Stability of Vortices in Ideal Fluids: the Legacy of Kelvin and Rayleigh

Th. Gallay
Institut Fourier, Université Grenoble Alpes, France

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Overview

Part I  Introduction to hydrodynamic stability theory
- The Rayleigh-Taylor instability
- Instabilities of shear flows
- Shear flows in stratified fluids

Part II  Vortices in ideal fluids: classical results
- Two-dimensional vortices
- Columnar vortices
- Howard and Gupta’s approach

Part III Spectral stability of inviscid columnar vortices
- Assumptions on the vorticity profile
- The spectral stability result
- A glimpse of the proof
Part I: What is Hydrodynamic Stability?

Hydrodynamic stability is the field which analyses the stability and the onset of instability in fluid flows.

Here is a nice, informal definition of stability:

“When an infinitely small variation of the present state will alter only by an infinitely small quantity the state at some future time, the condition of the system, whether at rest or in motion, is said to be stable but when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the system is said to be unstable.”

James Clerk Maxwell (1831–1879)

Hydrodynamic stability theory started in the 19th century. Progress has been slow, due to significant mathematical difficulties. Still today, most results apply to simple and idealized situations, typically laminar stationary flows given by explicit analytical expressions.
The Founding Fathers

Hermann von Helmholtz
(1821–1904)

George Gabriel Stokes
(1819–1903)

Sir William Thomson
Lord Kelvin
(1824–1907)

John William Strutt
Lord Rayleigh
(1842–1919)
Three Important References

XXIV. Vibrations of a Columnar Vortex.
By Sir William Thomson*.

THIS is a case of fluid-motion, in which the stream-lines are approximately circles, with their centres in one line (the axis of the vortex) and the velocities approximately constant, and approximately equal at equal distances from the axis. As a preliminary to treating it, it is convenient to express the equations of motion of a homogeneous incompressible inviscid fluid (the description of fluid to which the present investigation is confined) in terms of "columnar coordinates," \( r, \theta, z \)—that is, coordinates such that \( r \cos \theta = x, r \sin \theta = y \).

If we call the density unity, and if we denote by \( \dot{x}, \dot{y}, \dot{z} \) the velocity-components of the fluid particle which at time \( t \) is

* From the Proceedings of the Royal Society of Edinburgh, March 1, 1880.

On the Stability, or Instability, of certain Fluid Motions. By Lord Rayleigh, F.R.S., Professor of Experimental Physics in the University of Cambridge.

[Read February 12th, 1880.]

Proc. London Math. Soc. 11 (1880), 57-72

On the Dynamics of Revolving Fluids.
By Lord Rayleigh, O.M., F.R.S.

(Received December 8, 1916.)

The Rayleigh-Taylor Instability

\[ \partial_t \rho + u \cdot \nabla \rho = 0 \]

\[ \rho (\partial_t u + (u \cdot \nabla) u) = -\nabla p - \rho g e_z \]

\[ \text{div } u = 0 \]

\[ \rho(x, z, t) : \text{ fluid density} \]

\[ u(x, z, t) : \text{ velocity field} \]

\[ p(x, z, t) : \text{ internal pressure} \]

Equilibrium state: \( \rho = \bar{\rho}(z), \ u = 0, \ p = \bar{p}(z) \) (where \( \bar{p}' = -\bar{\rho}g \)).

Perturbed solutions: \( \rho = \bar{\rho}(z) + \tilde{\rho}, \ u = \tilde{u}, \ p = \bar{p}(z) + \tilde{p} \).
Linearized perturbation equations (after dropping the tildes):

\[
\bar{\rho}(z) \partial_t u_x = -\partial_x p, \quad \partial_t \rho + \bar{\rho}'(z) u_z = 0, \\
\bar{\rho}(z) \partial_t u_z = -\partial_z p - \rho g, \quad \partial_x u_x + \partial_z u_z = 0.
\]

Eigenvalue equation (\(\partial_t \leftarrow s\)) after horizontal Fourier transform (\(\partial_x \leftarrow ik\)):

\[
\bar{\rho}(z) s u_x = -ikp, \quad s \rho + \bar{\rho}'(z) u_z = 0, \\
\bar{\rho}(z) s u_z = -\partial_z p - \rho g, \quad iku_x + \partial_z u_z = 0.
\]

