in observing AOB and instability [9].

The phenomena of electromagnetically induced transparency (EIT), dispersion and related enhancement of nonlinearity in multi-level atomic systems have attracted great attention over the past two decades [10–12]. This is due to the induced atomic coherence in multi-level atomic systems which modifies the linear absorption and dispersion properties. It can also enhance the nonlinear optical processes such as four-wave mixing [13,14], harmonic generation [15,16], and two-photon absorption [17]. In recent past, enhancement in the third-order Kerr nonlinear index of refraction (coefficient of the intensity-dependent refractive index) was experimentally measured near the EIT condition, as well as near the more general coherent population trapping (CPT) conditions in

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three-level Λ-type rubidium atoms inside an optical cavity [18]. When both probe and coupling fields resonantly interact with their respective atomic transitions, i.e. when frequency detunings are zeros, then the coefficient of nonlinear index of refraction is zero at the exact EIT or CPT condition. It is interesting to note that if one of the frequency detunings is slightly nonzero, the value of nonlinear refractive index is greatly enhanced as compared to the situation when there is no coupling beam present (as in a two-level situation). Detailed studies of the variations of nonlinear refractive index with intensities and frequency detunings of both coupling and probe beams [18], along with bistable characters in this EIT system were reported [18]. The aims of these works were to experimentally demonstrate the controllability of AOB behaviors in the Λ-type system by varying the intensity and frequency detuning of the coupling laser beam. These additional experimental controls provided by the coupling laser beam do not exist in the previous systems using two-level atoms for AOB [1–3,7,8] or in the system with multi-Zeeman levels with only one laser beam [9]. Another advantage of the three-level atomic experimental setup is its simplicity with atoms in a vapor cell (no atomic beam or cooled atomic sample under vacuum is needed) in a two-photon Doppler-free configuration to overcome the first-order Doppler effect [12]. The enhanced nonlinearity here due to atomic coherence can be related to the effect reported in an earlier experiment with sodium vapor filled in a Fabry–Pérot cavity [19], where the two different frequencies for the atoms in a particular group of velocity distribution are provided by counter-propagating beams. Some interesting theoretical results for AOB have been reported for three-level atoms confined in an optical cavity under various conditions including the effect of spontaneously generated coherence [20,21].

Controlling AOB can have practical applications in all-optical switches, memories, transistors, and logic circuits, which have no need for optical–electronic–optical conversion of signal information. Due to the enhanced Kerr nonlinearity in the three-level atomic system owing to atomic coherence, the switching thresholds of such devices can reduce down considerably, and hence control becomes easier and more efficient, which is also essential in order to process optical signals faster and reliably at very low intensity levels of light. With such controlled linear and nonlinear optical properties of the three-level atomic system, the steady-state and dynamical behaviors of the composite systems with multi-level atoms inside an optical cavity can be well controlled experimentally. By adjusting the controlling beam frequency or intensity, the threshold values and the shape of the steady-state bistable curve can be controlled [22], and transitions from bistable to multistable hysteresis curves were observed [23]. With increased frequency detuning of the coupling beam the rotation of the hysteresis loop could be made from the forward direction (counterclockwise) to the backward direction (clockwise) [24], which is very unusual in such hysteresis cycles and also occurred in many other systems (magnetic, electrical, biological, etc.) due to energy dissipation considerations. This three-level atomic system also exhibits a very interesting behavior of a dynamical hysteresis cycle [25]. In a certain parameter range, the cavity output field of this composite system could become unstable, e.g. dynamic instability and chaotic behavior due to the competition between the optical pumping in the coupling transition and the saturation in the probe transition [26,27], again controllable by adjusting the coupling beam frequency detuning or intensity. The controllability of the hysteresis cycle in the three-level AOB allows one to study the phenomenon of stochastic resonance [28] in such systems. The fluctuations caused by the atomic coherence can lead to a noise-induced switching phenomenon [29] in such bistable systems with multi-level atoms, which we believe is not so conveniently available with the two-level atomic system.

Such controllable nonlinear optical processes in multi-level EIT systems, especially inside an optical cavity for studying AOB, chaos, stochastic resonance and noise-induced switching, will be the main focus of this review. We neither intend to review the entire field of EIT-related phenomena, nor do we try to include all the experiments of nonlinear optical processes done in multi-level atomic systems. This review has a limited scope of covering mainly the controlled nonlinear optical processes of three-level atoms inside an optical cavity giving rise to AOB and related phenomena, and compare them with two-level AOB systems.

In Sections 2 and 3, we review the two-level AOB theory and experiments, respectively. Section 4 discusses theoretical foundations and Section 5 describes experiments showing controllability of the AOB in a system with three-level atoms inside an optical ring cavity, emphasizing the enhanced nonlinear optical processes due to induced atomic coherence in such an intracavity EIT medium. In Section 6, steady-state optical instability and chaos arising in two-level AOB are discussed. Section 7 describes theoretical and experimental investigations of controlled optical dynamic instability and from period doubling to chaos in the cavity output field with three-level atoms inside an optical cavity. The phenomenon of stochastic resonance in AOB is explained in the three-level atomic system with a simple theoretical model. A few novel experiments well described by this generic model are
The above density-matrix equations describe the atomic dynamics of the system. The electric field seen by the atoms can be written as

\[ E = (E_p \exp(-i\omega pt) + c.c.). \]  

(2)

The electric field at frequency \( \omega_p \) (interacting with the atomic transition \(|1\rangle \) to \(|2\rangle \)) circulates inside the cavity as the cavity field. The induced atomic polarization responsible for AOB is given by

\[ P(\omega_p) = N\mu_{12}\rho_{12}, \]  

(3)

where \( N \) is the atom number density and \( \mu_{12} \) is the dipole moment of atoms. The field \( E_p \) enters into the cavity from the partially transparent mirror \( M_1 \) and drives the two-level atomic transition. Because of the boundary conditions for the optical ring cavity, the incident field \( E_p^\dagger \), the transmitted field \( E_p^T \) and fields at different locations in the cavity, \( E_p(0,t), E_p(L,t) \) obey the following boundary conditions [1]:

\[ E_p^T = T^{1/2}E_p(L,t), \]

\[ E_p(0,t) = T^{1/2}E_p^\dagger(t) + R \exp(-i\delta_0)E_p(L,t - \Delta t), \]  

(4)

where \( L \) is the length of the atomic sample and \( \Delta t = (2L + L)/c \) is the time taken by light to travel from mirror \( M_2 \) to \( M_1 \) via \( M_3 \) and \( M_4 \) (side arm length between \( M_2 (M_1) \) and \( M_3 (M_4) \)). The cavity frequency detuning is defined as \( \delta_0 = (\omega_{\text{cav}} - \omega_p)L_T/c \), where \( \omega_{\text{cav}} \) is the frequency of the nearest cavity mode to frequency \( \omega_p \), and \( L_T = 2(2L + L) \) represents the total length of the ring cavity.

The dynamical evolution of the field inside the cavity is governed by Maxwell’s equation (in slowly-varying envelope approximation)

\[ \frac{\partial E_p}{\partial t} + c \frac{\partial E_p}{\partial z} = 2\pi i\omega_p\mu_{12}P(\omega_p), \]  

(5)

with the boundary conditions as defined in Equation (4) above. In order to obtain polarization \( P(\omega_p) \) one needs to first solve the set of density-matrix Equations (1) in the steady-state limit, then using (4) to integrate Equation (5) in the steady-state limit over the length of the atomic sample. The steady-state boundary conditions can be written, at steady state, as

\[ E_p^T = T^{1/2}E_p(L), \]

\[ E_p(0) = T^{1/2}E_p^\dagger + R \exp(-i\delta_0)E_p(L). \]  

(6)

Alternatively, the steady-state behavior of AOB can be obtained by setting \( \partial \rho/\partial t = 0 \), and \( \partial E_p/\partial t = 0 \), so that the stationary field equation has the following expression

\[ \frac{\partial E_p}{\partial z} = -\chi(|E_p|^2)E_p, \]  

(7)

Figure 1. Schematic diagram of a unidirectional ring cavity having four mirrors (M1–M4) and an atomic vapor cell of length \( L \). The mirrors M3 and M4 are perfectly reflecting mirrors (\( R = 1 \) for each). The incident and the transmitted fields are represented by \( E_p^\dagger \) and \( E_p^T \), respectively.
where the complex dielectric susceptibility $\chi$ is defined as
\[ \chi = \chi_a + i\chi_d, \]  
(8)
in which the absorptive and the dispersive components are $\chi_a$ and $\chi_d$, respectively. It can be easily shown that the expression of $\chi$ has the following form
\[ \chi = A(1 - i\Delta)[1 + \Delta^2 + |E_p|^2/I_s]^{-1}, \]  
(9)
with unsaturated absorption coefficient $A = g_{12}^2\mu_{12}N/2\hbar\gamma_{21}$, atomic detuning parameter $\Delta = (\omega_{21} - \omega_p)/\gamma_{21} = \Delta_p/\gamma_{21}$, and the saturation intensity $I_s = \hbar^2\gamma_{21}\gamma_2/\mu_{12}^2$.

