Uniqueness and Stability for Shock Reflection problem

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Shock reflection by a wedge: Regular reflection
Shock reflection by a wedge: Mach reflection
Shock reflection by a wedge: Irregular Mach reflection.

Self-similar flow: \((\vec{u}, p, \rho)(x, t) = (\vec{u}, p, \rho)(\frac{x}{t})\).
Shock reflection

First described by E. Mach 1878. Reflection patterns: Regular reflection, Mach reflection.

J. von Neumann, 1940s: on transition between patterns


Analysis: Special models (Transonic small disturbance eq., pressure-gradient system, nonlinear wave eq.): Gamba, Rosales, Tabak, Canic, Keyfitz, Kim, Lieberman, Y. Zheng, G.-Q. Chen-W. Xiang.

Local existence results: S.-X. Chen.
Global shock reflection solutions for potential flow: G.-Q. Chen-F., Elling

Other self-similar reflection problems:

Prandtl Reflection: Elling-Liu, Bae-G.-Q. Chen-F

Shock interactions/reflection for Chaplygin gas: D. Serre

Shock reflection as a Riemann problem

Initial data: Constant (uniform) states (0) and (1):
State (0): velocity $\vec{u}_0 = (0, 0)$, density $\rho_0$, pressure $p_0$.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density $\rho_1$, pressure $p_1$.

$t > 0$: Self-similar solution of compressible Euler system:
$(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\xi)$, where $\xi = \frac{x}{t}$.
System of conservation laws in Multi-D

$$\partial_t u + \text{div}F(u) = 0 \ \text{in} \ \Omega \times \mathbb{R}^1,$$

$$\Omega \subset \mathbb{R}^n, \ u : \Omega \times \mathbb{R}^1 \rightarrow \mathbb{R}^m, \ F : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}.$$  

**System:** $m > 1$.  **Multi-D:** $n > 1$.

Systems in multiple dimensions, $n > 1$: Very little positive results are known about existence, stability, properties of general time-dependent solutions. Thus study **special solutions:** Riemann problem.

Riemann problem: piecewise-constant initial data

$$\Omega = \Omega_1 \cup \Omega_2 \cdots \cup \Omega_s,$$

$$u|_{\Omega_k \times \{t=0\}} = u_k^0 \ - \ \text{constant vector,} \ k = 1, \ldots, s.$$
Riemann Problem in Multi-D as Free Boundary Problem

For appropriate $\Omega_k$ and boundary cond. (BC), expect self-similar solutions $u(x, t) = U(\xi)$, where $\xi = \frac{x}{t} \in \Omega$.

Self-similar system in $\Omega$, for $U(\xi) = U\left(\frac{x}{t}\right)$:

$$\text{div} F(U) - (\xi \cdot \nabla)U = 0 \text{ in } \Omega$$

plus BC, and conditions at infinity (from initial condition)

Self-similar system can be hyperbolic, elliptic, mixed elliptic-hyperbolic, etc.

Solution may have some additional discontinuities – shocks, contact discontinuities, i.e. new subdomains: Free Boundary Problem
Uniqueness/nonuniqueness for Multi-D Riemann problems and for Shock Reflection

Riemann problem in whole space for 2D isentropic Euler system: Chiodaroli-DeLellis-Kreml (2015): For a class of initial data with flat shocks, entropy solutions are non-unique. BV self-similar solutions 2D, depending on one spatial variable (i.e. of special structure), are unique.

Shock reflection: Riemann problem in domain with boundary.