Scalar form of the eigenvalue equation (Rayleigh, 1916):

\[
-\partial_z (\bar{\rho}(z) \partial_z u_z) + k^2 \bar{\rho}(z) u_z - \frac{k^2 g}{s^2} \bar{\rho}'(z) u_z = 0.
\]

Rayleigh identity for eigenfunctions:

\[
\int \bar{\rho}(z) |\partial_z u_z|^2 \, dz + k^2 \int \bar{\rho}(z) |u_z|^2 \, dz - \frac{k^2 g}{s^2} \int \bar{\rho}'(z) |u_z|^2 \, dz = 0.
\]

\(\Rightarrow\) Sharp stability criterion: \(\bar{\rho}'(z) \leq 0\) for all \(z\) (Synge, 1933).
Instabilities of Shear Flows

\[ \partial_t \rho + u \cdot \nabla \rho = 0 \]
\[ \rho (\partial_t u + (u \cdot \nabla) u) = -\nabla p \]
\[ \text{div } u = 0 \]

\( \rho = 1 \) : fluid density
\( u(x, z, t) \) : velocity field
\( p(x, z, t) \) : internal pressure

Equilibrium state: \( \rho = 1, \ u = U(z) \ e_x, \ p = 0 \).

Perturbed solutions: \( \rho = 1, \ u = U(z) \ e_x + \tilde{u}, \ p = \tilde{p} \).
Linearized perturbation equations:

\[ \partial_t u_x + U(z) \partial_x u_x + U'(z) u_z = -\partial_x p, \]
\[ \partial_t u_z + U(z) \partial_x u_z = -\partial_z p, \]
\[ \partial_x u_x + \partial_z u_z = 0. \]

Eigenvalue equation (\( \partial_t \leftarrow s \)) after horizontal Fourier transform (\( \partial_x \leftarrow ik \)):

\[ \gamma(z) u_x + U'(z) u_z = -ikp, \quad \gamma(z) u_z + U'(z) u_z = -\partial_z p, \quad ik u_x + \partial_z u_z = 0, \]

where \( \gamma(z) = s + ikU(z) \) is the spectral function.

Scalar form of the eigenvalue equation (Rayleigh, 1880):

\[ -\partial_z^2 u_z + k^2 u_z + \frac{ikU''(z)}{\gamma(z)} u_z = 0. \]

Rayleigh identity for eigenfunctions:

\[ \int |\partial_z u_z|^2 \, dz + k^2 \int |u_z|^2 \, dz + ik \int \frac{\gamma(z)*U''(z)}{|\gamma(z)|^2} |u_z|^2 \, dz = 0. \]

⇒ Rayleigh’s stability criterion: \( U''(z) \neq 0 \) for all \( z \) (Rayleigh, 1880).
Shear Flows in Stratified Fluids

\[ \partial_t \rho + u \cdot \nabla \rho = 0 \]
\[ \rho (\partial_t u + (u \cdot \nabla) u) = -\nabla p - \rho g e_z \]
\[ \text{div} \ u = 0 \]

\( \rho(x, z, t) \) : fluid density
\( u(x, z, t) \) : velocity field
\( p(x, z, t) \) : internal pressure

Equilibrium state: \( \rho = \bar{\rho}(z) \), \( u = U(z) e_x \), \( p = \bar{p}(z) \) (where \( \bar{p}' = -\bar{\rho} g \)).

Perturbed solutions: \( \rho = \bar{\rho}(z) + \tilde{\rho} \), \( u = U(z) e_x + \tilde{u} \), \( p = \bar{p}(z) + \tilde{p} \).
Linearized perturbation equations:
\[
\bar{\rho}(z) \left( \partial_t u_x + U(z) \partial_x u_x + U'(z) u_z \right) = -\partial_x p, \quad \partial_t \rho + U(z) \partial_x \rho + \bar{\rho}'(z) u_z = 0, \\
\bar{\rho}(z) \left( \partial_t u_z + U(z) \partial_x u_z \right) = -\partial_z p - \rho g, \quad \partial_x u_x + \partial_z u_z = 0.
\]

Eigenvalue equation \((\partial_t \leftarrow s)\) after horizontal Fourier transform \((\partial_x \leftarrow i k)\):
\[
\bar{\rho}(z) \left( \gamma(z) u_x + U'(z) u_z \right) = -ikp, \quad \gamma(z) \rho + \bar{\rho}'(z) u_z = 0, \\
\bar{\rho}(z) \gamma(z) u_z = -\partial_z p - \rho g, \quad ik u_x + \partial_z u_z = 0,
\]
where again \(\gamma(z) = s + ikU(z)\) is the spectral function.