We consider $E_p$ as a complex quantity and define $E_p = \varepsilon_p(z) \exp(i\phi_p(z))$, which leads to the equations for the amplitude and phase as
\[ \frac{d\varepsilon_p}{d\tau} = -\chi_a(\varepsilon_p^2)\varepsilon_p, \quad \frac{d\phi_p}{d\tau} = -\chi_d(\varepsilon_p^2). \]  
(10)

By defining the quantities $P_1 = (E_1^2)$, and $P_T = |E_1|^2$ as incident and transmitted intensities and $\eta = \varepsilon_p(0)/\varepsilon_p(T)$, respectively. The expression for the cavity transmissivity is given by [1]
\[ \frac{Y}{P_T} = \frac{T^2}{(\eta - R)^2 + 4R\eta \sin^2\frac{1}{2}[\Phi_p(L) - \Phi_p(0) - \delta_0]}. \]  
(11)

By introducing normalized incident and transmitted intensities $Y = P_T/I_sT$ and $X = P_T/I_sT$, it is easy to show that [1]
\[ \frac{Y}{X} = \frac{T^2}{(\eta(X) - R)^2 + 4R\eta(X) \sin^2\frac{1}{2}[\Delta \ln(\eta(X)) - \delta_0]} \]  
(12)
where we have used the fact: $\Phi_p(L) - \Phi_p(0) = \Delta \ln(\eta(X))$. Note that the cavity transmissivity function $Y$ is a sensitive function of $\eta(X)$. When $X$ is very large then $\eta(X)$ approaches unity and the transmissivity becomes constant and equal to the empty cavity value ($\chi_a = \chi_d = 0$). For $\chi_d = 0$, we get the absorptive AOB [1]. When $\chi_a = 0$, then the dispersion dominates and we get the Kerr-type situation giving rise to the dispersive AOB [1]. The system can exhibit multistability depending upon the number of oscillations that the function $\sin[\frac{1}{2}(\Delta \ln(\eta(X)) - \delta_0)]$ makes and such oscillations are determined by the quantity $A\Lambda /(1 + \Lambda^2)$. The steady-state solution becomes particularly simple in the case of $A\Lambda \ll 1$. In the limits of $A\Lambda \to 0$, $T \to 0$, $\delta_0 \to 0$ and defining $C = A\Lambda/2T = \text{constant}$, $\theta = \delta_0/T = \text{constant}$, we get for a homogeneously-broadened system [1]
\[ Y = X \left\{ \left[1 + \frac{2C}{1 + \Lambda^2 + X} \right]^2 + \left(\theta - \frac{2CA}{1 + \Lambda^2 + X} \right)^2 \right\}. \]  
(13)

Clearly, there are two distinct regimes to achieve absorptive and dispersive AOB (Figure 2(a) and (b)). In the absorptive AOB case, atoms are driven on near resonance and in the dispersive case they are driven very far from resonance so a nonlinear refractive effect dominates. The system displays a hysteresis cycle in either case as a function of incident driving field intensity, however, the physics behind the formation of the two hysteresis loops is quite different. It is the saturation of atomic absorption that switches the cavity transmission from a low to a high state in the absorptive AOB, but the intensity-dependent index of refraction (Kerr-type) shifts the cavity resonances for the dispersive AOB. The study of the dynamics shows rich behaviors when both absorptive and dispersive contributions are important along with comparable decay rates for atomic polarization and the cavity mode.

Several experiments have been reported over the years for investigating the AOB in two-level atomic systems by various groups [1,3]. A very systematic experimental investigation in pure absorptive AOB in a collection of radiatively-broadened two-state atoms in atomic beams interacting with a single mode of cavity has been reported and this system was well characterized to make absolute comparisons with theory [31]. Non-resonant conditions for both cavity and atoms are also considered in these experiments where absorptive

![Figure 2](image-url)

Figure 2. Hysteresis loop of (a) a typical absorptive AOB and (b) a dispersive AOB.
and dispersive contributions were equally present and absolute comparisons with the theory could be made [32]. In the following we briefly describe these important experiments.

3. Two-level AOB: experimental observations of absorptive and absorptive-dispersive optical bistability

The earlier experiments on AOB in two-level atomic systems demand use of atomic beams to get rid of the first-order Doppler effect, which hinders the observation of AOB in a two-level atomic system. The experimental system consists of a set of ten well-collimated optically-prepumped atomic beams of sodium which intersect the axis of a high-finesse cavity at a right angle [31,32]. The atoms are prepared in the state $^2S_{1/2}, F = 2, m_F = 2$, which couples to the state $^2P_{3/2}, F = 3, m_F = 3$ (D$_2$ line of sodium) by a circularly-polarized light. Experiments were performed using confocal ring and out-of-confocal standing-wave cavity geometries. The separations of the mirrors were 5.0 cm for the confocal ring, having free spectral range (FSR) of 5.5 GHz, and 5.5 cm for the standing-wave cavity with 2.7 GHz FSR. Both cavities sustain a mode waist of 69 $\mu$m. The transmission coefficients of the mirrors were $T_1 = T_2 = (3.0 \pm 0.1) \times 10^{-3}$. The values of finesse $F$ of the confocal ring cavity and standing-wave cavity were 263 and 650, respectively. The intensity and frequency stabilized dye laser (linewidth of 500 kHz) was used in the experiment. The incident laser beam was mode-matched to the fundamental TEM$_{00}$ mode of the cavities with about 94% efficiency. The power transmitted through the cavity was detected by a calibrated photomultiplier tube. Measurement of small-signal absorption was done before each experiment. For the off-resonant experiments [32], the laser frequency was tuned away from the atomic transition and was continuously monitored using an independent interferometer to provide information on the laser frequency detuning with respect to the atomic line center. The calibrated line profile of optical-pumping fluorescence was also measured to determine the atomic frequency detuning. The atomic detuning was kept within $\pm 15$ MHz from line center. Cavity frequency detuning was achieved by varying the voltage applied to a PZT (piezoelectric transducer) which controls the position of the cavity input mirror of M1. The cavity detuning was measured by comparing the slope of the input–output response, high in the upper branch because atoms were unsaturated and the system was free from instabilities, with the resonant empty cavity slope. It was a direct measurement of the cavity detuning, provided one was on a sufficiently high section of the upper branch and did not depend on detector calibration, and also provided a sign of the detuning.

In the experiments the atomic density was varied gradually (by changing the temperature of the oven) and the input intensity modulated at 50 Hz. This scan rate was quite slow in comparison to any dynamic rates of the system and thus ensured the steady-state adiabatic behavior. The input–output power ($P_i$ versus $P_o$) measurements were done continuously. The experiments were repeated at several values of atomic and cavity detunings. In Figure 3, a detailed comparison of normalized input switching intensities as a function of cavity detuning parameter has been presented for the standing-wave cavity. This figure represents an absolute comparison between experimental measured data and theoretically calculated results for input switching points as a function of cavity detuning with fixed atomic detuning in the standing-wave cavity. The ‘phase diagram’ of this study is provided in Figure 4, which represents limits of hysteresis in the ($\Theta, C$) plane for a fixed atomic detuning in the ring cavity. The comparison with theory is also given here [32].

![Figure 3. Normalized input switching intensities $Y_1$ (rhombus) and $Y_2$ (dots) for the standing-wave cavity as a function of cavity detuning $\Theta$ with no atomic detuning $D = 0$ and $C = 15$. The solid line is the theoretical prediction for $Y$ scaled down by a factor 0.91, which is within our experimental uncertainties (from [32]). Reproduced with permission from Orozco et al. Phys. Rev. A 1987, 36, 3248. Copyright (1987) by the American Physical Society.](image-url)
The pure absorptive AOB hysteresis loop obtained in this system is shown in Figure 5. Figure 5(a) displays a plot of incident power versus transmitted power when no atomic beams were present, and Figure 5(b) shows the same input–output intensity plot in the presence of the atomic beams [31]. The lower and upper switching powers (as marked on Figure 5) were evaluated for several hysteresis curves under different parametric conditions, which are plotted in Figure 6 as a function of $C$. The relative uncertainties of both the switching powers and data determining $C$ were given in these curves. The experimental results were compared with the Gaussian-beam theory of optical bistability and found to be in very good agreement.

4. Three-level AOB: theory

The three-level atomic systems can be found in ladder-type, Λ-type, and V-type configurations. Here, we consider a closed three-level Λ-type system where

![Figure 4](image_url)

Figure 4. Phase diagram for hysteresis in the $(C, \Theta)$ plane for atomic detuning $\Delta = -0.75 \pm 0.25$ in the ring cavity. Both scales have overall uncertainties of $\pm 10\%$. The data point limits the existence of bistability as one detunes the cavity away from resonance. The curve represents the theoretical predictions without adjustment (from [32]). Reprinted with permission from Orozco et al. Phys. Rev. A 1987, 36, 3248. Copyright (1987) by the American Physical Society.

