Lecture 1
Consumer Choice

We'll begin with a few remarks about convexity. Convexity assumptions are ubiquitous in economics, usually thought of as summarizing some form of the "law of diminishing returns".

Definition 1.1. Let V be a vector space (it will usually be finite dimensional for us). A subset $S \subseteq V$ is *convex* if whenever $v_1, \ldots, v_n \in S$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_i \geqslant 0$ and $\sum \lambda_i = 1$, we have $\sum \lambda_i v_i \in S$. A function $f \colon V \to \mathbb{R}$ is *convex* if the set $\{(v, \lambda) : f(v) \geqslant \lambda\} \subseteq V \oplus \mathbb{R}$ is convex. The function f is *concave* if $-f$ is convex.

Let $S \subseteq V$ be convex with nonempty interior (we assume that V is a LCTVS, in the infinite dimensional case). Let $x \in \partial S$ be a boundary point of S. Then the Hahn-Banach theorem provides a continuous linear functional $\phi \in V^*$ such that $\phi(x) = 1$ and such that S is contained entirely in one of the half-spaces $\{v : \phi(v) \leqslant 1\}$ or $\{v : \phi(v) \geqslant 1\}$. This is called a *support hyperplane* for S at x. If ∂S is smooth at x then a support hyperplane is unique and is equal to the tangent hyperplane in the usual sense. In general support hyperplanes are not unique (e.g. consider a cube).

Let f be a smooth function on V. The *Hessian* of f is the symmetric bilinear form $V \otimes V \to \mathbb{R}$ defined by the matrix of second derivatives.

Proposition 1.2. *A smooth function f is convex if and only if the Hessian $D^2 f(x)$ is negative semidefinite for each $x \in V$.*

Proof. Begin by considering the 1-dimensional case.

□

Proposition 1.3. *Let $f \colon V \to \mathbb{R}$ be a concave function. Then the sets $\{v \in V : f(v) \geqslant \lambda\}$ (for fixed λ) are convex.* □

Economics is about rational decision-making constrained by budgets. Imagine that you are a consumer wandering through Wal-Mart. You have a certain sum of money in your pocket, and you want to spend it as "efficiently" as possible. How do you do this?

Let $V = \mathbb{R}^N$, where N is the number of different things that Wal-Mart sells. Each of these things is called a *commodity*, and a point of V (which might represent the contents of your shopping cart as you head to the checkout) is a *bundle of* *commodities* or just a *bundle*. V is in fact an ordered vector space, with positive cone given by $\{x : x_i \geqslant 0 \forall i\}$, and usually we require that a bundle lie in the positive cone (Wal-Mart won't allow you to trade 10,000 rolls of toilet paper for a flatscreen TV). Notice that our assumption that each point in V^+ is a bundle means we are assuming that all goods are infinitely divisible. In reality this is of course false (you can't buy half a TV) but it may be a usable approximation when large numbers of similar consumers are involved.

A *price vector* is an element of (the positive cone of) the dual space V^*. A price therefore assigns a positive real number to every bundle, which is called the *cost* of the bundle. Most shoppers operate under a *budget constraint*: they must choose their preferred bundle of commodities subject to $p(x) \leqslant R$, where p is some fixed price vector and R is their total budget.

How is this choice made? 19th-century utilitarian philosophers argued that the consumer derives *utility* from his/her purchases, and that the consumer will act to maximize *total utility* within budget constraints. What kind of thing should utility be? Imagine that we purchase just one kind of good (ie V is 1-dimensional). It is (perhaps) natural to suppose that the utility $U(x)$ is an *increasing* function of x (the more x we have, the more satisfying it is) but that it is *concave* (this is the "law of diminishing returns"). If the total utility of a bundle were the sum of the utilities of the component commodities, the total utility function would be defined on V^+, increasing, and concave. We'll generally make these assumptions about utility — of course there are other ways for such functions to arise as well as from the sums of componentwise utilities of individual commodities. Normally assume also that U is *strictly* convex on the strictly positive cone of V.