Scalar form of the eigenvalue equation (Taylor and Goldstein, 1931):
\[
-\partial_z (\bar{\rho}(z) \partial_z u_z) + k^2 \bar{\rho}(z) u_z + \frac{ik}{\gamma(z)} (\bar{\rho}U')'(z) u_z - \frac{k^2 g}{\gamma(z)^2} \bar{\rho}'(z) u_z = 0.
\]

Rayleigh identity for eigenfunctions:
\[
\int \left\{ \bar{\rho}(z) |\partial_z u_z|^2 + k^2 \bar{\rho}(z) |u_z|^2 + \frac{ik}{\gamma(z)} (\bar{\rho}U')'(z) |u_z|^2 - \frac{k^2 g}{\gamma(z)^2} \bar{\rho}'(z) |u_z|^2 \right\} \, dz = 0.
\]
Following Howard (1961), we apply the change of variables

\[ u_z = \gamma(z)^{1/2}v, \quad \text{where} \quad \gamma(z) = s + ikU(z). \]

The equation satisfied by \( v \) is

\[
-\partial_z(\bar{\rho}(z)\gamma(z)\partial_z v) + k^2\bar{\rho}(z)\gamma(z)v + \frac{ik}{2}(\bar{\rho}U')'(z)v + \left( \frac{\bar{\rho}\gamma'^2}{4\gamma} - \frac{k^2g\bar{\rho}'}{\gamma} \right)(z)v = 0.
\]

Multiplying both sides by \( v^* \), taking the real part, and integrating, we obtain

\[
\text{Re}(s) \int \left\{ \bar{\rho}(z)(|\partial_z v|^2 + k^2|v|^2) + \frac{k^2\bar{\rho}(z)U'(z)^2}{|\gamma(z)|^2}(\text{Ri}(z) - \frac{1}{4})|v|^2 \right\} dz = 0,
\]

where \( \text{Ri}(z) \) is the (local) \textbf{Richardson number}:

\[
\text{Ri}(z) = \frac{-\bar{\rho}'(z)g}{\bar{\rho}(z)} \frac{1}{U'(z)^2} = \left( \frac{\text{stratification}}{\text{shear}} \right)^2.
\]

which compares the restoring force due to the stratification with the potential instability due to the shear.

\[ \Rightarrow \text{Sufficient condition for stability (Miles-Howard)}: \text{Ri}(z) \geq 1/4 \text{ for all } z. \]
Part II: Two-dimensional Vortices

Evolution equations
\[ \partial_t u + (u \cdot \nabla) u = -\nabla p \]
\[ \text{div} \ u = 0 \]

Equilibrium state
\[ u = V(r) \, e_\theta \quad : \text{velocity field} \]
\[ p = P(r) \quad : \text{reduced pressure} \]
\[ rP'(r) = V(r)^2 \quad : \text{centrifugal balance} \]

Angular velocity: \[ \Omega(r) = \frac{V(r)}{r} \]
Vorticity: \[ W(r) = r\Omega'(r) + 2\Omega(r) \]

Rayleigh’s sufficient condition for stability: \[ W'(r) \neq 0 \text{ for all } r. \]
Examples of Vortex Profiles

• The Rankine vortex

\[ \Omega(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ 1/r^2 & \text{if } r \geq 1, \end{cases} \quad W(r) = \begin{cases} 2 & \text{if } r < 1, \\ 0 & \text{if } r > 1. \end{cases} \]

This corresponds to rigid rotation for \( r < 1 \) and irrotational flow for \( r > 1 \).