![Figure 5](image_url)

Figure 5. The $x$-$y$ oscilloscope traces of transmitted laser power $P_t$ as a function of incident laser power $P_i$ for zero atomic and cavity detunings. (a) No intracavity beams. (b) Intracavity beams with resonant absorption $\alpha_m = 1.13$ and with $P_i^{(1)} = 70 \mu W$, $P_i^{(2)} = 104 \mu W$ (from [31]). Reprinted with permission from Rosenberger et al. Phys. Rev. A 1983, 28, 2569. Copyright (1983) by the American Physical Society.

![Figure 6](image_url)

Figure 6. (a) Incident switching power $(P_i^{(1)}, P_i^{(2)})$ versus effective atomic cooperativity $C_e$ as determined from equation (1) of [31]. (b) Transmitted switching power $(P_t^{(1)}, P_t^{(2)})$ versus $C_e$. The overall determination of $C_e$ is uncertain to $\pm 15\%$. The curves are based on theoretical results of equation (3) [31] and plotted relative to the right-hand scale (from [31]). Reprinted with permission from Rosenberger et al. Phys. Rev. A 1983, 28, 2569. Copyright (1983) by the American Physical Society.
$E_i (i = 1, 2, 3) (E_2 > E_3 > E_1)$ represent the energy levels as shown in Figure 7(a). The probe laser (frequency $\omega_p$, amplitude $E_p$) interacts with the atomic transition $|1\rangle \rightarrow |2\rangle$ (frequency $\omega_{21}$) with a frequency detuning of $\Delta_p = \omega_{21} - \omega_p$. The coupling laser (frequency $\omega_C$, amplitude $E_C$) interacts with the other atomic transition $|2\rangle \rightarrow |3\rangle$ (frequency $\omega_{23}$) with a frequency detuning of $\Delta_C = \omega_{23} - \omega_C$. Under the semiclassical approach, Liouville’s equation of density operator in the dipole and rotating-wave approximations is given by [12]

$$
\dot{\rho}_{11} = \gamma_{31}(\rho_{33} - \rho_{11}) + \gamma_{21}\rho_{22} - \frac{i}{2}\Omega_{p}^*\rho_{21} + \frac{i}{2}\Omega_{p}\rho_{12},
$$

$$
\dot{\rho}_{22} = -\gamma_{31}\rho_{32} + \frac{\Delta_{C}}{8}\rho_{21} - \frac{i}{2}\Omega_{p}\rho_{12} + \frac{i}{2}\Omega_{C}\rho_{23},
$$

$$
\dot{\rho}_{33} = \gamma_{31}(\rho_{11} - \rho_{33}) + \gamma_{23}\rho_{22} - \frac{i}{2}\Omega_{C}^*\rho_{23} + \frac{i}{2}\Omega_{C}\rho_{32},
$$

$$
\dot{\rho}_{23} = -\gamma_{31}\rho_{23} + \frac{\Delta_{C}}{8}\rho_{22} + \frac{i}{2}\Omega_{p}\rho_{13},
$$

$$
\dot{\rho}_{31} = -\gamma_{31} + i(\Delta_{p} - \Delta_{C})\rho_{31} - \frac{i}{2}\Omega_{C}\rho_{21} + \frac{i}{2}\Omega_{p}\rho_{32}. 
$$

The populations of this closed atomic system satisfy the relation $\rho_{11} + \rho_{22} + \rho_{33} = 1$. The dipole moment matrix elements for the two transitions are considered to be real for the sake of simplicity. $\Omega_{p}$ and $\Omega_{C}$ are complex Rabi frequencies of the probe and coupling fields, respectively. The radiative decay rates from level $|2\rangle$ to $|1\rangle$ $(|3\rangle)$ is $\gamma_{21}$ ($\gamma_{23}$) and the non-radiative decay rate between levels $|3\rangle$ and $|1\rangle$ is $\gamma_{31}$. The EIT arises from the existence of atomic coherence between levels $|3\rangle$ and $|1\rangle$ induced by a two-photon process. As we will see, it is the presence of the coupling field $E_C$ which is responsible for the creation of $\rho_{31}$ in steady state.

Under the EIT condition in which the coupling field is strong and the probe field is weak, i.e. $\Omega_{C} \gg \Omega_{p}$, then in the steady state, almost all the atoms are in state $|1\rangle$ so that $\rho_{11} \approx 1, \rho_{22} \approx \rho_{33} \approx 0$ (to the first order in $\Omega_{p}/\Omega_{C}$). The two-photon Doppler-free configuration for coherence effects in such $\Lambda$-type system is that the probe beam and the coupling beam propagate collinearly [18]. Under this arrangement the first-order Doppler shifts of the probe laser and the coupling laser for the same group of atoms with velocity $v$ can be cancelled [18]. The effect of residual Doppler-broadening in this $\Lambda$-type EIT system can also be taken into account easily under the assumption of $\omega_p \approx \omega_C$. Hence, in the steady state [12] the susceptibility is given by

$$
\chi = \frac{i\epsilon\mu_{31}^{24}N_{0}\pi^{1/2}}{\hbar \omega_p} e^{-z}(1 - \text{erf}(z)),
$$

with $\text{erf}(z)$ being the error function and

$$
z = \frac{c}{\omega_p} \left[ \gamma_{31} + i(\Delta_{p} - \Delta_{C}) - \frac{\Omega_{C}^{2}}{4} \right].
$$

The probe electric field seen by the atoms can be written as

$$
E = (E_p \exp(-i\omega_p t)) + E_C \exp(-i\omega_C t) + c.c.
$$

The electric field at frequency $\omega_p$ (interacting with the atomic transition $|1\rangle$ to $|2\rangle$) circulates inside the cavity as the cavity field, but $E_C$ does not. The induced atomic polarization responsible for AOB is given by Equation (3). The dynamical evolution of the probe field inside the cavity is governed by Maxwell’s Equation (5) with the steady-state boundary conditions as defined above in Equation (6) for the two-level AOB case. In order to obtain polarization $P(\omega_p)$ one needs to first solve the set of density-operator Equations (14) for the three-level atomic system in this $\Lambda$-type configuration under the steady-state limit, then using Equation (4) one can integrate Equation (5) in the steady-state limit over the length of the intracavity medium. In this way the relationship between the cavity output field and the cavity input field is

\[ Image of Figure 7(a) and (b) showing the three-level atomic system configuration. \]
obtained, which is known as the hysteresis cycle of the three-level AOB under appropriate conditions. The numerically simulated AOB results are presented in Figure 7(b) with the procedure described above from a three-level system in Λ-type configuration (Figure 7(a)) under the parametric conditions mentioned in the caption. It is very clear from Figure 7(b) that the coupling field causes a lowering of the bistability threshold due to the enhanced nonlinearity and EIT effects, which provides controllability of the AOB hysteresis curve [33].

5. Three-level AOB: experimental observation

In the experiments on AOB with three-level atoms inside an optical cavity the main focus of attention was on the Λ-type configuration. Note that the experimental arrangement to observe AOB in the Λ-type system is slightly different from the ladder-type system. Here, both the probe and coupling beams propagate collinearly in the atomic medium to overcome the first-order Doppler effect. The probe and coupling lasers are orthogonally polarized. These two beams are combined by a polarization cube beam splitter (PBS) inside the cavity before entering the rubidium vapor cell, as shown in Figure 8.

The experiments were carried out in a three-level Λ-type system of \(^{87}\text{Rb}\) atoms using the \(D_{1}\) line of the \(5^2S_{1/2} \rightarrow 5^2P_{1/2}\) transitions as depicted in Figure 7(a).