Uniqueness for shock reflection can be considered in class of:

1. Time-dependent solutions: non-uniqueness for normal reflection, using technique of Chiodaroli-DeLellis-Kreml;
2. General self-similar solutions (??);

Potential flow system

Conservation of mass, Bernoulli’s law

\[ \rho_t + \text{div}(\rho \nabla \Phi) = 0, \]
\[ \Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = \text{const} \]

where:
\( \vec{u} = (u_1, u_2) \) – velocity
\( \Phi \) – velocity potential: \( \vec{u} = \nabla_x \Phi \).
\( \rho \) – density
\( p = \rho^\gamma \) – pressure
\( \gamma > 1 \) – adiabatic exponent (it is a given constant)

Compressible Euler system: Isentropic case

\[ \partial_t \rho + \text{div}(\rho \vec{u}) = 0, \]
\[ \partial_t (\rho \vec{u}) + \text{div}(\rho \vec{u} \otimes \vec{u}) + \nabla p = 0 \]
Regular reflection in self-similar coordinates $\xi = \frac{x}{t}$

Given:
State (0): velocity $\vec{u}_0 = (0, 0)$, density $\rho_0$, pressure $p_0$.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density $\rho_1$, pressure $p_1$.

Problem: Find self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\xi)$, where $\xi = \frac{x}{t}$, with asymptotic conditions at infinity determined by states (0) and (1), and satisfying $u \cdot \nu = 0$ on the boundary.
Potential flow: self-similar case

$$\Phi(\vec{x}, t) = t\psi(\xi, \eta), \quad \rho(\vec{x}, t) = \rho(\xi, \eta) \text{ with } (\xi, \eta) = \frac{\vec{x}}{t} \in \mathbb{R}^2.$$ 

Pseudo-potential: $$\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2).$$

Equation for $$\varphi$$:

$$\text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2\rho(|\nabla \varphi|^2, \varphi) = 0,$$

with $$\rho(|\nabla \varphi|^2, \varphi) = (K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla \varphi|^2))^{\frac{1}{\gamma - 1}}.$$ 

Equation is of mixed type:

**elliptic** \[|\nabla \varphi| < c(|\nabla \varphi|^2, \varphi, K),\]

**hyperbolic** \[|\nabla \varphi| > c(|\nabla \varphi|^2, \varphi, K),\]

where **sonic speed** $$c$$ is:

$$c^2 = \rho^{\gamma - 1} = K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla \varphi|^2).$$
Uniform states

Solutions with constant (physical) velocity \((u, v)\):

\[
\varphi(\xi, \eta) = -\frac{\xi^2 + \eta^2}{2} + u\xi + v\eta + \text{const.}
\]

Any such function is a solution.

Also (from formula) density \(\rho(\nabla \varphi, \varphi) = \text{const}\), thus sonic speed

\[
c = \rho \frac{\gamma - 1}{2} = \text{const}
\]

Then ellipticity region

\[
|\nabla \varphi(\xi, \eta)| = |(u, v) - (\xi, \eta)| < c
\]

is circle, centered at \((u, v)\), radius \(c\).
Shocks, RH conditions, Entropy condition

Shocks are discontinuities in the pseudo-velocity $\nabla \varphi$:

if $\Omega^+$ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are nonempty and open, and $S := \partial \Omega^+ \cap \Omega$ is a $C^1$ curve where $\nabla \varphi$ has a jump, then $\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution in $\Omega$ if and only if $\varphi$ satisfies potential flow equation in $\Omega^\pm$ and the Rankine-Hugoniot (RH) condition on $S$:

\[
[\varphi]_S = 0, \\
[\rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu]_S = 0,
\]

where $[\cdot]_S$ is jump across $S$.

Entropy Condition on $S$: density increases across $S$ in the flow direction.
Shock reflection as free boundary problem

\[
\text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2 \rho(|\nabla \varphi|^2, \varphi) = 0 \quad \text{in} \quad \Omega,
\]

\[
\rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu = \rho(|\nabla \varphi_1|^2, \varphi_1) \nabla \varphi_1 \cdot \nu \quad \bigg\} \quad \text{on} \quad P_1P_2
\]

\[\varphi = \varphi_1 \quad \text{on} \quad P_1P_4 \quad \text{(and prove} \quad D_\nu \varphi = D_\nu \varphi_2 \text{on} \quad P_1P_4)\]

\[\varphi_\nu = 0 \quad \text{on Wedge} \quad P_3P_4, \quad \text{Symmetry line} \quad P_2P_3,\]