• The Kaufmann-Scully vortex

\[ \Omega(r) = \frac{1}{1 + r^2}, \quad W(r) = \frac{2}{(1 + r^2)^2}. \]

• The Lamb-Oseen vortex

\[ \Omega(r) = \frac{1}{r^2} \left(1 - e^{-r^2}\right), \quad W(r) = 2e^{-r^2}. \]

This is the profile of a self-similar solution of the Navier-Stokes equations.

Remark: All vortex profiles are normalized so that \( \Omega(0) = 1 \), hence \( W(0) = 2 \).
Waterspouts and Tornadoes
Euler Equations in Cylindrical Coordinates

Velocity field

\[ u = u_r e_r + u_\theta e_\theta + u_z e_z \]

\[ \text{div} \, u = 0 \]

Vorticity

\[ \omega = \text{curl} \, u \]

\[ \omega = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z \]

\[ \frac{\partial}{\partial t} u_r + (u \cdot \nabla) u_r - \frac{u_\theta^2}{r} = - \frac{\partial}{\partial r} p, \]

\[ \frac{\partial}{\partial t} u_\theta + (u \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = - \frac{1}{r} \frac{\partial}{\partial \theta} p, \]

\[ \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \theta} u_\theta + \frac{\partial}{\partial z} u_z = 0. \]
Columnar Vortices

Columnar vortices are stationary solutions of the Euler equations of the form
\[ u = V(r) \mathbf{e}_\theta, \quad p = P(r), \quad \omega = W(r) \mathbf{e}_z, \]
where \( rP'(r) = V(r)^2 \) and \( W(r) = V'(r) + \frac{V(r)}{r} \).

To investigate their stability, we consider the linearized vorticity equations

\[
\begin{align*}
\frac{\partial}{\partial t} \omega_r + \Omega(r) \frac{\partial}{\partial \theta} \omega_r &= W(r) \frac{\partial}{\partial z} u_r, \\
\frac{\partial}{\partial t} \omega_\theta + \Omega(r) \frac{\partial}{\partial \theta} \omega_\theta &= W(r) \frac{\partial}{\partial z} u_\theta + r\Omega'(r) \omega_r, \\
\frac{\partial}{\partial t} \omega_z + \Omega(r) \frac{\partial}{\partial \theta} \omega_z &= W(r) \frac{\partial}{\partial z} u_z - W'(r) u_r,
\end{align*}
\]

where \( \Omega(r) = \frac{V(r)}{r} \). The velocity \( u \) is determined by solving the relations

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial \theta} u_z - \frac{\partial}{\partial z} u_\theta &= \omega_r, \\
\frac{\partial}{\partial z} u_r - \frac{\partial}{\partial r} u_z &= \omega_\theta, \quad \text{and} \quad \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) + \frac{\partial}{\partial \theta} u_\theta + \frac{\partial}{\partial z} u_z &= 0.
\end{align*}
\]
Fourier Decomposition

We consider solutions velocities and vorticities of the form

\[ u(r, \theta, z, t) = u_{m,k}(r, t) e^{im\theta} e^{ikz}, \quad \omega(r, \theta, z, t) = \omega_{m,k}(r, t) e^{im\theta} e^{ikz}, \]

where \( m \in \mathbb{Z} \) is the angular Fourier mode, \( k \in \mathbb{R} \) is the vertical wave number.

The linearized vorticity equations translate into

\[
\partial_t \omega = \mathcal{L}_{m,k} \omega, \quad \text{i.e.} \quad \begin{cases}
(\partial_t + i\Omega(r)) \omega_r &= W(r)iku_r, \\
(\partial_t + i\Omega(r)) \omega_\theta &= W(r)iku_\theta + r\Omega'(r)\omega_r, \\
(\partial_t + i\Omega(r)) \omega_z &= W(r)iku_z - W'(r)u_r.
\end{cases} \tag{3}
\]

In addition, the following relations hold:

\[
\begin{align*}
\omega_r &= \frac{im}{r}u_z - iku_\theta, \\
\omega_\theta &= iku_r - \partial_r u_z, \\
\omega_z &= \frac{1}{r} \partial_r (ru_\theta) - \frac{im}{r}u_r,
\end{align*}
\]

and

\[
\frac{1}{r} \partial_r (ru_r) + \frac{im}{r} u_\theta + iku_z = 0. \tag{4}
\]
The Eigenvalue equation

