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joint work with
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Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \]

Examples:
- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:
- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

\[ U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

Examples:
- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:
- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows
Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \quad \text{or} \quad U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

- **Challenges**: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, asymptotic regimes, etc.) are essential in many applications;

- **Goal**: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]
Well-Balanced (WB) Methods

\[ U_t + f(U)_x + g(U)_y = S(U) \]

- In many physical applications, solutions of the system are small perturbations of the steady states;
- These perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;

Goal:
- to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions;
- perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.
Asymptotic Preserving (AP) Methods

\[ U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

• Solutions of many hyperbolic systems reveal a multiscale character and thus their numerical resolution presence some major difficulties;

• Such problems are typically characterized by the occurrence of a small parameter by \( 0 < \varepsilon \ll 1 \);

• The solutions show a nonuniform behavior as \( \varepsilon \to 0 \);

• the type of the limiting solution is different in nature from that of the solutions for finite values of \( \varepsilon > 0 \).

Goal:

• asymptotic passage from one model to another should be preserved at the discrete level;

• for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as \( \varepsilon \to 0 \).
Finite-Volume Methods – 1-D

\[ U_t + f(U)_x = S \quad \left( = \frac{1}{\epsilon} S \right) \]

- \[ \bar{U}_k^n \approx \frac{1}{\Delta y} \int_{C_k} U(y, t^n) \, dy \] : cell averages over \( C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \)

- Semi-discrete FV method:

\[ \frac{d}{dt} \bar{U}_j(t) = -\frac{\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}_j \]

- \( \mathcal{F}_{j+\frac{1}{2}}(t) \): numerical fluxes
- \( \bar{S}_j \): quadrature approximating the corresponding source terms

- Central-Upwind (CU) Scheme:

\[
\{\overline{U}_j(t)\} \rightarrow \tilde{U}(\cdot, t) \rightarrow \left\{U^E_j, W_j(t)\right\} \rightarrow \left\{\mathcal{F}_{j+\frac{1}{2}}(t)\right\} \rightarrow \{\overline{U}_j(t + \Delta t)\}
\]

(Discontinuous) piecewise-linear reconstruction:

\[
\tilde{U}(y, t) := \overline{U}_j(t) + (U_x)_j(x - x_j), \quad x \in C_j
\]

It is conservative, second-order accurate, and non-oscillatory provided the slopes, \{(U_y)_k\}, are computed by a nonlinear limiter

Example — Generalized Minmod Limiter

\[
(U_y)_j = \text{minmod}\left(\theta \frac{\overline{U}_j - \overline{U}_{j-1}}{\Delta x}, \frac{\overline{U}_{j+1} - \overline{U}_{j-1}}{2\Delta x}, \theta \frac{\overline{U}_{j+1} - \overline{U}_j}{\Delta x}\right)
\]

where

\[
\text{minmod}(z_1, z_2, \ldots) := \begin{cases} 
\min_j\{z_j\}, & \text{if } z_j > 0 \ \forall j, \\
\max_j\{z_j\}, & \text{if } z_j < 0 \ \forall j, \\
0, & \text{otherwise},
\end{cases}
\]

and \(\theta \in [1, 2]\) is a constant
\[
\{\overline{U}_j(t)\} \rightarrow \widetilde{U}(\cdot, t) \rightarrow \left\{U_j^{E,W}(t)\right\} \rightarrow \left\{\mathcal{F}_{j+\frac{1}{2}}(t)\right\} \rightarrow \{\overline{U}_j(t + \Delta t)\}
\]

\(U_j^E\) and \(U_j^W\) are the point values at \(x_{j+\frac{1}{2}}\) and \(x_{j-\frac{1}{2}}\):

\[
\widetilde{U}(y,t) = \overline{U}_j + (U_x)_j(x - x_j), \quad x \in C_j
\]

\[
U_j^E := \overline{U}_j + \frac{\Delta x}{2}(U_x)_j
\]

\[
U_j^W := \overline{U}_j - \frac{\Delta x}{2}(U_x)_j
\]
\[ \{ \overline{U}_j(t) \} \rightarrow \tilde{U}(\cdot, t) \rightarrow \{ U^{E,W}_j(t) \} \rightarrow \{ F_{j+\frac{1}{2}}(t) \} \rightarrow \{ \overline{U}_j(t + \Delta t) \} \]

\[
\frac{d}{dt} \overline{U}_j = -\frac{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}}{\Delta x} + S_j
\]

where

\[
F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f(U^{E}_j) - a^-_{j+\frac{1}{2}} f(U^{W}_{j+1})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \alpha_{j+\frac{1}{2}} (U^{W}_{j+1} - U^{W}_j)
\]

\[
\alpha_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}
\]

\[a^+_{j+\frac{1}{2}} = \max \{ \lambda(U^{E}_j), \lambda(U^{W}_{j+1}), 0 \}, \quad a^-_{j+\frac{1}{2}} = \min \{ \lambda(U^{E}_j), \lambda(U^{W}_{j+1}), 0 \}\]