Solve for: Free boundary \(P_1P_2\) and function \(\varphi\) in \(\Omega\).
Expect equation elliptic in \(\Omega\).
Regular reflection, state (2)

\[\vec{u}_1\]

\[P_0\]

\[\phi = \text{pseudo-potential between the reflected shock and the wall}\]

\[\phi_1 = \text{pseudo-potential of state (1)}\]

Denote \(\nabla \phi(P_0) = (u_2, v_2)\). Since \(\phi_\nu = 0\) on wedge, then

\[v_2 = u_2 \tan \theta_w.\]

Rankine-Hugoniot conditions at reflection point \(P_0\), for \(\phi\) and \(\phi_1\): algebraic equations for \(u_2, \phi(P_0)\)
Regular reflection, state (2), detachment angle

If solution exists: Let

\[ \varphi_2(\xi, \eta) = -\left(\xi^2 + \eta^2\right)/2 + u_2\xi + v_2\eta + C, \]

where \( C \) determined by \( \varphi_2(P_0) = \varphi_1(P_0) \).

Existence of state (2) is necessary condition for existence of regular reflection

Given \( \gamma, \rho_0, \rho_1 \), there exists \( \theta_{\text{detach}} \in (0, \frac{\pi}{2}) \) such that:

state (2) exists for \( \theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2}) \),
state (2) does not exist for \( \theta_w \in (0, \theta_{\text{detach}}) \).

If \( \varphi_2 \) exist, then RH is satisfied along the line \( S_1 := \{ \varphi_1 = \varphi_2 \} \).
Weak and Strong State (2); Sonic angle

For each $\theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2})$ there exists two possible States (2): weak and strong, with $\rho_{2}^{\text{weak}} < \rho_{2}^{\text{strong}}$. We always choose weak state (2). For strong state (2), existence of global regular reflection solution is not expected, Elling (2011) confirms that.

There exist $\theta_{\text{sonic}} \in (\theta_{\text{detach}}, \frac{\pi}{2})$ such that:

State 2 is supersonic at $P_0$ for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$.
State 2 is subsonic at $P_0$ for $\theta_w \in (\theta_{\text{detach}}, \theta_{\text{sonic}})$. 
Von Neumann’s conjectures on transition between different reflection patterns

Recall: sonic angle $\theta_{sonic}$ and detachment angle $\theta_{detach}$ satisfy $0 < \theta_{detach} < \theta_{sonic} < \frac{\pi}{2}$.

**Sonic conjecture:**
Regular reflection for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, Mach reflection for $\theta_w \in (0, \theta_{sonic})$.

**Von Neumann’s detachment conjecture:**
Regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$, Mach reflection for $\theta_w \in (0, \theta_{detach})$.

We prove existence of regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$ for potential flow flow equation.
First discuss subsonic and supersonic regular reflections.
Supersonic regular reflection: State (2) is supersonic at $P_0$.

Structure of solution $\varphi$:

- $\varphi = \varphi_i$ in $\Omega_i$, $i=0,1,2$.
- $\varphi \in C^1(P_0P_2P_3)$, in particular $C^1$ across sonic arc $P_1P_4$.
- Shock $P_0P_2$ has flat part $P_0P_1$, curved part $P_1P_2$, and is $C^1$ across $P_1$.
- Equation is strictly elliptic in $\overline{\Omega} \setminus P_1P_4$. 
Subsonic regular reflection:  State (2) is subsonic at $P_0$.

Structure of solution $\varphi$:

- $\varphi = \varphi_i$ in $\Omega_i$, $i=0,1$.
- $\varphi \in C^1(P_0P_2P_3)$.
- $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at $P_0$.
- Shock $P_0P_2$ is $C^1$.
- Equation is strictly elliptic in $\overline{\Omega} \setminus \{P_0\}$. 

Subsonic regular reflection

- Incident shock $P_0P_1$.
- Reflected shock $P_0P_2$.
- Boundary $\Omega$. 