Given \( s \in \mathbb{C} \), \( m \in \mathbb{Z} \), and \( k \in \mathbb{R} \), the eigenvalue equation \((s - \mathcal{L}_{m,k})\omega = 0\) reads

\[
\begin{align*}
\gamma(r)\omega_r &= \text{i}kW(r)u_r, \\
\gamma(r)\omega_\theta &= \text{i}kW(r)u_\theta + r\Omega'(r)\omega_r, \quad \text{where} \quad \gamma(r) = s + \text{i}m\Omega(r). \\
\gamma(r)\omega_z &= \text{i}kW(r)u_z - W'(r)u_r,
\end{align*}
\]

Taking into account the relations (4) between the velocity and the vorticity, this systems reduces to a single equation for the radial velocity:

\[
-\partial_r \left( \frac{r^2 \partial_r^* u_r}{m^2 + k^2 r^2} \right) + \left\{ 1 + \frac{1}{\gamma(r)^2} \frac{k^2 r^2 \Phi(r)}{m^2 + k^2 r^2} + \text{i}mr \partial_r \left( \frac{W(r)}{m^2 + k^2 r^2} \right) \right\} u_r = 0, \quad (5)
\]

where we denote

\[
\partial_r^* = \partial_r + \frac{1}{r}, \quad \Phi(r) = 2\Omega(r)W(r). \quad \text{(Rayleigh function)}
\]

Remark: Eq. (5) has the same structure as the Taylor-Goldstein equation!
Stability of Rankine’s Vortex

In the (very) particular case of Rankine’s vortex, Lord Kelvin observed that the solution of the eigenvalue equation takes the explicit form

\[
    u_z(r) = \begin{cases} 
    A I_m(\beta r) & \text{if } r < 1, \\
    B K_m(kr) & \text{if } r > 1,
    \end{cases}
\]

where \( A, B \in \mathbb{C} \) and \( I_m, K_m \) are modified Bessel functions of order \( m \).

Matching conditions at \( r = 1 \) lead to the dispersion relation

\[
    \frac{I'_m(\beta)}{\beta I_m(\beta)} + \frac{2im}{(s+im)\beta^2} = \frac{K'_m(k)}{kK_m(k)}.
\]

This relation is difficult to study for general values of \( s \in \mathbb{C} \).

However, on the imaginary axis, Kelvin found countably many solutions \( s_n \in i\mathbb{R} \) which correspond to vibration modes of the columnar vortex.

He concluded, somewhat hastily, that Rankine’s vortex is linearly stable.
The Axisymmetric Case

When \( m = 0 \), we have \( \gamma(r) = s \) and the eigenvalue equation (5) reduces to

\[
- \partial_r \partial^*_r u_r + k^2 \left( 1 + \frac{\Phi(r)}{s^2} \right) u_r = 0,
\]

which is reminiscent of our analysis of the Rayleigh-Taylor instability. Eigenfunctions should satisfy the Rayleigh identity

\[
\int_0^\infty \left\{ |\partial^*_r u_r|^2 + k^2 \left( 1 + \frac{\Phi(r)}{s^2} \right) |u_r|^2 \right\} r \, dr = 0.
\]

Separating real and imaginary parts, we obtain:

**Rayleigh’s stability criterion**: if the Rayleigh function is nonnegative, i.e.

\[
\Phi(r) = 2\Omega(r)W(r) = \frac{1}{r^3} \frac{d}{dr} \left( r^2 \Omega(r) \right)^2 \geq 0,
\]

the vortex is spectrally stable with respect to axisymmetric perturbations.
The Two-dimensional Case

In the two-dimensional case \( k = 0 \), the eigenvalue equation (5) becomes

\[
-\partial_r (r^2 \partial_r^* u_r) + \left( m^2 + \frac{imrW'(r)}{\gamma(r)} \right) u_r = 0 ,
\]

which is reminiscent of our analysis of two-dimensional shear flows. Eigenfunctions should therefore satisfy the identity

\[
\int_0^\infty \left\{ |\partial_r (ru_r)|^2 + \left( m^2 + \frac{imrW'(r)}{\gamma(r)} \right) |u_r|^2 \right\} r \, dr = 0 .
\]