2-D extension is dimension-by-dimension
Non Well-Balanced Property – Example

\[ \begin{align*}
\rho_t + q_x &= 0, \\
q_t + f_2(\rho, q)_x &= -s(\rho, q)
\end{align*} \]

For steady-state solution: \( q = \text{Const} \) and \( \rho = \rho(x) \)

Implementing the CU scheme results in

\[
\frac{d\bar{\rho}_j}{dt} = -\frac{1}{\Delta x} \left[ a_{j+\frac{1}{2}}^{+} q_{j}^{E} - a_{j+\frac{1}{2}}^{-} q_{j+1}^{W} - a_{j-\frac{1}{2}}^{+} q_{j-1}^{E} + a_{j-\frac{1}{2}}^{-} q_{j}^{W} + \alpha_{j+\frac{1}{2}} (\rho_{j+1}^{W} - \rho_{j}^{E}) + \alpha_{j-\frac{1}{2}} (\rho_{j}^{W} - \rho_{j-1}^{E}) \right] \neq 0
\]

- The steady state would not be preserved at the discrete level;
- This would also true for the first-order version of the scheme;
- For smooth solutions, the balance error is expected to be of order \((\Delta x)^2\), but a coarse grid solution may contain large spurious waves.
Well-Balanced Methods
1-D 2 × 2 Systems of Balance Laws

\[
\begin{align*}
\rho_t + f_1(\rho, q)_x &= 0, \\
q_t + f_2(\rho, q)_x &= -s(\rho, q),
\end{align*}
\]

**Steady state solution:**

\[
f_1(\rho, q)_x \equiv 0, \quad f_2(\rho, q)_x + s(\rho, q) \equiv 0
\]

or

\[
K := f_1(\rho, q) \equiv \text{Const},
\]
\[
L := f_2(\rho, q) + \int_x s(\rho, q)d\xi \equiv \text{Const} \quad \forall x, t
\]

Numerical Challenges: to exactly balance the flux and source terms, i.e., to exactly preserve the steady states.

How to design a well-balanced scheme?
Well-Balanced Scheme

\[
\begin{align*}
\rho_t + f_1(\rho, q)_x &= 0, \\
q_t + f_2(\rho, q)_x &= -s(\rho, q)
\end{align*}
\]

- Incorporate the source term into the flux:

\[
\begin{align*}
\rho_t + f_1(\rho, q)_x &= 0, \\
q_t + (f_2(\rho, q)_x + R)_x &= 0,
\end{align*}
\]

\[R := \int_x s(\rho, q) d\xi\]

- Rewrite

\[
\begin{align*}
\rho_t + K_x &= 0, \\
q_t + L_x &= 0
\end{align*}
\]

where

\[K := f_1(\rho, q), \quad L := f_2(\rho, q)_x + R\]

- Define

conservative variables \(U = (\rho, q)^T\)
equilibrium variables \(W := (K, L)^T\)
Well-Balanced Scheme

\[ \dot{U}_t + f(U)_x = 0 \]

\[ U = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad f(U) = W := \begin{pmatrix} K \\ L \end{pmatrix} \]

Semi-discrete FV method:

\[ \frac{d}{dt}\overline{U}_j(t) = -\frac{\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t)}{\Delta x} \]

Two major modifications:

• Well-balanced reconstruction – performed on the equilibrium rather than conservative variables:

\[ \{\overline{U}_j(t)\} \rightarrow \tilde{U}(\cdot, t) \rightarrow \left\{W_{j}^{E,W}(t)\right\} \rightarrow \left\{U_{j}^{E,W}(t)\right\} \rightarrow \left\{\mathcal{F}_{j+\frac{1}{2}}(t)\right\} \rightarrow \{\overline{U}_j(t+\Delta t)\} \]

• Well-balanced evolution
Well-Balanced Reconstruction

Given: \( \overline{U}_j(t) = (\overline{\rho}_j, \overline{q}_j)^T \) – cell averages

Need: \( \mathbf{W}_j^{E,W} = (K_j^{E,W}, L_j^{E,W})^T \) – point values, where

\[
K := f_1(\rho, q), \quad L := f_2(\rho, q)x + R, \quad R := \int_s^x s(\rho, q)d\xi
\]

- Compute \( R_j = \int s(\rho, q)d\xi \) by the midpoint quadrature rule and using the following recursive relation:

\[
\begin{align*}
R_{1/2} & \equiv 0, \quad R_j = \frac{1}{2}(R_{j-1/2} + R_{j+1/2}), \\
R_{j+1/2} & = R(x_{j+1/2}) = R_{j-1/2} + \Delta x s(x_j, \overline{\rho}_j, \overline{q}_j)
\end{align*}
\]