Remark 1.4. Utility is not observable (there is no utility-meter you can bolt to my skull, to measure how much utility I'm getting out of a given bundle.) This is a fundamental problem. One can attempt to evade this by paying attention not to utility itself but to the foliation induced on V^+ by the level sets of U. These level sets are called *indifference hypersurfaces*, and the foliation is the *indifference foliation* — the idea being that if two bundles lie on the same indifference surface, the consumer will not prefer either one to the other. All predictions of the theory should be the same for two utility functions having the same indifference foliation. This amounts to an invariance under change of coordinates in utility-space.

Remarks on "revealed preference".

There are now two types of extremum problems that we can consider. Let a price-vector p be given

(a) The consumer may be given a budget R, and may try to maximize utility $U(x)$

subject to the budget constraint $px \leqslant R$. Let the (unique) point maximizing utility (i.e. preferred to any other feasible point) be denoted $\zeta(p, R)$.

(b) The consumer may specify a level of utility S and may attempt to minimize the cost px subject to the utility constraint $U(x) \geqslant S$. Let the (unique) point minimizing cost be denoted $\xi(p, S)$.

In either of these situations the extreme point x_0 has the property that the budget hyperplane $px = R$ passing through x_0 is tangent to the indifference hypersurface $U(x) = S$ passing through x_0. To put this another way, the differential $dU(x_0)\colon V \to \mathbb{R}$ is a multiple λp of the price vector. (If one formulates this as a constrained optimization problem in the manner of Calculus III, λ is a Lagrange multiplier.)

Traditionally this is expressed in the following way. The quantity $\partial U/\partial x_i$ is called the *marginal utility* of commodity i; one obtains (approximately) $\epsilon \partial U/\partial x_i$ units of extra utility from ϵ extra units of commodity i, the quantities of the other commodities being held fixed. The quantity

$$\frac{1}{p_i}\frac{\partial U}{\partial x_i}$$

is then the *marginal utility of the last dollar* spent on commodity i. The condition $dU = \lambda p$ is expressed in coordinate form by saying that all these marginal utilities are equal. In other words, a consumer who maximizes utility for a given budget (or who minimizes budget for a given utility) will adjust the expenditure on different commodities until the marginal utility of the last dollar spent on each commodity is the same. (Note that this condition is invariant under change of coordinates in utility-space.)

We focus attention for a moment on problem (b) above, that of minimizing the cost of a fixed amount S of utility. Let us consider the effect of varying the price p. For fixed S we now obtain a function $p \mapsto \xi(p, S)$, $V^* \to V$. The derivative of this function is a linear map $\sigma\colon V^* \to V$, i.e. a bilinear form on V^* (or an element of $V \otimes V$).

Theorem 1.5. *(Substitution theorem) The bilinear form σ thus defined (called the* substitution form*) is symmetric and negative (semi)definite.*

Notice that σ cannot be definite! This follows from the obvious fact that $\xi(p, S) = \xi(\lambda p, S)$ for any positive constant λ (the "veil of money"). We should really regard σ as a function defined on the Grassmannian of oriented 1-planes in

V^* (aka the unit sphere, if you want to fix a metric) and the domain of σ as a linear map is the tangent plane to this Grassmannian, which has dimension $N-1$. On the other hand, the range of σ also lies in an $N-1$-dimensional subspace, namely the tangent plane to the indifference surface. The correct notion of nondegeneracy would be to say that *this* linear map is invertible. See the remark 1.6 below.)

Proof. We fix S and regard ξ as a function of p. Let $R(p) = p\xi(p)$, the minimum cost of a bundle of utility S when prices are p. Then

$$dR = pd\xi + dp \cdot \xi = dp \cdot \xi$$

because $pd\xi = 0$ since $d\xi$ lies in the tangent plane to the surface $U = S$. It follows that

$$dp \wedge d\xi = 0$$

considered as a 2-form on V^*. But this 2-form is just the antisymmetrization of $\sigma(dp)$, so σ is symmetric.