Taking the imaginary part, we deduce that

\[
m \Re(s) \int_0^\infty \frac{W'(r)}{\gamma(r)^2} |u_r|^2 r^2 \, dr = 0 .
\]

\( \Rightarrow \) **Sufficient condition** for spectral stability in the two-dimensional case:

\[
W'(r) \neq 0 \quad \text{(monotonicity of the vorticity profile)}
\]
Howard and Gupta’s approach

In the general case where \( m \neq 0 \) and \( k \neq 0 \), partial results can be obtained using the approach of Howard and Gupta (1962):

**Proposition 1:** Assume that \( W'(r) \leq 0 \) and \( W(r) \to 0 \) as \( r \to \infty \). Then

i) The eigenvalue equation (5) has no nontrivial solution with \( s = m(a - ib) \) if

\[
a \neq 0 \quad \text{and} \quad b(1 - b) \leq 0.
\]

ii) The eigenvalue equation (5) has no nontrivial solution with \( \text{Re}(s) \neq 0 \) if

\[
\frac{k^2}{m^2} \frac{\Phi(r)}{\Omega'(r)^2} \geq \frac{1}{4} \quad \text{for all} \quad r > 0.
\]

**Remark:** The “Richardson” condition (6)

- is never satisfied for the Lamb-Oseen vortex;
- is satisfied for the Kaufmann-Scully vortex iff \( m^2 \leq 4k^2 \).
On the hydrodynamic and hydromagnetic stability of swirling flows

By LOUIS N. HOWARD
Mathematics Department, Massachusetts Institute of Technology†

AND A. S. GUPTA
Mathematics Department, Indian Institute of Technology, Kharagpur†

(Received 10 May 1962)

Some general stability criteria for non-dissipative swirling flows are derived, and extended to the case of an electrically conducting fluid in the presence of axial magnetic field and current. In particular it is shown that the analogy between a rotating and a stratified fluid holds in this case, and that an important determinant of stability is a ‘Richardson number’ based on the analogue of the density gradient and the shear in the axial flow.

From Section 3: “Remarks on the non-axisymmetric case”

The overall conclusion of this consideration of the non-axisymmetric case is thus essentially negative: the methods used to derive the Richardson number and semicircle results in the axisymmetric case reproduce the known results of Rayleigh for two-dimensional perturbations and pure axial flow, but seem to give very little more. In fact the present situation with regard to non-axisymmetric perturbations seems to be very unsatisfactory from a theoretical point of view.
Localization of the Unstable Spectrum

spectral parameter $s = m(a - ib)$

Kelvin modes $\rightarrow$ no spectrum

essential spectrum $\rightarrow$

$\Omega(r)$

$r$ $\bar{r}$

$b$

$|a|$
Our assumptions are formulated in terms of the vorticity profile $W$:

**Assumption H1:** The vorticity profile $W : [0, \infty) \rightarrow \mathbb{R}_+$ is a $C^1$ function such that $W'(0) = 0$, $W'(r) < 0$ for all $r > 0$, and

$$\Gamma := \int_0^\infty W(r) r \, dr < \infty.$$

**Remark:** If $\Omega$ denotes the angular velocity, we then have

$$\Omega(r) = \frac{1}{r^2} \int_0^r W(s)s \, ds,$$

and

$$\Phi(r) = 2\Omega(r)W(r) \geq 0.$$

**Assumption H2:** The Richardson function $J : (0, \infty) \rightarrow \mathbb{R}_+$ defined by

$$J(r) = \frac{\Phi(r)}{\Omega'(r)^2}, \quad r > 0,$$

satisfies $J'(r) < 0$ for all $r > 0$ and $rJ'(r) \rightarrow 0$ as $r \rightarrow \infty$. 
The Main Result

Given $m \in \mathbb{Z}$ and $k \in \mathbb{R}$, we introduce the enstrophy space

$$X_{m,k} = \left\{ \omega \in L^2(\mathbb{R}^+, r \, dr)^3 \left| \frac{1}{r} \partial_r (r \omega_r) + \frac{\text{im}}{r} \omega_\theta + ik \omega_z = 0 \right\},$$

equipped with the norm

$$\|\omega\|_{L^2}^2 = \int_0^\infty |\omega(r)|^2 r \, dr,$$

where $|\omega|^2 = |\omega_r|^2 + |\omega_\theta|^2 + |\omega_z|^2$.