- Compute the point values of \( K \) and \( L \) at \( x_j \) from the cell averages, \( \overline{\rho}_j \) and \( \overline{q}_j \):

\[
K_j = f_1(\overline{\rho}_j, \overline{q}_j), \quad L_j = f_2(\overline{\rho}_j, \overline{q}_j) + R_j
\]
Well-Balanced Reconstruction

- Apply the minmod reconstruction procedure to \( \{K_j, L_j\} \) and obtain the point values at the cell interfaces:

\[
K_j^E = K_j + \frac{\Delta x}{2}(K_x)_j, \quad L_j^E = L_j + \frac{\Delta x}{2}(L_x)_j,
\]
\[
K_j^W = K_j - \frac{\Delta x}{2}(K_x)_j, \quad L_j^W = L_j - \frac{\Delta x}{2}(L_x)_j,
\]

- Finally, equipped with the values of \( K_j^{E,W}, L_j^{E,W} \) and \( R_{j \pm \frac{1}{2}} \), solve

\[
K_j^E = f_1(\rho_j^E, q_j^E), \quad L_j^E = f_2(\rho_j^E, q_j^E) + R_{j + \frac{1}{2}},
\]
\[
K_j^W = f_1(\rho_j^W, q_j^W), \quad L_j^W = f_2(\rho_j^W, q_j^W) + R_{j - \frac{1}{2}}
\]

for \( U_j^{E,W} = (\rho_j^{E,W}, q_j^{E,W})^T \).
Well-Balanced Evolution

\[
\frac{d}{dt} \bar{U}_j = -\frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{\Delta x}
\]

where

\[
\mathcal{F}^{(1)}_{j+\frac{1}{2}} = a_{j+\frac{1}{2}}^+ K^E_j - a_{j+\frac{1}{2}}^- K^W_{j+1}
\]

\[
+ \alpha_{j+\frac{1}{2}} \left( \rho^W_{j+1} - \rho^E_j \right) \mathcal{H} \left( \frac{|K_{j+1} - K_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j K_j, K_{j+1}} \right),
\]

\[
\mathcal{F}^{(2)}_{j+\frac{1}{2}} = a_{j+\frac{1}{2}}^+ L^E_j - a_{j+\frac{1}{2}}^- L^W_{j+1}
\]

\[
+ \alpha_{j+\frac{1}{2}} \left( q^W_{j+1} - q^E_j \right) \mathcal{H} \left( \frac{|L_{j+1} - L_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j \{L_j, L_{j+1}\}} \right),
\]
Proof of the Well-Balanced Property

Theorem. The central-upwind semi-discrete schemes coupled with the well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the corresponding steady states exactly.
Example – Gas dynamics with pipe-wall friction

\[
\begin{aligned}
\rho_t + q_x &= 0, \\
q_t + \left( c^2 \rho + \frac{q^2}{\rho} \right)_x &= -\mu \frac{q}{\rho} |q|,
\end{aligned}
\]

- \( \rho(x, t) \) is the density of the fluid
- \( u(x, t) \) is the velocity of the fluid
- \( q(x, t) \) is the momentum
- \( \mu > 0 \) is the friction coefficient (divided by the pipe cross section)
- \( c > 0 \) is the speed of sound

Equilibrium variables:

\[
K(x, t) = q(x, t) \quad L(x, t) = \left( c^2 \rho + \frac{q^2}{\rho} \right)(x, t) + R(x, t),
\]

\[
R(x, t) = \int_x^x \mu \frac{q(\xi, t)}{\rho(\xi, t)} |q(\xi, t)| \, d\xi
\]

Steady states: \( K \equiv \text{Const}, \quad L \equiv \text{Const} \)
Numerical Tests

• Steady state initial data:

\[ K(x, 0) = q(x, 0) = K^* = 0.15 \quad \text{and} \quad L(x, 0) = L^* = 0.4, \]

in a single pipe \( x \in [0, 1] \)

• Perturbed initial data:

\[ K(x, 0) = K^* + \eta e^{-100(x-0.5)^2}, \quad L(x, 0) = L^* = 0.4, \quad \eta > 0 \]

in a single pipe \( x \in [0, 1] \)

We compare the WB and NWB methods ...
### Numerical Test – Steady state initial data

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.94E-18</td>
<td>7.77E-18</td>
</tr>
<tr>
<td>200</td>
<td>9.71E-19</td>
<td>9.71E-18</td>
</tr>
<tr>
<td>400</td>
<td>1.66E-18</td>
<td>9.57E-18</td>
</tr>
<tr>
<td>800</td>
<td>2.18E-18</td>
<td>1.18E-17</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K$ rate</th>
<th>$L$ rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.29E-06</td>
<td>-</td>
</tr>
<tr>
<td>200</td>
<td>3.30E-07</td>
<td>1.9668</td>
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<tr>
<td>400</td>
<td>8.34E-08</td>
<td>1.9843</td>
</tr>
<tr>
<td>800</td>
<td>2.09E-08</td>
<td>1.9965</td>
</tr>
</tbody>
</table>
Numerical Test – Perturbed initial data

perturbation of q, \( \eta = 10^{-3} \)

