Topological Hall Effect

The geometric or topological Hall effect (THE) is the name for the Hall effect induced by chiral spin texture [1,2].

Hall conductivity undergoes integral transitions at critical magnetic fields—"an unambiguous signature of the THE" [1].

THE from a gauge transformation

Consider 2D electron gas (2DEG) subject to smoothly varying magnetization $M(r) = Mn(r)$

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 - g \sigma \cdot M$$

$g$ - coupling constant

$\sigma$ - vector of Pauli matrices

$M$ - constant amplitude of magnetization

$n(r)$ - unit vector in direction of magnetization (assumed to vary slowly in $r$)

Gauge Transformation

- Apply gauge transformation $T(\mathbf{r})$ such that the direction of magnetization is along the axis of spin quantization. I.e.

$$T^\dagger(\mathbf{r})[\mathbf{\sigma} \cdot \mathbf{n}(\mathbf{r})]T(\mathbf{r}) = \sigma_z$$

- Note: $T(\mathbf{r})$ depends explicitly on the components of $\mathbf{n}(\mathbf{r})$.

- Gauge transformation must be unitary, to preserve normalization.

$$T^\dagger(\mathbf{r})T(\mathbf{r}) = I$$

- These two conditions completely determine the form of $T(\mathbf{r})$

Transform Kinetic Energy

- Applying the same transformation to kinetic energy:
  \[ T^\dagger \nabla^2 T = \nabla^2 + 2T^\dagger (\nabla T) \cdot \nabla + T^\dagger (\nabla^2 T) \]
  \[ = [\nabla + T^\dagger \nabla T]^2 \]

- Recall that a charged particle subject to a vector potential \( A(r) \) has a modified momentum operator [4]
  \[ p \rightarrow p - eA \]
  \[ \nabla \rightarrow \nabla - i e \hbar A \]

- "extra" term in kinetic energy operator can be written as a vector potential
  \[ - \frac{\hbar^2}{2m} [\nabla + T^\dagger \nabla T]^2 = - \frac{\hbar^2}{2m} [\nabla - \frac{ie}{\hbar} A(r)]^2 \]
  \[ A := 2i \phi_0 T^\dagger \nabla T \quad \phi_0 := \frac{\hbar}{2|e|} \text{(flux quantum)} \]

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Transformed Hamiltonian

- After applying transformation:

\[
\hat{H}^\prime = -\frac{\hbar^2}{2m} \left[ \nabla - \frac{ie}{\hbar c} A(r) \right]^2 - gM\sigma_z
\]

- Note: As \( T(r) \) depends explicitly on \( n(r) \), so does \( A(r) \)

- Want to consider case when spin-flip transitions are negligible; spin polarization adiabatically follows \( n(r) \)

  - Corresponds to condition: \( \frac{\varepsilon_F}{\varepsilon_0 \frac{1}{k_F\xi}} \ll 1 \)

  - \( \varepsilon_F \) - Fermi energy; \( \varepsilon_0 = 2gM \) - spin splitting; \( k_F \) - magnitude of Fermi wave vector; \( \xi \) - characteristic length of variation of \( n(r) \)

Topological Field

- Assume the 2DEG is half metallic, with Fermi energy located in spin-up subband.
  - Allows us to project this Hamiltonian onto the spin-up subspace
  - Then spin can be neglected, reducing Hamiltonian to that of a spinless charged particle

\[
\hat{H} = -\frac{\hbar^2}{2m}[\mathbf{v} - ie\frac{\mathbf{a}(r)}{\hbar c}]^2 + V(r)
\]

\[
\mathbf{a} = \pi \phi_0 \frac{n_x \nabla n_y - n_y \nabla n_x}{1 + n_z}
\]

- Topological field is defined as

\[
B_t = (\mathbf{\nabla} \times \mathbf{a}) \cdot \hat{z} = \frac{\phi_0}{4\pi} \epsilon_{\mu \nu \lambda} n_n (\partial_x n_n) (\partial_y n_n)
\]