To see that it is negative let p_0, p_1 be two points of V^* and ξ_0, ξ_1 the corresponding values of ξ. By the minimization properties we have

$$p_0\xi_1 \geqslant p_0\xi_0, \quad p_1\xi_0 \geqslant p_1\xi_1.$$

Hence

$$(p_0 - p_1)(\xi_0 - \xi_1) \leqslant 0.$$

If $p_1 - p_0 = \delta p$ this expression is (to first order) $\sigma(\delta p, \delta p)$. Thus σ is negative. □

Here is the economic interpretation. For a fixed value of p, σ is a square matrix whose entries σ_{ij} give the change in the amount of the i'th good purchased when the j'th price is changed infinitesimally. The symmetry of this matrix is a nontrivial prediction of the theory.

The diagonal elements of a negative matrix, σ_{ii}, must be negative. Interpretation: if the price of a good increases, less of it is bought. (But note that this is under the unworldly and unobservable condition of *constant utility*.)

The off-diagonal elements σ_{ij} may have either sign. If the sign is positive, then goods i and j are *substitutes* (if coffee becomes more expensive, I buy more tea). If the sign is negative then the goods are *complements* (if coffee becomes more expensive, I buy less cream). Exercise: If there are only two goods, show that they must be substitutes.

Remark 1.6. This analysis is related to some very classical topics in differential geometry. Let M be an embedded hypersurface in $\mathbb{R}^N$ (e.g. an indifference surface). Let n be a fixed unit normal vector field to M. For vector fields X, Y tangent to M one defines the *second fundamental form*

$$II(X,Y) = \langle \nabla_X, Y, n\rangle = -\langle Y, \nabla_X n\rangle.$$

A calculation shows that II is tensorial (it depends only on the values of X and Y at the given point) and symmetric.

Now we want to analyze the more realistic situation where maximization takes place subject to a budget constraint rather than a utility constraint. In the first instance we can consider a fixed price vector p and investigate $\zeta(p,R)$ as a function of the total budget R. (This function is called an *Engel curve*.) Intuition may at suggest that $\zeta(p,R)$, or more exactly each of its components, is likely to be an increasing function of R (as my budget increases, I buy more champagne). But this is not invariably the case (as my budget increases, I buy less box wine — I'm drinking champagne instead!) If $\zeta_j(p,R)$ is a decreasing function of R (in some range), one says that j is an *inferior good* (in that range).

Now consider the general case where ζ varies as a function both of p and R. Consider then $\zeta\colon \Omega = V^* \times \mathbb{R} \to V$ and, as above, we are interested in studying small changes, i.e. in $d\zeta$ which is a V-valued 1-form on Ω. Let us use the notation σ for the substitution form, which is a $V \otimes V$-valued function on Ω (a tensor field), and let α denote the derivative of ζ with respect to R for fixed prices (also a function on Ω), which is a V-valued function. We have

Proposition 1.7. *On Ω one has*

$$d\zeta = \sigma \cdot dp + \alpha(dR - \zeta \cdot dp).$$

(Here p, R are the coordinate functions on V^*, $\mathbb{R}$ respectively, so that dp is a V^*-valued 1-form and dR is an ordinary 1-form. The dot denotes contraction of V with V^*.)

This is called the *Slutsky equation*. The interpretation is that the change in the optimal bundle arising from a change of prices has two components: the first (the *substitution effect*) arises from exchanging goods at constant utility (the previous discussion) and the second (the *income effect* or *wealth effect*) arises because the change in prices may change the amount of utility available.

Proof. This is an identity for 1-forms, i.e. linear functionals on $V^* \oplus \mathbb{R}$. We check it on a basis. When applied to basis vectors of the form $(0, Y)$, $dp = 0$ and the equation just follows from the definition of α. On the other hand consider applying it a basis vector of the form (X, Y) that is tangent to the graph $p \mapsto p\xi(p, S)$ (that gives the minimum cost of a bundle at constant utility). On this curve $d\zeta(X, Y) = \sigma(X) = \sigma \cdot dp(X, Y)$ (by the substitution theorem), and on the other hand $dR(X, Y) = \zeta(X) = \zeta \cdot dp(X, Y)$ (as shown in the proof of the substitution theorem). So the Slutsky equation is verified in this case too. These two kinds of vectors span $V^* \oplus \mathbb{R}$ so we are done. $\square$

Can this be interpreted as the Maurer-Cartan equation for a suitable connection?