**Theorem 1:** Consider a columnar vortex with a vorticity profile $W$ satisfying assumptions H1, H2 above. Given $m \in \mathbb{Z}$ and $k \neq 0$, let $\mathcal{L}_{m,k}$ be the generator of the linearized evolution (3). Then the spectrum of $\mathcal{L}_{m,k}$ in the enstrophy space $X_{m,k}$ satisfies

$$\sigma(\mathcal{L}_{m,k}) \subset i \mathbb{R}.$$

Theorem 1 asserts that all eigenvalues of the linearized operator $\mathcal{L}_{m,k}$ outside the essential spectrum are on the imaginary axis (Kelvin's vibration modes).
Spectral Stability

In addition to Theorem 1, one can show that, for any \( s \in \mathbb{C} \) with \( \text{Re}(s) \neq 0 \),

\[
\sup_{m \in \mathbb{Z}} \sup_{k \in \mathbb{Z} \setminus k_0} \|(s - \mathcal{L}_{m,k})^{-1}\| < \infty, \quad \text{for any} \quad k_0 > 0.
\]

As a consequence, if we introduce the space

\[
\dot{L}^2_{\text{per},h} = \left\{ \omega \in L^2(\mathbb{R}^2 \times T_h)^3 \mid \text{div}\,\omega = 0, \quad \int_0^h \omega(x_1, x_2, x_3) \, dx_3 = 0 \right\},
\]

where \( T_h = \mathbb{R}/(\mathbb{Z}h) \) and \( h > 0 \) is the vertical period, we have as a corollary:

**Theorem 2:** Under the assumptions of Theorem 1, let \( \mathcal{L} \) denote the linearized operator (1). Then, for any \( h > 0 \), the spectrum of \( \mathcal{L} \) in the space \( \dot{L}^2_{\text{per},h} \) satisfies

\[
\sigma(\mathcal{L}) = i\mathbb{R}.
\]

According to Theorem 2, columnar vortices satisfying H1, H2 are spectrally stable with respect to perturbations with no particular symmetry.
A Glimpse of the Proof

Given $m \in \mathbb{Z}$ and $k \in \mathbb{R}$, we recall that

$$
\mathcal{L}_{m,k}\omega = \begin{pmatrix}
-\text{i}m\Omega(r)\omega_r + W(r)\text{i}u_r \\
-\text{i}m\Omega(r)\omega_\theta + W(r)\text{i}u_\theta + r\Omega'(r)\omega_r \\
-\text{i}m\Omega(r)\omega_z + W(r)\text{i}u_z - W'(r)u_r
\end{pmatrix} = A_m\omega + B_{m,k}\omega.
$$

Proposition 2: Fix $m \in \mathbb{Z}$ and $k \neq 0$.

1) The linear operator $A_m$ defined above is bounded in $X_{m,k}$ with spectrum

$$
\sigma(A_m) = \left\{ z \in \mathbb{C} \mid z = -\text{i}mb \text{ for some } b \in [0, 1] \right\},
$$

which is purely continuous if $m \neq 0$ and reduces to a single eigenvalue if $m = 0$.

2) The linear operator $B_{m,k}$ defined above is compact in $X_{m,k}$.

It follows that $\sigma(A_m)$ is the essential spectrum of $\mathcal{L}_{m,k}$. The rest of the spectrum consists of isolated eigenvalues with finite multiplicity, which can accumulate only on the essential spectrum.
The Homotopy Argument

We fix henceforth $m \in \mathbb{Z}$ and $k \neq 0$.

By proposition 1, the linear operator $L_{m,k}^1$ associated with the vorticity profile 

$$W_1(r) = \frac{2}{(1 + 4k^2r^2/m^2)^2} = \frac{2m^4}{(m^2 + 4k^2r^2)^2},$$

has no eigenvalue outside the imaginary axis.

Given another profile $W_0$ satisfying assumption H1, we consider the family 

$$W_t = (1 - t)W_0 + tW_1, \quad t \in [0, 1],$$

which interpolates between $W_0$ and the reference profile $W_1$.