Diagram:

- Incident shock $P_0P_1$.
- Reflected shock $P_0P_2$.
- Boundary $\Omega$. 

Equation:

$$
\begin{align*}
\varphi & = \varphi_i \text{ in } \Omega_i, \ i=0,1. \\
\varphi & \in C^1(P_0P_2P_3). \\
\varphi & = \varphi_2, \ D\varphi = D\varphi_2 \text{ at } P_0. \\
\text{Shock } P_0P_2 & \text{ is } C^1. \\
\text{Equation is strictly elliptic in } \overline{\Omega} \setminus \{P_0\}. 
\end{align*}
$$
Existence of regular reflection solutions

Theorem 1. (G.-Q. Chen-F.). If \( \rho_1 > \rho_0 > 0, \gamma > 1 \) then a regular reflection solution \( \varphi \) exists for all wedge angles \( \theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2}) \). Here I skip some details related to ”attached shocks”. The type of reflection (supersonic or subsonic) for each \( \theta_w \) is determined by the type of State 2 at the reflection point \( P_0 \) for \( \theta_w \). Moreover, solution satisfies the following additional properties:
1) Equation is elliptic for $\varphi$ in $\Omega$, ellipticity degenerates near sonic arc $P_1P_4$.

2) $\varphi$ is $C^{1,1}$ near and across the sonic arc $P_1P_4$;

3) Reflected shock is $C^{2,\beta}$, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$ between $\vec{e}_\eta = (0, 1)$ and $\vec{e}_{S_1} = P_0P_1$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$. 

Properties of solution: subsonic case

1) Equation is elliptic for \( \varphi \) in \( \Omega \), except for the sonic wedge angle (then ellipticity degenerates at \( P_0 \)).

2) \( \varphi \) is \( C^{2,\alpha} \) inside \( \Omega \), and \( C^{1,\alpha} \) near and up to the reflection point \( P_0 \), and \( \varphi = \varphi_2 \), \( D\varphi = D\varphi_2 \) at \( P_0 \);

3) Reflected shock is \( C^{2,\alpha} \) away from \( P_0 \) and \( C^{1,\alpha} \) up to \( P_0 \), and a graph for a cone of directions \( Con(\vec{e}_\eta, \vec{e}_{S_1}) \);

4) \( \varphi_2 \leq \varphi \leq \varphi_1 \) in \( \Omega \), and \( \partial_e(\varphi_1 - \varphi) < 0 \) if \( e \in Con(\vec{e}_\eta, \vec{e}_{S_1}) \).
Stability of normal reflection as $\theta_w \to \pi/2$

Furthermore, the solutions $\varphi$ converge in $W_{loc}^{1,1}$ to the solution of the normal reflection as $\theta_w \to \pi/2$. 
Proof Th. 1:

By degree theory. Yields existence of "admissible solutions":

Admissible solutions:

1. Have structure supersonic or subsonic reflections depending on $\theta_w$. Recall: this includes ellipticity in $\Omega$ and regularity of $P_0P_2$ and of $\varphi$ in $P_0P_2P_3$;
2. $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$;
3. satisfy nonstrict monotonicity $\partial_e (\varphi_1 - \varphi) \leq 0$ in $\Omega$ for any $e \in Con(e_\eta, e_{S_1})$. 
Convexity of shock, uniqueness, stability

**Theorem 2. (Chen-F.-W. Xiang)** For admissible solutions, shock is strictly convex in its relative interior. Moreover, regular reflection solution satisfying (1)-(2) have cone of monotonicity (3) if and only if the shock is convex.

Based on convexity of shock, we prove:

**Theorem 3. (Chen-F.-Xiang)** Admissible solutions are unique (and exist, by Thm. 1).

**Corollary. (Chen-F.-Xiang)** Regular reflections solutions with convex shocks are unique (and exist by Thms. 1, 2).

Stability (from uniqueness and compactness):

**Theorem 3. (Chen-F.-Xiang)** Let \( \varphi_i \) be a sequence of admissible solutions (or solutions with convex shocks) for wedge angles \( \theta_i \), and let \( \theta_i \to \theta^* \in (\theta^{\text{detach}}, \pi/2] \). Then \( \varphi_i \to \varphi^* \) in \( W^{1,p}_{loc} \) for any \( p \in [1, \infty) \), where \( \varphi^* \) is the unique admissible solution for wedge angle \( \theta^* \).
Outline of proof of uniqueness

We prove uniqueness of admissible solutions (thus with convex shock).