Consider now the variation in the amount of a given good (say number 1) purchased, in terms of its price (total budget being fixed). According to the Slutsky equation the rate of variation is $\sigma_{11} - \alpha_1\zeta_1$. The first term is negative, in accord with intuition, but the second term will be positive if the good is inferior. If the second term is so positive that the sum is positive, the good is called a *Giffen good*: the amount purchased by the consumer is an *increasing* function of the price. It is questionable whether such things exist in real life. The classic example is to consider the cheapest and worst of a class of things, such that the class as a whole is in some sense a necessity of life, but the cheap version is so undesirable that people would buy the more expensive one if they could afford it. For instance (quoting Marshall's principles of economics, 1895, which introduces the idea):

> As Mr.Giffen has pointed out, a rise in the price of bread makes so large a drain on the resources of the poorer labouring families and raises so much the marginal utility of money to them, that they are forced to curtail their consumption of meat and the more expensive farinaceous foods: and, bread being still the cheapest food which they can get and will take, they consume more, and not less of it.

In "Inferior Goods, Giffen Goods, and Shochu" (Baruch and Kannai, 2001), econometric evidence is presented that in a certain range, shochu (the cheapest grade of Japanese rice wine) behaves as a Giffen good. A more recent proposal suggests that it is plausible that under certain circumstances gasoline or other fuels may behave in this way for some subpopulations. Even if aggregate decreases as price rises, this does not rule out Giffen-like behavior for certain groups of people.

The mention of populations leads to the following kind of question. Suppose that instead of *one* consumer we have a large number of different consumers, with different budgets but otherwise identical indifference foliations. To be precise, let $\rho(R)$ be the density of consumers with budget R (normalized so that $\int_0^\infty \rho(R)dR = 1$). We can define the *aggregate demand* for goods at a given price level p to be the vector

$$A(p) = \int_0^\infty \zeta(p, R)\rho(R)dR.$$

One might suppose that aggregate demand might behave more "smoothly" than individual consumer demand.

Theorem 1.8. *(Hildenbrand) If $\rho(R)$ is a monotone decreasing function of R, then $dA\colon V^* \to V$ is a negative form (i.e. $dA(X, X) \leqslant 0$) even though it may not be symmetric. Thus, there are no "aggregate Giffen goods". (In fact, there are not even any "aggregate inferior goods".)*

Proof. We shall assume that ρ is smooth, though this isn't really necessary. The derivative dA is the sum of the aggregated values of the two terms in the Slutsky equation. The aggregated substitution-effect term is certainly negative (because the substitution form σ is negative *pointwise*). Thus, it suffices to show that the aggregated income-effect term is negative also.

Let $X = (X^i)$ be a vector in V^* and let us consider $I(X, X)$, where I is the aggregate income-effect term

$$I(X, Y) = -\int_0^\infty \alpha(p, R)(X)\zeta(p, R)(Y)\rho(R)dR.$$

Observe that (by definition) $\alpha(p, R)(X) = \partial/\partial R\zeta(p, R)(X)$. Thus

$$I(X, X) = -\int_0^\infty \tfrac{1}{2}\frac{\partial}{\partial R}\left(\zeta(p, R)(X)^2\right)\rho(R)dR$$
$$= \int_0^\infty \left(\zeta(p, R)(X)^2\right)\rho'(R)dR \leqslant 0$$

on integration by parts. The final inequality follows since ρ is decreasing. This proves the theorem. □

Of course the assumption that ρ is monotone decreasing is not very realistic. The shochu paper mentioned above shows that the conclusion may not hold for a strongly peaked ρ.

Lecture 2

Exchange economies

In this lecture we'll discuss some fundamental ideas of "free market" economic theory. To keep matters simple, we will discuss only the case of a so-called "exchange economy". In such an economy, there are a number of agents who start with a certain allocation of goods. The problem is for them to exchange these goods among themselves in such a way as to arrive at a new allocation which is in some sense "optimal" (or, more modestly, "unimprovable"). An exchange economy omits some notable features of the real economy, such as the production of goods from other goods, the market for labor, etc. But it still is complex enough to illustrate the operation of the classical theorems.