- initial state
- WB, N=100
- NWB, N=100
- NWB, N=1600

perturbation of q, \( \eta = 10^{-6} \)

- initial state
- WB, N=100
- NWB, N=100
- NWB, N=3200
Example – 1-D Saint-Venant System of Shallow Water

\[
\begin{aligned}
    & h_t + (hu)_x = 0 \\
    & (hu)_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x = -ghB_x
\end{aligned}
\]

- Steady-state solutions (moving):

\[
hu = \text{Const}, \quad \frac{u^2}{2} + g(h + B) = \text{Const}
\]

- Stationary steady-state solutions (lake at rest):

\[
u = 0, \quad h + B = \text{Const}
\]
Shallow Water Equations

\[
\begin{align*}
    h_t + (hu)_x &= 0 \\
    (hu)_t + (hu^2 + \frac{g}{2}h^2)_x &= -ghB_x
\end{align*}
\]

- **Well-balanced scheme** should exactly balance the flux and source terms so that the steady states are preserved:
  
  - Steady-state solutions *(moving)*:
    
    \[ hu = \text{Const,} \quad \frac{u^2}{2} + g(h + B) = \text{Const} \]
  
  - Stationary steady-state solutions *(lake at rest)*:
    
    \[ u = 0, \quad h + B = \text{Const} \]
Well-Balanced Methods – Some References

- Shallow water models (preserving “lake at rest” steady states):
  - LeVeque (1998) – incorporating the source term into the Riemann solver
  - Jin (2001) – well-balanced source term averaging
  - Perthame, Simeoni (2001) – kinetic scheme
  - Gallouët, Hérard, Seguin (2003) – Roe-type scheme
  - Russo (2005) staggered central scheme
  - Xing, Shu (2005, 2006) – WENO schemes
  - Noelle, Pankratz, Puppo, Natvig (2006) – high-order schemes
  - Lukácová-Medvidová, Noelle, Kraft (2007) FVEG scheme
  - Berthon, Marche (2008) – relaxation schemes
  - Fjordholm, Mishra, Tadmor (2008, 2011) energy stable schemes
  - Abgrall, Audusse, Bristeau, Castro, Chertock, Dawson, Donat, Epshteyn, George, Karni, Klingenberg, Mohammadian, Parés, Ricchiuto, ...
Well-Balanced Methods – Some References

- **Shallow water models (preserving moving steady states):**
  - Russo, Khe (2009, 2010) – staggered central schemes
  - Xing (2014) – discontinuous Galerkin method

- **Shallow water models (positivity preserving schemes):**
  - Perthame, Simeoni (2001) – kinetic scheme
  - Kurgnov, Petrova (2007) – central-upwind scheme with continuous piecewise linear bottom reconstruction
  - Berthon, Marche (2008) – relaxation schemes
  - Bollermann, Noelle, Lukáčová-Medvid’ová (2011) – special time-quadrature for the fluxes
  - Bollermann, Chen, Kurganov, Noelle (2013): well-balanced reconstruction of wet/dry fronts
Shallow Water Equations — Moving Steady States

\[
\begin{align*}
&\begin{cases}
ht + (hu)_x = 0 \\
(hu)_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x = -ghB_x
\end{cases}
\end{align*}
\]

We incorporate the source term in the discharge equation into its flux term:

\[
\begin{align*}
&\begin{cases}
h_t + q_x = 0, \\
q_t + \left( hu^2 + \frac{g}{2}h^2 + R \right)_x = 0,
\end{cases}
\end{align*}
\]

where

\[
K := q, \quad L := hu^2 + \frac{g}{2}h^2 + R, \quad R(x, t) := g \int_x^x h(\xi, t)B_x(\xi) \, d\xi
\]

General (moving-water) steady state can be expressed in terms of \( q \) and \( K \):

\[
K \equiv \text{Const}, \quad L \equiv \text{Const}
\]
**Numerical Tests**

- Three sets of initial and boundary conditions:
  - **Supercritical flow with**
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ h(0, t) = 2, \quad q(0, t) = 24; \]
  - **Subcritical flow with**
    \[ h(x, 0) = 2 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ q(0, t) = 4.42, \quad h(25, t) = 2; \]
  - **Transcritical flow without a shock with**
    \[ h(x, 0) = 0.66 - B(x), \quad q(x, 0) \equiv 0, \]
    \[ q(0, t) = 1.53, \quad h(25, t) = 0.66. \]
- **Discontinuous bottom topography:**
  \[ B(x) = \begin{cases} 
  1, & \text{if } 0.25 \leq x \leq 0.75, \\
  0, & \text{otherwise},
\end{cases} \]
Euler Equations with Gravity