![Figure 8. The experimental arrangement used for studying optical bistability and instability in \(^{87}\text{Rb}\) atomic vapor. M1–M3 are mirrors of optical ring cavity, PZT is the piezoelectric transducer, LD1 and LD2 are coupling and probe lasers, respectively; PB1–PB5 are polarizing cubic beam splitters; BS is a beam splitter; EOM is an electro-optic modulator; λ/2 is a half-wave plate; FR is a Faraday Rotator; D1 is a detector; APD is an avalanche photodiode detector; LS is a lock-in signal from reference Fabry–Pérot cavity, SAS is a saturation absorption spectroscopy setup (from [24]). Reproduced with permission from Joshi et al. Phys. Rev. A 2004, 70, 041802(R). Copyright (2004) by the American Physical Society.](image-url)
frequency of the optical ring cavity was locked onto the transmission peak of the reference laser beam. The EOM was then switched-on, which was in the path of the probe laser beam (LD2), and a triangular modulation with a duration of 5 ms was applied to the cavity input intensity. In the absence of the coupling beam no AOB could be found, but an optical-transistor-like behavior was observed. Interestingly, in the presence of the coupling beam one couldn’t find AOB, if both $\Delta C = 0$ and $\Delta P = 0$, even with a quite high $P_C$. As soon as one of the frequency detunings ($\Delta P$ or $\Delta C$) was set to a nonzero value (a few MHz), AOB became visible for a wide range of $P_C$. Figure 9 displays several experimentally observed AOB curves controlled by the coupling beam ($\Delta C$ and $P_C$) keeping other parameters (like atom number density, cavity detuning, and $\Delta P$) fixed. A typical AOB curve for the three-level $\Lambda$-type system (under the conditions of $\Delta P = 50$, $\Delta C = 51.5$ MHz, $P_C = 1.5$ mW, and cavity detuning of about 40 MHz) is displayed in Figure 9(a). Control of the AOB curve by simply changing the parameter $\Delta C$ to 25.7 MHz is shown in Figure 9(b). It is very clear from these figures that both switching thresholds ($Y_1$ and $Y_2$) and width ($Y_2 - Y_1$) of AOB could be controlled by simply changing $\Delta C$ alone, keeping $P_C$ fixed. The definition of switching threshold intensities $Y_1$ (the lower one) and $Y_2$ (the upper one) means the places where $dY/dX = 0$ in the AOB hysteresis curve. In Figure 9(c) and (d), $P_C = 8.4$ mW was fixed, but $\Delta C$ was tuned to 85.6 and 256.8 MHz, respectively. This caused significant changes in the shape of the AOB hysteresis curve. So, the AOB range and the switching intensity thresholds ($Y_1$ and $Y_2$) could be controlled very effectively by either $P_C$ or $\Delta C$. Intensity threshold values in AOB are strongly dependent on $\Delta p$, cavity detuning, atomic number density (or cooperativity parameter), $P_C$, and $\Delta C$. When the EIT and/or CPT are exactly filled (i.e. $\Delta P = 0$, $\Delta C = 0$ or $\Delta P + \Delta C = 0$), then the bistable behavior should be governed by the absorption. Since absorption is greatly reduced at such EIT or CPT condition, so the absorptive AOB was also not detectable. As one sets $\Delta C \neq 0$ and $\Delta P = 0$ (Figure 9(a) and (b)) the switching thresholds come down drastically from the situation where $\Delta C = 0$ and $\Delta P = 0$ with no AOB. Such an observation of AOB at quite low intracavity field intensity and low coupling intensity was a result of the enhanced Kerr nonlinearity near resonance due to induced atomic coherence by the coupling beam interacting with the nearby transition slightly off-resonance (i.e. $\Delta C \neq 0$). This kind of AOB is a mixture of absorptive and dispersive types where the dispersion (or phase detuning) is greatly enhanced by the Kerr nonlinearity, as measured in [18]. With the increase of $\Delta C$ to larger values (Figure 9(c)) the Kerr nonlinearity in the system reduces down, and hence the switching intensity $Y_2$, as well as the width of the hysteresis cycle are increased. For a very large coupling detuning ($\Delta C = 256.8$ MHz, Figure 9(d)) the nonlinearity sharply reduced in such a three-level system and the nonlinear refraction due to frequency

![Figure 9](https://example.com/figure9.png)

**Figure 9.** The input–output intensity characteristics of the optical cavity with $\Delta P = 0$. (a) is for $P_C = 1.5$ mW, $\Delta C = 51.4$ MHz; (b) is for $P_C = 1.5$ mW, $\Delta C = 25.7$ MHz; (c) is for $P_C = 8.4$ mW, $\Delta C = 85.6$ MHz; and (d) is for $P_C = 8.4$ mW, $\Delta C = 256.8$ MHz (from [22]). Reproduced with permission from Joshi et al. *Phys. Rev. A* 2003, 67, 041801(R). Copyright (2003) by the American Physical Society.
detuning dominated, which was similar to the two-level dispersive OB situation. As a result, the shape and width of the OB curve are modified considerably. These measured results are consistent with the previous work on the measurements of nonlinear refractive index in this system [18]. The cooperativity parameter $C$, traditionally defined as $C = (\alpha_0/4\pi)F$ ($\alpha_0$ is the single-pass absorption) in two-level AOB systems [1–3, 7, 8], could be estimated with the measured atomic absorption and cavity finesse under the EIT condition in the experiment.

The point of emphasis here is the flexibility of the AOB control in the three-level atomic system with experimental parameters, which is not available in the two-level systems. The special control parameters are the frequency detuning $\Delta_C$ and power $P_C$ of the coupling beam. In Figure 10, the switching threshold intensity ratios $Y_2/Y_1$ are plotted as a function of $\Delta_C$ for two different coupling intensities, where curve A is for $P_C=8.4\,\text{mW}$ and curve B for $P_C=1.5\,\text{mW}$, respectively [22]. When $\Delta_C$ is small, the ratio $Y_2/Y_1$ is also small, clearly indicating a lower OB threshold because of the enhanced Kerr nonlinearity near resonance. With an increased $\Delta_C$, $Y_2/Y_1$ also increases because the nonlinearity reduces down substantially. Further increase in $\Delta_C$ again reduces $Y_2/Y_1$, which is due to the effect of dispersion caused by $\Delta_C$ alone (at larger $\Delta_C$, the Kerr nonlinearity drops down appreciably and the system behaves like a two-level atomic system).

Next, the dependence of switching thresholds for a fixed $\Delta_C$ on the variation of $P_C$ is depicted in Figure 11, in which curves A and B represent switching threshold ratio $Y_2/Y_1$ for $\Delta_C=42.8\,\text{MHz}$ and 17.1 MHz, respectively. Curve A rises with increasing $P_C$ which is most likely due to the decrease in the Kerr nonlinearity with increasing coupling power $P_C$ for this value of $\Delta_C$. However, we find an opposite behavior in curve B which slightly descends with the increase of $P_C$ and this is again because of the increasing nonlinearity with the increase of $P_C$ at this particular frequency detuning $\Delta_C$ [22].

The observed AOB behaviors with such a three-level EIT system inside an optical cavity are quite different from the previously studied two-level atomic media. Due to the induced atomic coherence near EIT resonance, the linear absorption and dispersion as well as nonlinearity, are all greatly altered. The advantages of using such a three-level system as an intracavity medium are its simplicity of operation and controllability. By propagating the coupling and probe fields collinearly through the vapor cell containing the three-level $\Lambda$-type rubidium atoms, there is no need to use atomic beams or cold atomic samples to overcome the first-order Doppler effect.

The interesting open question here is to find the usual optimal conditions for achieving the desired AOB behaviors for both fundamental studies and practical applications. The experimental demonstrations of controlled AOB in the three-level atomic system with the coupling beam intensity and frequency detuning provides an important step towards this goal. By controlling the threshold intensities and the width of the AOB curve with the intensity and frequency detuning of the coupling beam, one can build more efficient all-optical switches and logic-gate devices for optical computing and quantum information processing. Due to the reduced absorption and enhanced Kerr nonlinearity via atomic coherence in this intracavity EIT system, the switching threshold value ($Y_2$) is

![Figure 10. Ratio of switching threshold intensities $Y_2/Y_1$ as a function of $\Delta_C$. Curve A is for $P_C=8.4\,\text{mW}$ and curve B is for $P_C=1.5\,\text{mW}$ (from [22]). Reproduced with permission from Joshi et al. Phys. Rev. A 2003, 67, 041801(R). Copyright (2003) by the American Physical Society.](image1)

![Figure 11. Ratio of switching threshold intensities $Y_2/Y_1$ as a function of $P_C$. Curve A is for $\Delta_C=42.8\,\text{MHz}$ and curve B is for $\Delta_C=17.1\,\text{MHz}$ (from [22]). Reproduced with permission from Joshi et al. Phys. Rev. A 2003, 67, 041801(R). Copyright (2003) by the American Physical Society.](image2)
greatly reduced to make the optical switches more efficient with the possibility to operate at much lower light intensities. Complete understanding of these interesting OB phenomena will require a proper and complete theoretical calculation matching with the detail of the experimental condition, such as multi-Zeeman sub-levels and various Raman processes even near the EIT resonances.

6. Two-level AOB: instability

The single-mode dynamical instability in a two-level AOB system was first studied within the framework of a plane-wave model [7]. It was then analyzed using the Gaussian field model along with its experimental feasibility [7]. Experimental demonstration of a single-mode instability came into existence in an experiment quite similar to the one used for showing AOB in two-level system. Such a single-mode instability could be considered as a passive counterpart of the Lorentz–Haken model. Hence, the appropriate investigation of optical instabilities required a simple theoretical model which could be realized experimentally [7,8].

6.1. Theoretical model of dynamical instability in a two-level system

Using plane-wave theory of AOB, a variety of single-mode pulsation phenomena were predicted [1,7]. This is in addition to the dynamical effects due to multimode fields in the cavity. Under the exact resonance conditions for the atoms and cavity ($\Delta = 0, \Theta = 0$) there was no positive-slope instability [1,7]. The system was destabilized when there was a mismatch between the incident field frequency and the cavity frequency, which led to instability. Such instability can develop either in the presence or absence of bistability. It was observed that instability could develop easily when the cavity detuning has an opposite sign with the atomic detuning. For the bistability cooperativity parameter $C = 50$, the spontaneous oscillation is periodic and displays a frequency equal to the cavity detuning $\Theta = \omega_{\text{cav}} - \omega_p$. Increasing $C$ to the value around 300, the oscillation shows a period doubling to chaotic behavior [7]. These results are plotted in Figure 12 by numerically solving Equations (1) and (5) (which are similar to equations (2.20) of [7]) under the parametric conditions shown in the caption. However, in experiments such predicted behavior was not observed since the realistic situation was far away from this plane-wave theory. A linear stability analysis of Equations (1) and (5) around an arbitrary steady state give a fifth-degree polynomial equation for the exponential rate constants which governs the linearized evolution [7]. The explicit form of the equation is a bit cumbersome and will not be discussed here. One can describe the Routh–Hurwitz criterion using coefficients of these equations.