Heuristic idea:
By Th. 1, when $\theta_w \to \frac{\pi}{2}$, admissible solutions converge to normal reflection. Also we have uniform estimates for admissible solutions. Then use the method of continuity:

Suppose $\varphi$, $\hat{\varphi}$ are two admissible solutions for some $\theta^*_w \in (\theta^d_w, \frac{\pi}{2})$. Then it is sufficient to:

1. Construct continuous families $\theta_w \mapsto \varphi(\theta_w)$, $\theta_w \mapsto \hat{\varphi}(\theta_w)$ for $\theta_w \in [\theta^*_w, \frac{\pi}{2})$, with $\varphi(\theta^*_w) = \varphi$, $\hat{\varphi}(\theta^*_w) = \hat{\varphi}$,

2. Show ”local uniqueness”: if two admissible solutions for same $\theta_w$ are close in $C^1$, then they are equal.

Both are achieved if we can linearize FBP at an admissible solution, and linearization is ”good” so that we can construct solutions for nearby wedge angles by Implicit Function Theorem.
Outline of proof of uniqueness

Rigorously, cannot use linearization for supersonic reflections: elliptic degeneracy near sonic arc requires very detail control of $D^2 \varphi$ on sonic arc $P_1P_4$ to show well-posedness of linearization. We do not have this control at one point: $P_1$, where shock meets sonic arc.

Then we use a ”nonlinear version of linearization”: apply degree theory with ”small” iteration set, consisting of functions close to the background solution (in appropriate norm). To apply degree theory, we need to show (in particular) that fixed point of iteration map cannot occur on the boundary of the iteration set. This is done using local uniqueness theorem.

We use convexity of shock for proof of local uniqueness theorem.
Proof of uniqueness: fixed points do not exist on boundary of iteration set, by local uniqueness

Given admissible solution $\varphi^*$ for wedge angle $\theta^*_w$, define iteration set:

$$K = \{(\varphi, \theta_w) : |\theta_w - \theta^*_w| < \delta, \ "\text{dist}"(\varphi, \varphi^*) < \varepsilon\}.$$  

Fix small $\varepsilon$. Suppose for any $\delta > 0$ there exists a fixed point (admissible solution)

$$(\varphi, \theta_w) \in \partial K.$$  

Then, by compactness of admissible solutions there exist a sequence $(\varphi^{(i)}, \theta_w^{(i)}) \in \partial K$ such that

$$(\varphi^{(i)}, \theta_w^{(i)}) \to (\varphi^{**}, \theta^*_w),$$

where $\varphi^{**}$ is an admissible solution for the "background" angle $\theta^*_w$, and

$$"\text{dist}"(\varphi^*, \varphi^{**}) = \varepsilon.$$  

Contradiction to local uniqueness if $\varepsilon$ is small.
Proof of uniqueness: Role of convexity (heuristic)

When linearize FBP, variations of shock locations introduce an additional zero-order term in the oblique boundary condition derived from RH condition \( \rho D \varphi \cdot \nu = \rho_1 D \varphi_1 \cdot \nu \). This term has the ”correct” sign if shock is convex:

**Linerization of RH conditions:** shock is \( \eta = f(\xi) \) with \( \Omega \subset \{ \eta < f(\xi) \} \) after rotating coordinates. Then RH:

\[
\varphi^\varepsilon(\xi, f^\varepsilon(\xi)) = \varphi_1(\xi, f^\varepsilon(\xi));
\]

\[
\left( (\rho(|D \varphi^\varepsilon|^2, \varphi^\varepsilon) D \varphi^\varepsilon - \rho_1 D \varphi_1) \cdot (D \varphi_1 - D \varphi^\varepsilon) \right)(\xi, f^\varepsilon(\xi)) = 0,
\]

where we use that \( \nu = \frac{D \varphi_1 - D \varphi^\varepsilon}{|D \varphi_1 - D \varphi^\varepsilon|} \). Here \( \varphi^\varepsilon = \varphi + \varepsilon \delta \varphi + \ldots \), same for \( f^\varepsilon \). Taking \( \frac{d}{d\varepsilon} \) at \( \varepsilon = 0 \) in 1st condition and using \( \partial_\eta(\varphi_1 - \varphi) < 0 \) by entropy condition:

\[
\delta f = \frac{1}{|\partial_\eta(\varphi_1 - \varphi)|} \delta \varphi.
\]
Proof of uniqueness: Role of convexity (heuristic)

Now take $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 2nd RH condition

$$
\left( (\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon) \right)(\xi, f^\varepsilon(\xi)) = 0,
$$

Get two terms. First, linearization of oblique condition:

$$
\frac{d}{d\varepsilon} \left[ \left( (\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon) \right) \right]_{\varepsilon=0} (\xi, f(\xi)) = a\partial_\nu \delta \varphi + b\partial_\tau \delta \varphi + c\delta \varphi,
\text{ where } a(\xi) \geq \lambda > 0, \ c(\xi) \leq -\lambda < 0
$$

Second term comes from the perturbation of shock location:

$$
\partial_\eta \left[ \left( (\rho(|D\varphi|^2, \varphi)D\varphi - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi) \right) \right] \delta f
$$

$$
= A(\varphi_1 - \varphi)_{\tau\tau} \delta f = \frac{A}{|\partial_\eta (\varphi_1 - \varphi)|} (\varphi_1 - \varphi)_{\tau\tau} \delta \varphi,
$$

where $A > 0$. Convexity of shock equivalent to $(\varphi_1 - \varphi)_{\tau\tau} \leq 0$, and then the coefficient of $\delta \varphi$ has "correct" sign.
Regularity near sonic arc and linearization.

We discuss why linearization does not work for supersonic reflections.

Estimates near sonic arc. Flatten sonic arc: introduce coordinates

\[ x = c_2 - r, \quad y = \theta - \theta_w, \]

where \((r, \theta)\) are polar coordinates centered at \(O_2 = (u_2, v_2)\).

Then \(\Omega \cap \mathcal{N}(\Gamma_{sonic}) \subset \{x > 0\}\) and \(\Gamma_{sonic} \subset \{x = 0\}\), where \(\Gamma_{sonic}\) is arc \(P_1P_4\).
Estimates near sonic arc

Let $\psi = \varphi - \varphi_2$. Self-similar potential flow equation rewritten for $\psi$ in $(x, y)$-coordinates is:

$$[2x - (\gamma + 1)\psi_x]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0 \quad \text{in } \Omega \subset \{x > 0\}.$$  

plus "small" terms. Full equation is homogeneous. Also,

$$\psi > 0 \quad \text{in } \Omega, \quad \leftarrow \text{recall: } \varphi \geq \varphi_2$$

$$\psi = 0 \quad \text{on } \Gamma_{\text{sonic}} = \partial \Omega \cap \{x = 0\}.$$  

Equation is elliptic in $x > 0$ if

$$\psi_x < \frac{2}{\gamma + 1}x.$$  

Then ellipticity degenerates in $x$-direction near sonic arc $\{x = 0\}$. 

Regularity in $\Omega$ near sonic arc (supersonic case)

Theorem 4. (Bae-Chen-F., 2009)
1) For every $P$ in sonic arc $(P_1P_4]$ (i.e. excluding $P_1$)

$\varphi \in C^{2,\alpha}(\overline{\Omega \cap B_R(P)})$, for some small $R > 0$, any $\alpha \in (0, 1)$.