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= -\rho \phi_x \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= -\rho \phi_y \\
E_t + (u(E + p))_x + (v(E + p))_y &= -\rho (u\phi_x + v\phi_y)
\end{align*}
\]

- \( \rho \) is the density
- \( u, v \) are the \( x \)- and \( y \)-velocities
- \( E \) is the total energy
- \( p \) is the pressure; \( E = \frac{p}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2) \)
- \( \phi \) is the gravitational potential
Well-Balanced Methods – Some References

- **Euler equations with gravitational fields:**
  - N. Botta, R. Klein, S. Langenberg, and S. Lützenkirchen (2004) – well-balanced finite-volume methods, which preserve a certain class of steady states for nearly hydrostatic flows
  - Y. Xing and C.-W. Shu (2013) – higher order finite-difference methods for the gas dynamics with gravitation
  - ...
Euler Equations with Gravity

\[
\begin{cases}
\rho t + (\rho u)_x + (\rho v)_y = 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho \phi_x \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho \phi_y \\
E_t + (u(E + p))_x + (v(E + p))_y = -\rho (u\phi_x + v\phi_y)
\end{cases}
\]

Multiply the first (density) equation by \(\phi\) and add to the last (energy) equation to obtain ...

\[
\begin{cases}
\rho t + (\rho u)_x + (\rho v)_y = 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho \phi_x \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho \phi_y \\
(E + \rho \phi)_t + (u(E + \rho \phi + p))_x + (v(E + \rho \phi + p))_y = 0
\end{cases}
\]
Steady States

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= -\rho\phi_x \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= -\rho\phi_y \\
(E + \rho\phi)_t + (u(E + \rho\phi + p))_x + (v(E + \rho\phi + p))_y &= 0
\end{align*}
\]

Plays an important role in modeling model astrophysical and atmospheric phenomena in many fields including supernova explosions, (solar) climate modeling and weather forecasting.

Steady state solution:

\[
\begin{align*}
 u &\equiv 0, \quad v \equiv 0, \quad K_x = p_x + \rho\phi_x \equiv 0, \quad L_y = p_y + \rho\phi_y \equiv 0 \\
 K &:= p + Q, \quad Q(x, y, t) := \int_x^x \rho(\xi, y, t)\phi_x(\xi, y) \, d\xi \\
 L &:= p + R, \quad R(x, y, t) := \int_y^y \rho(x, \eta, t)\phi_y(x, \eta) \, d\eta
\end{align*}
\]
2-D Well-Balanced Scheme

- Incorporate the source term into the flux:

\[
K := p + Q, \quad Q(x, y, t) := \int_{\xi}^{y} \rho(\xi, y, t)\phi_x(\xi, y), d\xi
\]

\[
L := p + R, \quad R(x, y, t) := \int_{\eta}^{y} \rho(x, \eta, t)\phi_y(x, \eta), d\eta
\]

\[
\begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
E + \rho \phi
\end{pmatrix}_t + \begin{pmatrix}
\rho u \\
\rho u^2 + K \\
\rho uv \\
u(E + \rho \phi + p)
\end{pmatrix}_x + \begin{pmatrix}
\rho v \\
\rho uv \\
\rho v^2 + L \\
v(E + \rho \phi + p)
\end{pmatrix}_y = \begin{pmatrix}0 \\ 0 \\ 0 \end{pmatrix}
\]

- Define

  \textbf{conservative variables: } \mathbf{U} := (\rho, \rho u, \rho v, E)^T

  \textbf{equilibrium variables: } \mathbf{W} := (\rho, K, L, E + \rho \phi)^T

- Solve by the well-balanced scheme ...
Well-Balanced Scheme

- Define

  conservative variables: \( U := (h, hu, hv)^T \)

  equilibrium variables: \( W := (u, v, K, L)^T \)

  fluxes in the \( x \)- and \( y \)-directions: \( f(U, B) \) and \( g(U, B) \)

- Assume that at time \( t \) the cell averages are available

\[
\bar{U}_{j,k}(t) := \frac{1}{\Delta x \Delta y} \int \int_{C_{j,k}} U(x, y, t) \, dx \, dy,
\]

- Solve by the well-balanced scheme

\[
\{\bar{U}_{j,k}(t)\} \rightarrow \tilde{U}(\cdot, t) \rightarrow \left\{ W_{j,k}^{E,W,N,S}(t) \right\} \rightarrow \left\{ U_{j,k}^{E,W,N,S}(t) \right\} \\
\rightarrow \left\{ F_{j+\frac{1}{2},k}(t), G_{j,k+\frac{1}{2}}(t) \right\} \rightarrow \{\bar{U}_{j,k}(t + \Delta t)\}
\]
Example — 2-D Isothermal Equilibrium Solution