6.2. Experimental results of single-mode instability in the two-level system

The experimental arrangement used for observing instabilities in the two-level atomic system is quite similar to what has been described for observing AOB in this two-level sodium atomic system in Section 3. As the instability only appears when both the optical cavity and atoms are out of resonance such measurements of the atomic and the cavity detunings were very crucial in such experiments [7,8]. In the experiment the dye laser beams were first aligned with atomic beams to ensure their perpendicularity so that the first-order Doppler-broadening could be minimized. For this purpose the retro-reflected light beam from a corner cube prism was used. The calibration of the weak-signal absorption was done and related to the optical-pumping fluorescence. After this alignment and calibration, the atomic density was varied by changing the temperature of the atomic oven and the input intensity was modulated at a frequency of about 50 Hz. This modulation frequency was many orders of magnitude smaller than the dynamic rates of the system. The powers going inside the cavity, as well as coming out from the cavity, were continuously monitored. The cavity detuning and atomic detuning were defined by $\Delta = (\omega_{21} - \omega_p)/\gamma_\perp$, $\Theta = (\omega_{\text{cav}} - \omega_p)/\kappa$, where $\omega_p$ was the frequency of the incident/probe field, $\omega_{\text{cav}}$ was the frequency of the cavity mode, and $\omega_{21}$ was the atomic transition frequency; $\kappa$ was the cavity decay rate and $\gamma_\perp$ ($\gamma_\perp$ of Equation (1)) was the polarization decay rate. When the ratio of input turning points (lower and upper switching powers) in the resonant absorptive case was around two, both the atomic and cavity detunings were changed from zero with $\Delta, \Theta < 0$. With the development of instability the input–output characteristics changed drastically. This was because the mean value of the pulsating state was displaced above the corresponding time-independent steady state. There was a beautiful match between the theoretical prediction and the experimental results. By changing the cavity and laser detunings at a given value of $C$, the instability boundary in the $(\Theta, \Delta)$ plane was explored. This led to absolute comparisons with theoretical predictions with no adjustable parameters. The instability boundaries were explored in the detuning space $(\Theta, \Delta)$ of the system. The comparison of theory with experiments has been presented in Figure 13 for a ring cavity whose atomic density is
characterized by $C$ [7,8]. The measurements were done by keeping the atomic detuning fixed but changing the cavity detuning only. The transit broadening was considered as a homogeneous process responsible for polarization decay rates and hence the saturation intensity. Although this assumption was not well justified, it seems to work well in matching the theoretical and experimental results in the resonant absorptive bistability case. For small cavity and atomic detunings this approximation gave good quantitative agreements with experiments [7,8]. The inhomogeneous broadening was due to imperfect collimation of the atomic beams, which could be neglected. In Figure 13 the absolute quantitative comparisons of the ring-cavity experimental results with theory are presented [7,8]. However, the theoretical results based on the plane-wave theory disagree with the experimental data because the instability domain is wider. There was an uncertainty in getting uniform field also,
however, for the parameters selected for this study it was in good agreement with the other work on instabilities as mentioned in \[8\] without assuming the uniform field limit.

In these experiments the ranges of parameters used were \(0 < |\Delta| < 6, 0 < |\Theta| < 40, C < 350\) \[8\]. The decay rate of the resonator was two times smaller than the polarization decay rate and no higher-order bifurcation of the self-pulsing states was identified in the upper branch of the bistable curve. Sometimes, circulations of fields in both directions in the ring cavity were observed due to some kind of gain-assisted generations of counter-propagating waves. Such counter-propagating waves had higher-order transverse structures (beyond the TEM\(_{00}\) mode). It was possible to correlate the multiple branches in the hysteresis loop with the appearance of counter-propagating beams. A typical hysteresis loop is presented in Figure 14 for the unstable region and such an effect is due to the hard-mode excitation, i.e. the system becomes unstable due to finite size perturbation. The explanation of such

7. Instabilities in three-level AOB: comparison of experiment and theory

The experimental setup to observe and control instability in AOB of a \(\Lambda\)-type three-level atomic system is the same as described above for observing and controlling AOB. The atomic transitions in \(^{87}\)Rb are shown in Figure 7(a). The first step required for this experiment was to lock the frequencies of both the coupling and the probe lasers to respective atomic transitions with the help of Fabry–Pérot cavities. First the coupling beam was locked to the resonant frequency of the coupling transition (5\(S_{1/2}, F' = 2 \rightarrow 5P_{1/2}, F_0 = 2\) followed by the probe beam which was locked to the resonant frequency of the probe transition (5\(S_{1/2}, F = 1 \rightarrow 5P_{1/2}, F' = 2\)). Once these lasers got frequency locked then applied voltages on the PZTs of the two Fabry–Pérot cavities could be varied to bring the laser output frequencies to any desired values. The ring cavity was scanned across its resonance by applying a ramp voltage to the PZT mounted on mirror M\(_3\). Transmitted light from the cavity was measured by an avalanche photodiode (APD). Without the coupling (control) beam, the transmission profile of the ring cavity was basically symmetric in shape. In the presence of the coupling beam the cavity transmission profile became asymmetric or oscillatory due to dynamical instability. These dynamic oscillations in the cavity transmission were due to the enhanced nonlinear refractive index which depends sensitively on the experimentally controllable parameters, i.e. intensities and frequency detunings of the coupling/probe beam. The measurements of the dynamic instability as functions of these experimental parameters are discussed in the following.

Qualitative comparisons between the experimental observations (left column) and theoretical calculations (right column) of the optical dynamic oscillation in the cavity transmission were done as presented in
Figure 15 for three different probe frequency detunings. For this purpose the other parameters set in the experimental measurements and theoretical calculations were: coupling power $P_c = 11 \text{ mW}$, probe input power $P_{p}^{in} = 2.8 \text{ mW}$, and $\Delta_c = 0$, with probe frequency detunings to be in Figure 15(a) and (d) $\Delta_p = 25 \text{ MHz}$; in (b) and (e) $\Delta_p = 35 \text{ MHz}$; and in (c) and (f) $\Delta_p = 45 \text{ MHz}$. The theory matched quite well with the

![Figure 15](attachment:figure15.png)

Figure 15. Experimentally observed and theoretically calculated cavity transmission profiles from an optical ring cavity containing three-level Λ-type atoms for three different values of coupling frequency detuning. The left column is the experimental observations and the right column is the corresponding theoretical calculations. The parameters used in the experiment and theoretical calculation are: $P_c = 11 \text{ mW}$, $P_{p}^{in} = 2.8 \text{ mW}$, $\Delta_c = 0$, and (a) and (d) $\Delta_p = 25 \text{ MHz}$; (b) and (e) $\Delta_p = 35 \text{ MHz}$; (c) and (f) $\Delta_p = 45 \text{ MHz}$ (from [26]). Reproduced with permission from Yang et al. *Phys. Rev. A* **2004**, *70*, 033807. Copyright (2004) by the American Physical Society.
experimental results. The time period of the oscillation decreased with increasing probe frequency detuning. The oscillation finally stopped when the probe frequency detuning reached \( \Delta_p = 60 \text{ MHz} \) (not displayed here).

The numerical simulation had been carried out using the atomic equations as described by Equation (14) in Section 4 along with cavity field equations as described in the following [26]. By defining

\[
E_p = \left( \frac{\hbar \omega_p}{2 \pi n_0 c S_p} \right)^{1/2} \alpha_p, \tag{18}
\]

where \( \varepsilon_0 \) is the free-space permittivity and \( c \) is the speed of light in vacuum; \( S_p \) is the area of the transverse section of the probe light beam. In terms of its real and imaginary parts \( \alpha_1 \) and \( \alpha_2 \), the equations for the cavity field \( \alpha_p \) can be reformulated as:

\[
\tau \dot{\alpha}_1 = \tau_2 \alpha_p^{\text{in}} - \gamma_c \alpha_1 - \frac{2\pi}{\lambda} v_{\text{cav}} \alpha_2 - \left[ 1 + \frac{2}{3} (1 + w_p + w_c) \right] \alpha_2 - f(\alpha_1^2 + \alpha_2^2) \alpha_2, \tag{19}
\]

\[
\tau \dot{\alpha}_2 = -\gamma_c \alpha_2 + \frac{2\pi}{\lambda} v_{\text{cav}} \alpha_1 + \left[ 1 + \frac{2}{3} (1 + w_p + w_c) \right] \alpha_1 + f(\alpha_1^2 + \alpha_2^2) \alpha_1, \tag{20}
\]

where

\[
\rho_{22} = \frac{1 + w_p + w_c}{3}, \tag{21}
\]

\[
\rho_{33} = \rho_{11} = w_p - w_c, \tag{22}
\]

where \( \gamma_c \) is the cavity decay rate and \( v_{\text{cav}} \) is the cavity scanning speed. The parameters \( e \) and \( f \) are related to the linear and the nonlinear phase shifts of the cavity. \( \tau_2 \) is the transmission coefficient of mirror M2.