2) $D^2\varphi$ has a jump across sonic arc $P_1P_4$:

$D_{rr}\varphi|_{\Omega} - D_{rr}\varphi_2 = \frac{1}{\gamma+1}$ on arc $(P_1P_4]$.

Remark: $(\varphi - \varphi_2)_r = (\varphi - \varphi_2)_{r\theta} = (\varphi - \varphi_2)_{\theta\theta} = 0$ on $P_1P_4$

3) $D^2\varphi$ in $\Omega$ does not have a limit at $P_1$. 
Linearization

Linearization of

\[ [2x - (\gamma + 1)\psi_x]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0 \quad \text{in} \quad \Omega \subset \{x > 0\} \]

is

\[ [2x - (\gamma + 1)\psi_x]V_{xx} + \frac{1}{c_2}V_{yy} - (1 + (\gamma + 1)\psi_{xx})V_x = 0. \]

in \( \Omega \subset \{x > 0\} \). Can show that \( 0 \leq (\gamma + 1)\psi_x \leq 2 - \delta \). Thus equation is elliptic in \( x > 0 \), and ellipticity degenerates near sonic arc \( P_1P_4 = \{x = 0\} \cap \partial\Omega \) since \( |\psi_x| \leq Cx \). Also, need to prescribe Dirichlet data on \( P_1P_4 \). Because of elliptic degeneracy, this can be done under conditions on coefficient of \( V_x \) (e.g. if the coefficient is non-positive). Thus we need some precise \( L^\infty \) bounds of \( \psi_{xx} \). Such estimates are not available near \( P_1 \).
Linearization vs. Iteration Problem

(Main and some other terms of) potential flow equation:

\[
[2x - (\gamma + 1)\psi_x + O(D\psi, \psi, x)]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0
\]

in \( \Omega \subset \{ x > 0 \} \). (Main terms of) linearization are:

\[
[2x - (\gamma + 1)\psi_x]V_{xx} + \frac{1}{c_2}V_{yy} - (1 + (\gamma + 1)\psi_{xx})V_x = 0.
\]

Using iteration procedure (map from \( \phi \) to \( \psi \)): Iteration equation for \( \psi \) defined by ellipticity cutoff and plugging \( \phi \) into "non-main" parts of coeff's. Main and some non-main terms are:

\[
[2x - (\gamma + 1)x\eta \left( \frac{\psi_x}{x} \right) + O(D\phi, \phi, x)]\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0,
\]

\( \eta \) is cutoff function. Coefficients do not depend on \( D^2\phi \).
Instability under non-symmetric perturbations

\[ t = 0 \]

Incident Shock

\[ \vec{u}_1 = (u_1, \varepsilon) \]
\[ \rho_1, p_1 \]

(1)

Reflected Shock

\[ \vec{u}_0 = (0, 0) \]

\[ \rho_0, p_0 \]

(0)

Now velocity of state (1) \( \vec{u}_1 = (u_1, \varepsilon) \) is slightly non-parallel to wedge axis.

**Theorem (J. Hu)** Let \( \varepsilon \neq 0 \) and small. Let wedge angle \( \theta_w \) is close to \( \frac{\pi}{2} \). Then there does not exist a solution of regular reflection structure (thus strictly subsonic near \((0, 0)\)).
Instability under non-symmetric perturbations

The reason is rigidity of potential flow in subsonic domains: it is shown that, if such solution exists, then velocity is not continuous up to $(0, 0)$. Also, by subsonicity condition, velocity is bounded. Then $\varphi$ is Lipschitz. However, for uniformly elliptic equations near corner, with Neumann condition, Lipschitz solutions are $C^{1,\alpha}$, a contradiction.

We expect that for Euler system this instability does not happen, i.e. that subsonic self-similar solutions which are discontinuous near the corner, exist.
Open problems

- **Uniqueness/nonuniqueness** in classes of solutions, as discussed.
- Prove existence results for Euler system. One of difficulties is in vorticity estimates near stagnation points, noticed by D. Serre for isentropic Euler system.
- Mach reflection...