[Xing, Shu; 2013]

• The ideal gas with $\gamma = 1.4$; domain $[0, 1] \times [0, 1]$
• The gravitational force is $\phi_y = g = 1$
• The steady-state initial conditions are

  \[
  \rho(x, y, 0) = 1.21e^{-1.21y}, \quad p(x, y, 0) = e^{-1.21y}, \quad u(x, y, 0) \equiv v(x, y, 0) \equiv 0
  \]

• Solid wall boundary conditions imposed at the edges of the unit square
A small initial pressure perturbation:

\[ p(x, y, 0) = e^{-1.21y} + \eta e^{-121((x-0.3)^2+(y-0.3)^2)}, \quad \eta = 10^{-3} \]
WB : 50 × 50, 200 × 200

NWB : 50 × 50, 200 × 200
Shallow Water System with Coriolis Force

\[
\begin{aligned}
&\left\{
\begin{array}{l}
    h_t + (hu)_x + (hv)_y = 0 \\
    (hu)_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x + (huv)_y = -ghB_x + fhv \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2} h^2 \right)_x = -ghB_y - fhu
\end{array}
\right.
\end{aligned}
\]

- $h$: water height
- $u, v$: fluid velocity
- $B$: bottom topography
- $g$: gravitational constant
- $f$: Coriolis parameter; $f \equiv 0 \implies$ Saint Venant system of shallow water.
Steady States

\[
\begin{cases}
ht + (hu)_x + (hv)_y = 0 \\
(hu)_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x + (huv)_y = -ghB_x + fhv \\
(hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2} h^2 \right)_y = -ghB_y - fhv
\end{cases}
\]

- “Lake at rest”: \( u \equiv 0, \ v \equiv 0, \ h + B \equiv \text{Const} \)
- Geostrophic equilibria ("jets in the rotational frame") are both stationary and constant along the streamlines:

\[
\begin{align*}
&u \equiv 0, \ vy \equiv 0, \ hy \equiv 0, \ By \equiv 0, \ K \equiv \text{Const} \\
&v \equiv 0, \ ux \equiv 0, \ hx \equiv 0, \ Bx \equiv 0, \ L \equiv \text{Const}
\end{align*}
\]

Here,

\[
K := g(h + B - V) \quad \text{and} \quad L := g(h + B + U)
\]

are the potential energies defined through the primitives of the Coriolis force \((U, V)^T\):

\[
V_x := \frac{f}{g}v \quad \text{and} \quad U_y := \frac{f}{g}u
\]
Example — 2-D Stationary Vortex


\[ h(r, 0) = 1 + \varepsilon^2 \begin{cases} 
\frac{5}{2} (1 + 5\varepsilon^2) r^2 \\
\frac{1}{10} (1 + 5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2} r^2 + \varepsilon^2 (4 \ln (5r) + \frac{7}{2} - 20r + \frac{25}{2} r^2) \\
\frac{1}{5} (1 - 10\varepsilon + 4\varepsilon^2 \ln 2), 
\end{cases} \]

\[ u(x, y, 0) = -\varepsilon y \Upsilon(r), \quad v(x, y, 0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 
5, & r < \frac{1}{5} \\
\frac{2}{r} - 5, & \frac{1}{5} \leq r < \frac{2}{5} \\
0, & r \geq \frac{2}{5}, 
\end{cases} \]

Domain: \([-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2} \]

Boundary conditions: a zero-order extrapolation in both \(x\)- and \(y\)-directions

Parameters: \( B \equiv 0, \quad f = 1/\varepsilon \) and \( g = 1/\varepsilon^2 \) with \( \varepsilon = 0.05 \)
Asymptotic Perserving Methods
Low Froude Number Flow Regimes

\[
\begin{aligned}
    & h_t + (hu)_x + (hv)_y = 0 \\
    & (hu)_t + \left( hu^2 + \frac{gh^2}{2} \right)_x + (huv)_y = -ghB_x + fhv \\
    & (hv)_t + (huv)_x + \left( hv^2 + \frac{gh^2}{2} \right)_x = -ghB_y - fhu
\end{aligned}
\]

- \( h \): water height
- \( u, v \): fluid velocity
- \( g \): gravitational constant
- **Recall from the previous example:**
  - \( B \equiv 0 \) – bottom topography
  - \( f = 1/\varepsilon \) – Coriolis parameter
Dimensional Analysis

Introduce

\[ \hat{x} := \frac{x}{\ell_0}, \quad \hat{y} := \frac{y}{\ell_0}, \quad \hat{h} := \frac{h}{h_0}, \quad \hat{u} := \frac{u}{w_0}, \quad \hat{v} := \frac{v}{w_0}. \]