The variation of the measured period of oscillation as a function of probe beam frequency detuning \( \Delta_p \), is plotted in Figure 16(a) where for the sake of comparison the calculated theoretical curve is also given. There is reasonable agreement between the experimentally measured data and the theoretically calculated results. The parameters used for this plot are \( \Delta_c = 0 \), \( \Delta_p = 35 \text{ MHz} \), \( P_p^{\text{in}} = 3.0 \text{ mW} \), and \( T = 70^\circ \text{C} \). Figure 17(b) depicts the experimental and theoretical curves for the oscillation period versus cavity input power with \( \Delta_c = 0 \), \( \Delta_p = 35 \text{ MHz} \), \( P_c = 11.0 \text{ mW} \), and \( T = 70^\circ \text{C} \). The experimental parameters were all measured directly and no fitting parameters are used for making such comparisons with the theoretical calculations.

The physical origin of such instability is due to two competing dynamic processes in the system, i.e. optical pumping from state \( 5S_{1/2} F = 2 \) to state \( 5P_{1/2} F' = 2 \) by the coupling field and the nonlinear saturation effect in the transition from state \( 5S_{1/2} F = 1 \) to state \( 5P_{1/2} F' = 2 \) due to the cavity probe field [26]. This kind of

![Figure 16](image-url)

**Figure 16.** Comparison between experimentally measured and theoretically calculated oscillation period versus (a) probe frequency detuning \( \Delta_p \) and (b) coupling frequency detuning \( \Delta_c \). The parameters used for the plots are: (a) \( \Delta_c = 0 \), \( P_c = 11.0 \text{ mW} \), and \( P_p^{\text{in}} = 3.0 \text{ mW} \) and (b) \( \Delta_p = 0 \), \( P_c = 11.0 \text{ mW} \) and \( P_p^{\text{in}} = 3.0 \text{ mW} \) for both the experiment and theoretical calculation (from [26]). Reproduced with permission from Yang et al. *Phys. Rev. A* **2004**, *70*, 033807. Copyright (2004) by the American Physical Society.
instability in AOB is unique for the three-level atomic system and can not be observed in a two-level system. The period of oscillation is determined by the relative strengths of these two dynamic processes that are governed by the intensities and frequency detunings of the coupling and probe beams. The main difference between the roles of the power level of the lasers and the frequency detunings is that both frequency detunings play the same role while the probe and coupling powers play the opposite role. More explicitly the larger probe laser power enhances the nonlinear saturation but larger coupling power produces the situation of optical pumping. On the other hand, both coupling and probe frequency detunings can actually increase the optical saturation effect.

If the coupling frequency detuning is large, the process of optical pumping gets reduced, and consequently the optical saturation process becomes more dominant. Enhancement in optical saturation with increasing probe frequency detuning in a certain range can be attributed to EIT absorption behaviors, i.e. reduction in absorption at exact resonance because of EIT, but large absorption with certain nonzero probe detunings [26]. In Figures 16 and 17 such behaviors can be clearly observed. The enhanced Kerr nonlinearity of the multi-level atomic medium due to induced atomic coherence and the feedback mechanism through the optical ring cavity are responsible for the optical dynamic instability in this system with three-level atoms in the optical cavity.

The discrepancy in the two curves of oscillation period versus coupling and probe powers (Figure 16) is due to the neglected spatial distribution (Gaussian beam profile, focusing, and the propagation effect in the cell) of the cavity field in the theoretical calculation, which has a bigger impact for the intracavity field intensity and its interaction with the atoms. Also, there is uncertainty in experimentally determining the intracavity intensity from the cavity input power due to the inaccuracy in estimating the cavity field beam size. In the optical pumping, contributions from the Zeeman sub-levels have not been taken into account in the theoretical calculation. The second-order residual Doppler effect was also not taken into account in these calculations. One of the limitations in the experiments was the relatively low output power of the coupling laser, i.e. maximum 12 mW as cavity input. However, the theory can provide some clues on what could be expected at higher coupling beam powers. The period of oscillation increases initially, up to a maximum value, as the coupling power increases, and then decreases toward zero with further increase of the coupling power.

From self-pulsing behavior in the cavity output the system can reach a chaotic regime via the period-doubling route. To achieve such a situation experimentally, the parameters are so adjusted that the cavity output profile shows self-pulsing oscillation with a scanning cavity. Then, the cavity scan was turned off and the optical ring cavity was frequency locked. At this stage, a gradual variation of $A_c$ towards zero brought drastic changes in cavity output profile.

7.1. Observation of chaotic dynamics in the three-level system

The cavity transmission profiles observed experimentally for four different coupling frequency detunings are shown in Figure 18. The probe and coupling beam...
The parameters used in the experiment were $P_c = 19.7 \text{ mW}$ (measured just before it reaches the PBS), $P_{in} = 16.6 \text{ mW}$ (cavity input), $\Delta_P = 0$, $\Delta_D$ (cavity detuning) = 0, and $T = 85^\circ \text{C}$. These parameters were fixed during measurements. For a large value of $\Delta_c$ (>100 MHz), the system remained at the steady state in the upper branch of the bistable curve, so the cavity output is stable. Decreasing $\Delta_c$ to around 60 MHz, the cavity output showed an oscillation, i.e. the system went into the instability region. A further decrease of $\Delta_c$ causes the oscillation pattern to be the same but the oscillation period and oscillation amplitude changed continuously [27] when $\Delta_c$ reached 20 MHz, yet another set of peaks appeared in between the original peaks (Figure 18(b)) and the oscillation period doubled. When $\Delta_c$ was reduced down to 10 MHz, the oscillation period quadrupled, as shown in Figure 18(c), and eventually, the observed oscillations behaved chaotically as $\Delta_c$ was reduced further, to below 7 MHz (Figure 18(d)), where the linear dispersion and Kerr nonlinearity are very large and optimized [27]. These experimentally measured results agree very well with the theoretical calculations. The theoretical results were obtained by numerically solving the same equations as used for the instabilities. The power spectra corresponding to these time series are shown in Figure 19. Clearly, the spectrum changes from sharp spikes (typical for a periodic system) to a broadband with random components (typical for a chaotic system) as $\Delta_c$ decreases. Thus, the route from ‘period-doubling to chaos’ has been exhibited in the experiment confirmed by the spectral analysis.

To establish the chaotic nature of the observed data, the largest Lyapunov exponent (LE) has been calculated as a function of the coupling frequency detuning $\Delta_c$, as shown in Figure 20. The solid line is calculated theoretically. The system was stable for large $\Delta_c$ values where the calculated LE showed negative value. When $\Delta_c$ approached the value of 60 MHz, the calculated LE showed a value very close to zero, indicating that the system went into the instability region and the cavity output was showing a regular oscillation (under regular oscillation, LE should be zero; however, since there are errors in the computation, a small residual positive number remains). As $\Delta_c$ reduced to below 20 MHz, the LE increased quickly and reached a high value of about 0.5 when below $\Delta_c = 7$ MHz. A large positive LE was a clear indication that the system was behaving chaotically. Numerically computed LEs from the time series recorded in the experiment are shown as solid triangles in Figure 20. They show good agreement with the theoretically calculated results without any adjustable parameters.

This experiment was further extended to vary other parameters such as probe detuning, cavity detuning, probe power and coupling power. No period-doubling phenomenon could be observed when changing the probe and coupling laser powers at large coupling detunings, possibly due to the limited output powers of
the two diode lasers. Also, in order to observe the period-doubling route to chaos by changing frequency detunings, the output powers of the two lasers need to be quite high and the Rb cell needs to operate at a high temperature (>80°C).

8. Stochastic resonance in atomic optical bistability

The phenomenon of stochastic resonance (SR) is seemingly counterintuitive in nature. It occurs when adding a certain amount of noise to the input of a system and increases the output signal-to-noise ratio (SNR) for a signal passing through the nonlinear medium. Alternatively, it means that noise can induce a resonance-like effect in multi-state nonlinear systems. The SR has been identified in climatic cycles, electronic and magnetic systems, optical systems, and biological/neuronal systems [34]. In the recent past the SR phenomenon was observed in a simple electronic circuit (Schmidt trigger), a semiconductor diode laser, a bidirectional ring dye laser, thermally induced optical bistability in semiconductors [35], and neurophysiological systems [36].

Very recently, SR was experimentally demonstrated [28] in a three-level AOB system with \( \Lambda \)-type configuration (Figure 7(a)) confined in an optical ring cavity. The AOB system with enhanced Kerr nonlinearity due to atomic coherence was driven by a periodic signal and a Gaussian white noise source with variable strength, and displayed an improved output SNR, a characteristic signature of SR. There was a good qualitative agreement of the measured results with the predictions of the theoretical generic model for the SR phenomenon.