All isolated eigenvalues of the linear operator $L_{m,k}^t$ depend continuously on $t$, and must therefore merge into the essential spectrum as $t$ varies from 0 to 1.

Our strategy is to show that such a merger is impossible!
Assume that \( s_n = m(a_n - ib_n) \) is a sequence of eigenvalues of the linearized operator \( L_{m,k} \) such that
\[
\begin{align*}
an &> 0, \quad 0 < b_n < 1, \quad a_n \to 0, \quad b_n \to \bar{b} \in [0,1].
\end{align*}
\]

Define \( \bar{r} \in [0,\infty] \) such that \( \Omega(\bar{r}) = \bar{b} \), see the figure on page 23.

Using the Richardson function \( J(r) = \Phi(r)/\Omega'(r)^2 \), we distinguish several cases:

**Case 1:** \( J(\bar{r}) > m^2/(4k^2) \)

Howard and Gupta’s identity for the eigenfunction \( u_n(r) = \gamma_n(r)^{1/2}v_n(r) \):
\[
\int_0^\infty \left\{ \mathcal{A}(r) |\partial_r^* v_n|^2 + |v_n|^2 + \frac{\mathcal{A}(r)\Omega'(r)^2}{a_n^2 + (\Omega(r) - b_n)^2} \left( \frac{k^2}{m^2} J(r) - \frac{1}{4} \right) |v_n|^2 \right\} r \, dr = 0,
\]
where \( \mathcal{A}(r) = r^2/(m^2 + k^2 r^2) \).

The integrand becomes singular near \( r = \bar{r} \) as \( n \to \infty \), and the leading terms are all positive, which gives a contradiction for large \( n \).
Case 2: \( J(\bar{r}) < m^2/(4k^2) \)

The (suitably normalized) eigenfunctions \( u_n(r) \) satisfy

\[
u_n(r) \xrightarrow{n \to \infty} u_\infty(r) = \alpha_- \varphi_-(r) + \alpha_+ \varphi_+(r), \quad \text{for all } r \neq \bar{r},
\]

where \( \alpha_-, \alpha_+ \in \mathbb{C} \) and \( \varphi_-, \varphi_+ \) are solutions of the limiting equation

\[
-\partial_r (A(r) \partial_r^* u_r) + \left\{ 1 - \frac{k^2}{m^2} \frac{A(r) \Phi(r)}{(\Omega(r) - b)^2} + \frac{r}{\Omega(r) - b} \partial_r \left( \frac{W(r)}{m^2 + k^2 r^2} \right) \right\} u_r = 0,
\]

which is real valued and has a regular singular point at \( r = \bar{r} \).

One has \( \varphi_{\pm}(r) \approx (r - \bar{r})^{d_{\pm}} \) as \( r \to \bar{r} \), where the exponents satisfy

\[
d(d - 1) + \frac{k^2}{m^2} J(\bar{r}) = 0, \quad 0 < d_- < \frac{1}{2} < d_+ < 1. \quad (7)
\]

Under assumption H2, both functions \( \varphi_-, \varphi_+ \) are unbounded as \( r \to \infty \), hence both coefficients \( \alpha_-, \alpha_+ \) must be nonzero. In view of (7), this implies that \( u_\infty(r) \) cannot be real (up to a global phase factor) on \((0, \bar{r})\), which is a contradiction.

The remaining cases can be treated using (more or less) similar ideas.
Some Open Questions

• Is assumption H2 necessary or just technical? Are columnar vortices spectrally stable under the sole assumption that the vorticity profile is a monotone function of the radius?

• Are columnar vortices linearly stable under assumptions H1, H2? Can one prove that, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\|e^{tc}\| \leq C e^{\varepsilon |t|}, \quad \text{for all } t \in \mathbb{R}?$$

• Are columnar vortices nonlinearly stable in any reasonable sense? Can we use a variational method to prove nonlinear stability, as in the two-dimensional case?

• Do the results above extend to the slightly viscous case? Is the Lamb-Oseen vortex a stable self-similar solution of the Navier-Stokes equations when three-dimensional perturbations are considered?
Selected References


Thank you for your attention!