Image credit: https://www.tcd.ie/physics/research/groups/magnetism/research/spinelec/
Topological Field and Berry Phase

- $B_t$ acts on the electrons in the spin-up subband like an ordinary magnetic field, resulting in a Lorentz-type force

$$B_t = \frac{\phi_0}{4\pi} \epsilon_{\mu \nu \lambda} n_\mu (\partial_x n_\nu)(\partial_y n_\lambda)$$

- Integral of $B_t$ over an area $S$ enclosed by a contour $C$ is proportional to Berry phase

- Relevant definition of Berry phase:
  - Spherical angle spanned by $S$ inside contour $C$ in $n$-space

- Topological field is defined completely by the texture of magnetization


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Connecting to Conductivity

- Consider mesoscopic system of 2DEG subject to stray field of periodic 2D lattice of magnetic nanocylinders.
- Net topological flux per unit cell must be integer multiple of $\phi_0$. Thus, average topological field is:

$$\bar{B}_t = \frac{I\phi_0}{s_0}$$

$I$ = integer $\quad s_0$ = area of unit cell

- Using semiclassical Drude theory of magnetotransport:

$$\sigma_{xx}^s = \frac{n_s e^2 \tau_s}{m} \quad \sigma_{xy}^s = s\sigma_{xx}^s \frac{e\bar{B}_t \tau_s}{m} \quad s = \begin{cases} +1 & \text{for } \uparrow \\ -1 & \text{for } \downarrow \end{cases}$$

Simulating Topological Hall

Setup: triangular lattice of uniformly magnetized nanocylinders, with magnetization $M_s$ along the z-axis, placed over top a 2D diluted magnetic semiconductor film (the 2DEG).

An additional magnetic field $B_{ext}$ is applied uniformly to the film, parallel to the axis of the cylinders.
Topology Depends on External Field

Color corresponds to $B_0$ in the film, scaled by $M_s$, with $B_{ext}=0$.

"Topology" here refers to the distribution of polarized regions, separated and enclosed by contours along which $E_0=0$ in film.
$B_{ext} = 0$

$B_z = 0$ in film

$(+/−)$ sign of $B_z$ in film

$\chi_t = + \chi_o$

$\frac{\chi_t}{\chi_o} = \left(\text{# of lines enclosing } +\right) - \left(\text{# of lines enclosing } -\right) / \text{unit cell}$. 
\[ \gamma_k = + \gamma_0 \]

\[ 0 > B_{ext} > B, \]

when \( B_{ext} = B_1 \), regions enclosing positively-oriented spin dipoles will vanish.

\[ \gamma_k = 0 \] for \( B_{ext} \leq B \).
$B_{\text{ext}} = B_z > 0$

The regions enclosing positive fields all enlarge until they meet.

At this point, the topology is changing non-trivially.

\[
\begin{array}{c}
\text{triangular lattice} \rightarrow \text{honeycomb lattice}
\end{array}
\]
\[ \gamma_t = -2\gamma_0 \]

\[ B_2 < B_{\text{ext}} < B_3 \]

There are twice as many negative, enclosed regions as the positive regions enclosed previously.

When \( B_{\text{ext}} = B_3 \), the regions enclosing negative field will vanish, and \( \gamma_t = 0 \).
Topological Changes in Hall Conductivity Induced by External Field

The graph shows the change in Hall conductivity, $\sigma_{xy}(B)$, normalized by the zero-field conductivity, $\sigma_{xy}^0$, as a function of the magnetic field $B$. The graph exhibits discrete jumps at specific field values, labeled $B_1$, $B_2$, and $B_3$. These jumps correspond to topological transitions in the system.

The slope of the graph at ordinary Hall effect conditions is indicated, showing a linear relationship until a transition occurs.

The notation $\sigma_{xy}(B)/\sigma_{xy}^0$ is used to denote the normalized Hall conductivity.
Topological field within thin film, in units of ".