The generic model for SR is described by [34]

\[
\dot{x}(t) = -V'(x) + A_c \cos(\Omega_c t + \phi_c) + \xi(t),
\]  

where \( V(x) \) is the reflection symmetric potential

\[
V(x) = -\frac{a}{2} x^2 + \frac{b}{4} x^4.
\]
Here, $A_s$, $\Omega_s$, and $\phi_s$ are the signal amplitude, signal frequency, and a simple phase factor, respectively. The parameter $\xi(t)$ denotes Gaussian white noise with a zero mean and correlation function $\langle \xi(t)\xi(0) \rangle = 2D_n\delta(t)$, where $D_n$ characterizes the noise amplitude. The double-well structure of a standard two-state system is described by $V(x)$, which is characterized by parameters $a$ and $b$. Equation (23) represents an overdamped, driven anharmonic oscillator with third-order nonlinearity if there is no noise term. Adding a periodic signal to the system together with a source of noise, the SNR of the output shows a maximum at a certain noise strength $D_n$, producing a resonance-like feature in the SNR as a function of $D_n$, and thus exhibiting the SR phenomenon [34]. The output SNR can exceed the input SNR under specific conditions, which can be very useful in certain applications.

The experimental setup to demonstrate the SR phenomenon is quite similar to the AOB experiments, as shown in Figure 8, except that an additional Gaussian white noise source has been added to EOM. The energy levels of the $D_1$ line of the $^{87}$Rb atom are employed to form the required three-level system in the $\Lambda$-type configuration (Figure 7(a)). At the beginning of the experiment a proper optical bistability (here it is a refractive AOB hysteresis curve) was established. For the refractive (two-level) AOB hysteresis curve, the parameters $a$, and $b$ in Equation (24) are representing the difference between the cavity resonance frequency and the optical frequency of the probe laser field, and the third-order nonlinear susceptibility of the medium, respectively [37]. For the three-level AOB system, both $a$ and $b$ are related to the probe laser frequency detuning $\Delta_P$, the coupling laser frequency detuning $\Delta_C$ and the coupling power $P_C$, etc. [28]. The power going inside the cavity was then set in such a way that it falls almost in the middle of the observed AOB hysteresis loop. At this stage a small sinusoidal voltage signal with modulation frequency 150 Hz was applied to the EOM along with a noise voltage (the Gaussian broadband noise) generated by an arbitrary waveform generator. APD detectors were used to detect input, as well as output, optical signals of the cavity. The input and output waveforms with a time duration of 1 s were stored digitally and then a fast-Fourier transform was performed on the digitized data to obtain a power spectrum. By repeating the above process several times an averaged power spectrum was obtained. The SNR in evaluating the SR is defined by the ratio of the magnitudes of the power spectrum at the signal frequency, and the noise level without the input signal. In these experiments the bandwidth was limited by the frequency response of the EOM.

Figure 21. (a) Experimentally measured output SNR as a function of input noise amplitude. The frequency of the sinusoidal signal is 150 Hz. Here, curves A, B and C are for three different values of signal amplitudes corresponding to 10, 30 and 45 $\mu$W, laser power variation about the point of operation of cavity input laser, respectively. The $x$-axis represents twice the standard deviation of noise amplitude divided by the width of the bistable region. (b) The theoretical predictions from the generic model of SR (from [28]). Reproduced with permission from Joshi et al. Phys. Rev. A 2006, 74, 013817. Copyright (2006) by the American Physical Society.

Figure 21(a) shows the measured output SNR as a function of the noise strength for this system. The SNR in a linear scale is on the $y$-axis while normalized noise strength is on the $x$-axis, which is twice the standard deviation of the noise strength divided by the width of the bistable region. The marking of the $x$-axis is done in this way because the maximal SNR is normally found to occur at noise variance approximately half the barrier height of the double-well potential (or at half the width of the AOB hysteresis cycle in this case). In the experiment the AOB hysteresis cycle could be controlled very effectively and thus allowed the
observation of the SR phenomenon for several different OB curves. In Figure 21 the experimental parameters are \( T = 68^\circ \text{C} \), \( P_c = 12 \text{ mW} \), \( \Delta_c = 150 \text{ MHz} \), \( \Delta_p = 100 \text{ MHz} \), and cavity detuning \( \Delta_p = 50 \text{ MHz} \). Curves A, B, and C are for three different values of the signal amplitude as mentioned in the caption. There is a common trend in all three curves, i.e. the SNR first increases with increasing noise strength, reaches a maximum, and then reduces down as the noise strength further increases. For several different experimental conditions similar AOB results were obtained. The basic underlying mechanism for the SR phenomenon can be understood in a very simplified manner using a double-well potential system.

A Fokker–Planck equation corresponding to the overdamped motion of a particle in a double-well potential is quite suitable to describe the AOB phenomenon. The upper and lower branches of the AOB hysteresis curve can be attributed to the two wells of the double-well potential. A periodic signal to the cavity input field causes the depths of these two wells to alter periodically and the external noise works in cooperation with the signal to transfer a particle from one well to another well. For a small noise strength the hopping rate between wells is low. With increasing noise strength the hopping events increase and the SNR becomes maximum when noise strength reaches an optimal value for a given signal amplitude. However, for too large noise strengths the hopping rate becomes so large that the signal is lost in the background noise. The model of Equation (24), in which

\[
\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial x} p \right) - \gamma x p + \gamma \delta \delta_0 \delta_1 \text{SNR},
\]

reaches a maximum, and then reduces down as the noise strength further increases. For several different experimental conditions similar AOB results were obtained. The basic underlying mechanism for the SR phenomenon can be understood in a very simplified manner using a double-well potential system.

The hopping rate \( T^+ (T^-) \) from \( x_+ \) to \( x_- \) (\( x_- \) to \( x_+ \)) can be obtained from the familiar Kramers equation: \( \sim \exp[-2U_{\text{ad}}/D] \), in which \( U_{\text{ad}} = U^0(1 \pm \delta_0 \cos(\Omega t)) \) defines the time-dependent potential barrier, \( U^0 \) is the barrier height, \( \delta_0 \) is a quantity related to the signal amplitude and \( D \) is the noise variance. Using Equation (25) one can obtain the autocorrelation function and power spectrum, from which the SNR is given by [28,34]

\[
\text{SNR} = \left( K^2/D^2 \right) \exp[-2U^0/D].
\]  

In the above equation, the parameter \( K \) is related to the signal intensity potential barrier and its width, and can be estimated using the expressions for barrier height \( a^2/4b \) and barrier width \( 2(a/b)^{1/2} \) [28,34]. With the help of the experimentally observed AOB, the parameters \( a \) and \( b \) (defined in Equation (24)) can be approximately obtained. For different signal amplitudes, the calculated SNRs are depicted in Figure 21(b), which have qualitatively good agreements (curves A, B, and C are \( K = 2.737, 5.477, \) and 7.303, respectively) with the experimental results for similar parameters as depicted in Figure 21(a).

For the AOB system under consideration the adiabatic approximation is valid provided \( \Omega \ll a \) (\( 'a' \) defined in Equation (24)). The modified Kramers’ rate could be appropriately realized for the signal frequency slower than the rate for the probability to equilibrate within a well. This adiabaticity condition is reasonably achieved for low values of \( \Omega \), provided the dispersive AOB condition is satisfied. The SR phenomenon was observed in this AOB system for a wide range of experimental parameters, which show the generality and ruggedness of this interesting phenomenon for a variety of physical circumstances. The variation of \( G \) (gain in SNR) as a function of noise strength (Figure 22) is similar to what was observed for the output SNR in Figure 21, i.e. there is a peak in the response of the system at a certain noise strength. Here

![Figure 22: Experimentally measured SNR gain G as a function of input noise amplitude. The curves A (dotted), B (dashed), and C (solid) are for the three signal amplitudes as in Figure 21 (from [28]). Reproduced with permission from Joshi et al. Phys. Rev. A 2006, 74, 013817. Copyright (2006) by the American Physical Society.](image-url)
$G>1$ implies occurrence of a strong cooperative phenomenon such that the incoherent noise power can feed into the coherent output signal. This cooperative phenomenon could be further assisted due to the coherence generated by the EIT medium inside the optical cavity.

The passive three-level AOB system is an ideal two-state system to study this SR phenomenon since the bistable curve, therefore the shape of the double-well potential, can be easily changed by the coupling beam parameters, which can be used to explore various properties of this interesting SR phenomenon. Since the linear and Kerr nonlinear properties can all be modified in such a three-level AOB system, one can carefully investigate the effects of noise transfer in this nonlinear bistable system, as well as tunneling between the bistable states.

9. Optical switching in the three-level AOB system

9.1. Deterministic switching: controlled switching between bistable states

The output intensity of an optical cavity stays on the lower branch of a bistable curve (Figure 23(a)), when the input intensity is increased from zero or a lower value outside the bistable region. When the input intensity reaches the threshold value $Y_2$, it jumps to the upper branch of the bistable hysteresis curve. On the other hand, with a decrease of the input intensity when the system is in the upper branch, the output intensity stays on the upper branch up to a threshold value $Y_1$, when it then jumps down. Thus, the AOB can be utilized to implement an optical switch in the following manner. Initially, the input intensity is set in the middle-region of the bistable curve such that the cavity output intensity is at the lower branch (point A) of the curve. By applying an intensity pulse with its peak value higher than the threshold value $Y_2$ (Figure 23(b)), this brings the output intensity to the upper branch (point C). At the end of the pulse, the cavity output intensity stays on the upper branch at position B. By applying a negative pulse, the output intensity comes down to the lower branch A (through point D) again. In this way an all optical binary switch is realized.