Substituting them into the SWE and dropping the hats in the notations, we obtain the dimensionless form:

\[
\begin{align*}
\left\{
\begin{array}{l}
h_t + (hu)_x + (hv)_y = 0, \\
(hu)_t + \left( hu^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_x + (huv)_y = \frac{1}{\varepsilon} hv, \\
(hv)_t + (huv)_x + \left( hv^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_y = -\frac{1}{\varepsilon} hu,
\end{array}
\right.
\end{align*}
\]

in which

\[ \text{Fr} := \frac{w_0}{\sqrt{gh_0}} = \varepsilon \]

is the reference Froude number.
Explicit Discretization

Eigenvalues of the flux Jacobian:

\[
\begin{align*}
\{ u \pm \frac{1}{\varepsilon} \sqrt{h}, u \} \quad \text{and} \quad \{ v \pm \frac{1}{\varepsilon} \sqrt{h}, v \}
\end{align*}
\]

This leads to the CFL condition

\[
\Delta t_{\text{expl}} \leq \nu \cdot \min \left( \frac{\Delta x}{\max_{u,h} \{ |u| + \frac{1}{\varepsilon} \sqrt{h} \}}, \frac{\Delta y}{\max_{v,h} \{ |v| + \frac{1}{\varepsilon} \sqrt{h} \}} \right) = \mathcal{O}(\varepsilon \Delta_{\text{min}}).
\]

where \( \Delta_{\text{min}} := \min(\Delta x, \Delta y) \)

- \( 0 < \nu \leq 1 \) is the CFL number
- Numerical diffusion: \( \mathcal{O}(\lambda_{\text{max}} \Delta x) = \mathcal{O}(\varepsilon^{-1} \Delta x) \).
- We must choose \( \Delta x \approx \varepsilon \) to control numerical diffusion and the stability condition becomes

\[
\Delta t = \mathcal{O}(\varepsilon^2)
\]
Low Froude Number Flows

Low Froude number regime \((0 < \varepsilon \ll 1) \Rightarrow\) very large propagation speeds

Explicit methods:
- very restrictive time and space discretization steps, typically proportional to \(\varepsilon\) due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:
- uniformly stable for \(0 < \varepsilon < 1\);
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of \(\varepsilon\)
Some References

- Harlow, Welch; 1965
- Chorin; 1967
- Harlow, Amsden; 1971
- Klainerman, Majda; 1981
- Turkel; 1987
- Abarbanel, Duth, Gottlieb; 1989
- Gustafsson, Stoor; 1991
- Klein; 1995
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- Guillard, Viozat; 1999
- Guillard, Murrone; 2004
- Kadioglu, Sussman, Osher, Wright, Kang; 2005
Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991], [Golse, Jin, Levermore; 1999].

Idea:
- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as $\varepsilon \to 0$.

Figure 7: Properties of AP-schemes

0.4 Outline

The present work is a review of several Asymptotic-Preserving schemes, constructed in the kinetic and fluid framework. Inevitably, the choice of the model problems is related with the author's knowledge and with the concept of providing the reader with the most important features of AP-schemes. These schemes can be designed for several other singularly perturbed problems, that admit asymptotic behaviours/regimes.

An overview of the subject of this manuscript is:
- Chapter 1 deals with the Boltzmann equation in the drift-diffusion limit
- Chapter 2 discusses the Vlasov-Poisson system in the quasi-neutral limit
- Chapter 3 treats the subject of the Vlasov equation in the high-field limit and considering variable Larmor radii
- Chapter 4 introduces an Asymptotic-Preserving scheme for a highly elliptic potential equation
- Chapter 5 deals finally with a highly anisotropic, nonlinear, degenerate parabolic temperature equation.
Though the existing AP schemes work perfectly well for many simpler models, their applicability to the full Euler system is rather limited: They work very well for large ($\varepsilon \sim 1$) and intermediate ($\varepsilon \sim 10^{-1}$) values of $\varepsilon$, but may become inefficient for smaller Mach numbers.
Hyperbolic Flux Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

\[
\begin{align*}
(\alpha) + 2(1 - \alpha)(hu)_x + (1 - \alpha)(hv)_y = 0, \\
(\alpha) + (hu^2 + \frac{1}{2} h^2 - a(t)h) + (hu) + \frac{1}{\varepsilon^2} h_x = \frac{1}{\varepsilon} hv, \\
(\alpha) + (hv^2 + \frac{1}{2} h^2 - a(t)h) + \frac{1}{\varepsilon^2} h_y = -\frac{1}{\varepsilon} hu.
\end{align*}
\]