Experimental demonstration of such controlled optical switching was reported in [38] and results are depicted in Figure 24. For a typical bistable curve the initial input power in the experiment was set at 1.73 mW. The positive and negative pulses were formed by applying pulse voltages to the EOM. The amplitudes of the intensity pulses were chosen to be just a little bigger than the half width of the bistable hysteresis cycle. The initial output intensity was set at the lower branch (state A). As can be seen from Figure 24, the output intensity was brought to the upper branch (state B in Figure 24(b)) by the positive
pulse (Figure 24(a)) and stays there; then the negative pulse brought it down to the lower-branch value (state A) again. The states A and B were very stable and the extinction ratio of this switching action reaches about 20:1.

9.2. Noise-induced switching: fluctuating atomic coherence in a three-level AOB system

Noise-induced switching caused by fluctuating atomic coherence was experimentally demonstrated in the AOB system consisting of three-level atoms in the Λ-type configuration in an optical ring cavity [29] as described in the above experiments of controlling AOB and SR. The physical origin behind such a noise-induced switch could be attributed to a fluctuating nonlinear phase. This nonlinear phase originated due to enhanced Kerr nonlinearity induced by atomic coherence when the EIT system fluctuates due to the noise in the laser parameters. More precisely any fluctuations in the interacting field’s intensity/frequency, or fluctuations in the optical cavity length, would cause changes in the induced atomic coherence in such an EIT medium, and consequently in the refractive index of the medium, which leads to a change in the round-trip cavity phase and therefore induces the switching. This phenomenon is physically different from the periodic nature of dynamic instability [26] explained using deterministic equations, as discussed in Section 7.

The experimental arrangement is very similar to the one shown in Figure 8 for controllability of AOB, except a white-noise source was added to the EOM. Several kinds of AOB hysteresis loops could be observed by changing intensities and frequency detunings of the coupling/probe beams, Rb cell temperature, and the cavity frequency detuning $\Delta_\nu$. The EOM scans the intensity of the cavity input field back and forth (like an intensity modulator). To see the typical bistable curve, the EOM was switched on with a triangular ramp voltage. After establishing the proper refractive optical bistability, the EOM scanning was switched off, and set in such a way that the biased cavity input intensity lies almost in the middle of the observed AOB hysteresis loop. The typical linewidth of the diode laser used in the experiment was in the range of several MHz. Though the lasers used here were extended cavity type, they did possess residual phase (frequency) and intensity noises responsible for the fluctuations in the nonlinear refractive index causing the switching phenomenon. Measurements with lasers far-off resonant from the atomic absorption lines suggest that the magnitude of the intensity fluctuation was about 10–100 times smaller than the frequency fluctuation responsible for such switching. Any frequency jitters in the ring cavity were essentially associated with the linewidth of the third laser for locking the cavity, which was of the same order of magnitude as that of the probe laser (i.e. several MHz). Hence, such cavity frequency jitter was taken into account along with the frequency fluctuation of the probe laser circulating in the cavity. The cavity output was monitored by an APD detector for observing switching phenomenon due to the fluctuating refractive index of the medium. In order to obtain an averaged response of the optical cavity output, the above process has been repeated many (about 50) times in a fixed time interval. A typical switching time-trace of the cavity output field observed experimentally is shown in Figure 25(a) for the experimental conditions of cavity temperature: $T = 68^\circ C$, coupling laser

![Figure 25](image_url)
power: $P_C = 12\, \text{mW}$, probe laser frequency detuning: $\Delta_p = 50\, \text{MHz}$, and cavity detuning $\Delta_C = 50\, \text{MHz}$. In Figure 25(a), (b), and (c), the curves are for three different AOB hysteresis curves with decreased widths by selecting coupling laser frequency detunings: $\Delta_C = 55$, 35, and 25 MHz, respectively, keeping all other parameters unchanged. From these curves one can see that if the operating point of the input intensity on the hysteresis curve is moved towards the upper threshold of the AOB curves (this was done by reducing the width of AOB) the occurrence of switching events to the upper metastable state in the same time interval increases, provided it is still far from the upper threshold so that a critical slowing down does not dominate the dynamical process. At each impulse from the noise peak there is a sharp spike up to the upper state, which quickly decays down to the lower state owing to the asymmetric potential $U(\phi)$ [29]. The intracavity EIT medium causes a cavity linewidth narrowing, thus a small change in refractive index will easily cause the cavity mode to shift by a linewidth leading to switching from no cavity transmission (lower branch) to cavity transmission (upper branch). However, the frequency stabilization factor (proportional to $(\omega_p l)/(\partial (\text{Re}(n))/\partial \omega_p)$, where $n$ is total refractive index) pulls the system back to its original state, leading to the observed behaviors as shown in Figure 25.

In Figure 26, measured occurrences of the cavity output intensity switching to the upper branch of the AOB hysteresis curve as a function of AOB widths are displayed. The experimental parameters are shown in the caption. The operating point of the input power level is kept at the middle of each AOB curve. Clearly, with the increase of the AOB width, the occurrence of the switching events diminishes. This is because the noise spikes are less likely to be strong enough compared to the increase in the half width of the AOB curve. Hence, the average time required for switching increases which is in good agreement with the theoretical result [29].

In order to quantify such switching phenomenon further three different AOB hysteresis curves of different widths are selected in Figure 27. The experimental parameters are shown in the caption. The system can sit on different operating points on each of the hysteresis curves (shown in the insets of Figure 27) and observe the occurrences of switching phenomenon, which are plotted as a function of relative position of operating point within the AOB curve. When the width of the hysteresis curve is small (Figure 27(a)) the occurrence of the switching events at low operating points (far from threshold) is large in comparison to the hysteresis curves of Figure 27(b) and (c) with wider widths. The noise spikes were strong enough to bring the required phase change for switching the cavity field to the upper state for the AOB curve with narrower width. As the operating point gets near to the upper threshold point, the switching occurrences increase in all the curves and show similar behavior.

10. Conclusion and outlook
The two-level AOB has been well studied over the years and many experimental demonstrations have been reported from time to time. Many ingenious ideas were applied not only to observe AOB in a two-level system, but also the instabilities and predictions to observe chaos in the two-level AOB were also performed. However, there was a lack of good experimental control in all those experiments. The situation improved remarkably with the introduction of a three-level EIT medium in the AOB system. The physical properties of these systems such as absorption, dispersion and nonlinearities can easily be controlled due to the induced coherence in these media, which can then be used to control all the phenomena associated with three-level AOB. The three-level $\Lambda$-type EIT media were used for the first time to show such a control in three-level AOB. This system allowed one to generate a variety of hysteresis loops in such a system by easily available experimental control. This medium was further used to experimentally observe the optical dynamic instability and chaos inside an optical ring cavity. In such work the experimental demonstration of optical instability and chaotic behavior over large
ranges of the parameter space are found to be consistent with the theoretically calculated results of this model. The optical instability and chaos could be independently controlled by adjusting experimental parameters, such as the powers and frequency detunings of the coupling and cavity fields. Such control offers many advantages over the previous experimental systems where the experimental tunability is limited by the constraints of using only one laser beam to facilitate the two competing nonlinear processes. Such three-level EIT systems were also facilitated to observe the phenomenon of SR when the system was simultaneously driven by a periodic signal and randomize noise that led to an enhanced output SNR confirming the nonlinear cooperative effects between signal and noise in such a system. There was a large gain in SNR, which was due to the enhanced Kerr nonlinearity induced by the atomic coherence in such a system. The phenomenon of noise-induced switching in the three-level AOB system was also observable due to the controllability in such a system. This phenomenon was a result of a three-level EIT medium driven by a cavity field with intrinsic fluctuations in its frequency and intensity, causing atomic coherence and thus enhanced nonlinearity to fluctuate, leading to the random switching between bistable states of the AOB system. Such a system could be potentially useful in generating pseudo random numbers required in quantum communication and cryptography. The three-level EIT medium inside an optical cavity thus provides a means to carry out systematic studies of dynamic behaviors in such an interesting system, which cannot only help our fundamental understanding of atom–cavity field interactions, but also provide useful information for potential applications of multi-level EIT systems in all-optical switches, all-optical buffering, and optical solitons, which are key elements for all-optical networking.

This review article is a tribute to L.M. Narducci for his sustained and invaluable contributions in quantum optics along with many other colleagues, who pioneered several investigations in the area of two-level AOB [4–7]. Such interesting investigations include, but are not limited to, the study of transmission and fluorescent spectra, analogy with first-order phase transition, self-pulsing, instabilities, etc. in AOB of two-level systems [4–7]. Later on he worked with multilevel systems and studied many interesting phenomena in emission and absorption properties of three-level systems, effects of quantum interference, and inversion-less amplification in four-level systems with random pumping [39].

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References