This system can be written in the following vector form:

\[
U_t + \tilde{F}(U)_x + \tilde{G}(U)_y + \hat{F}(U)_x + \hat{G}(U)_y = S(U)
\]

non-stiff terms | stiff terms | source terms

How to choose parameters \(\alpha\) and \(a(t)\)?
Hyperbolic Flux Splitting

\[ U_t + \tilde{F}(U)_x + \tilde{G}(U)_y + \hat{F}(U)_x + \hat{G}(U)_y = S(U) \]

non-stiff terms  
nonlinear part  
stiff terms  
source terms  
linear part

Need to ensure: \( U_t + \tilde{F}(U)_x + \tilde{G}(U)_y = 0 \) is both nonstiff and hyperbolic

Eigenvalues of the Jacobians \( \partial \tilde{F}/\partial U \) and \( \partial \hat{G}/\partial U \): 

\[
\left\{ \begin{array}{l}
u \pm \sqrt{(1 - \alpha)v^2 + \alpha \frac{h - a(t)}{\varepsilon^2}}, \quad v \\
u \pm \sqrt{(1 - \alpha)u^2 + \alpha \frac{h - a(t)}{\varepsilon^2}}, \quad u
\end{array} \right. 
\]

We then take 

\[ \alpha = \varepsilon^2 \quad \text{and} \quad a(t) = \min_{(x,y)\in\Omega} h(x, y, t) \]
Discretization of the Split System

\[ U^{n+1} = U^n + \Delta t \left( \tilde{F}(U)_x^n + \Delta t \tilde{G}(U)_y^n \right) \]

nonlinear part, explicit

\[ + \hat{F}(U)_{x}^{n+1} + \hat{G}(U)_{y}^{n+1} = S(U)^{n+1} \]

linear part, implicit

- Nonstiff nonlinear part is treated using the second-order central-upwind scheme
- Stiff linear part reduces to a linear elliptic equation for \( h^{n+1} \) and straightforward computations of \( (hu)^{n+1} \) and \( (hv)^{n+1} \)

\[
\Delta t \leq \nu \cdot \min \left( \frac{\Delta x}{\max_{u,h} \left\{ |u| + \sqrt{(1 - \alpha)u^2 + \alpha \frac{h - a(t)}{\varepsilon^2}} \right\}}, \frac{\Delta y}{\max_{v,h} \left\{ |v| + \sqrt{(1 - \alpha)v^2 + \alpha \frac{h - a(t)}{\varepsilon^2}} \right\}} \right)
\]
Proof of the AP Property

**Theorem.** The proposed hyperbolic flux splitting method coupled with the described fully discrete scheme is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number $\varepsilon \to 0$.

**Remark.** In practice, the fully discrete scheme is both second-order accurate in space and time as we increase a temporal order of accuracy to the second one by implementing a two-stage globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2). The proof holds as well.
Example — 2-D Stationary Vortex


\[ h(r, 0) = 1 + \varepsilon^2 \begin{cases} 
\frac{5}{2} (1 + 5\varepsilon^2) r^2 \\
\frac{1}{10} (1 + 5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2} r^2 + \varepsilon^2 (4 \ln(5r) + \frac{7}{2} - 20r + \frac{25}{2} r^2) \\
\frac{1}{5} (1 - 10\varepsilon + 4\varepsilon^2 \ln 2),
\end{cases} \]

\[ u(x, y, 0) = -\varepsilon y \Upsilon(r), \quad v(x, y, 0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 
5, & r < \frac{1}{5} \\
\frac{2}{r} - 5, & \frac{1}{5} \leq r < \frac{2}{5} \\
0, & r \geq \frac{2}{5};
\end{cases} \]

Domain: \([-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2}\]

Boundary conditions: a zero-order extrapolation in both \(x\)- and \(y\)-directions

Numerical Tests:

- Experimental order of convergence
- Comparison of non-AP and AP methods for various values of \(\varepsilon\)
Experimental order of convergence

$L^\infty$-errors for $h$ computed using the AP scheme on several different grids for $\varepsilon = 0.1$ (left) and $10^{-3}$
Comparison of non-AP and AP methods, $\varepsilon = 1$
Comparison of non-AP and AP methods, $\varepsilon = 0.1$
Comparison of non-AP and AP methods, $\varepsilon = 0.01$
## Comparison of non-AP and AP methods, CPU times

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\varepsilon = 1$</th>
<th>$\varepsilon = 0.1$</th>
<th>$\varepsilon = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AP</td>
<td>Explicit</td>
<td>AP</td>
</tr>
<tr>
<td>40 x 40</td>
<td>0.18 s</td>
<td>0.16 s</td>
<td>0.06 s</td>
</tr>
<tr>
<td>80 x 80</td>
<td>1.57 s</td>
<td>1.32 s</td>
<td>0.29 s</td>
</tr>
<tr>
<td>200 x 200</td>
<td>24.11 s</td>
<td>21.36 s</td>
<td>5.36 s</td>
</tr>
</tbody>
</table>
Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$

Smaller times: $200 \times 200$, larger times: $500 \times 500$
LIFE IS LIKE MATH

IF IT GOES TOO EASY SOMETHING IS WRONG
THANK YOU!