More formally, assume that we have a vector space V of bundles of commodities, as in the previous lecture. Assume that there exists a set $\mathscr{A}$ of economic agents, and that each $\alpha \in A$ starts with a certain bundle of commodities $w_\alpha \in V^+$ (the *endowment*). They exchange commodities among themselves to arrive at new bundles $a_\alpha \in V^+$ (the *allocation*) subject to the law of conservation of commodities,

$$\sum_{\alpha \in \mathscr{A}} a_\alpha = \sum_{\alpha \in \mathscr{A}} w_\alpha .$$

What are the most satisfactory allocations?

In answering this question we assume that each agent α has their own utility function U_α, as in the previous lecture. It is standard to assume that no meaning can be attached to comparisons between U_α and U_β, for $\beta \neq \alpha$: "there are no interpersonal comparisons of utility". (In other words, we can't trade off the utility I get from listening to bluegrass against the utility you get from listening to opera.) This may seem an admirably fair-minded assumption, but it prevents one from formulating any notion of the utility attached to "social welfare" and therefore, perhaps, contributes to the idea that such welfare is an illusion. (Margaret Thatcher - "there is no such thing as society".)

Let M denote the manifold of all possible allocations. Each U_α is a smooth function on M. The *Pareto cone* at $x \in M$ is the subset of the tangent space $T_x M$ defined by

$$\{X \in T_x M : dU_\alpha(X) > 0 \quad \forall \alpha \in \mathscr{A}\}.$$

In other words, my shifting the allocation x a little in the direction of the Pareto cone, the utility of *every* agent can be increased. An allocation x is *Pareto efficient* if its Pareto cone is empty (so that there is no local way to improve everyone's utility; any change that makes one agent better off makes another worse off.)

Remark 2.1. "Efficient" is one of the most loaded words in economics; after all, who wants to be inefficient? One should underline therefore that a Pareto efficient allocation may be very far from what we would generally regard as desirable. Supposing for example that there is only one good (e.g., money). Since the utility-functions are assumed to be increasing, each agent's utility will be increased by having more money and decreased otherwise. But the total amount of money is constant, so no change can increase everyone's wealth and thus *every* allocation is Pareto efficient, including the allocation "all of you give all of your money to me".

A classical toy model to think about allocation and efficiency is called the "Edgeworth Box" after F.Y. Edgeworth (1845-1926). See figure below.

In this model, we consider an economy with two goods (say, apples and bananas) and two agents (Alice and Bob). We normalize so that the total amount of each good in the initial allocation is 1 unit.

Because there are only two agents, the manifold of allocations (M above) can be taken to be the unit square (the "box"). A point (x, y) of the square represents the allocation of x apples, y bananas to Alice and $1 - x$ apples, $1 - y$ bananas to Bob. There are two foliations of the unit square: Alice's indifference foliation, whose leaves are convex, and Bob's indifference foliation, measured the other way, whose leaves are concave. At any point (x, y), the Pareto cone is contained between the tangent lines to Alice's and Bob's indifference foliations at that point. (See figure.)

It follows that the Pareto efficient allocations are represented by those points where Alice's and Bob's indifference foliations are *tangent* to one another. For each leaf of Alice's indifference foliation (that is, for each fixed value of Alice's utility function) there exists by convexity exactly one Pareto efficient allocation on that leaf. As Alice's utility varies, these allocations vary on a curve, called by Edgeworth the *contract curve*. One of Edgeworth's major contributions was (apparently) to show that the notion of efficiency does not single out a unique best possible allocation.

The notion of efficiency depends only on the *total* amounts of each good in the initial endowment, not on how these are distributed between Alice and Bob. Suppose, however, that the initial endowment is (x_0, y_0). Alice's and Bob's indifference curves through this point delineate a lozenge-shaped region called the *feasible region* — any allocation in this region will be viewed by both Alice and

Bob as an improvement over the initial endowment. It is unlikely that both will be happy with a redistribution taking the allocation outside the feasible region. The part of the contract curve that lies within the feasible region is said to consist of the *core* allocations—those Pareto efficient allocations which improve both Alice and Bob's view of their position. (With more than two agents the definition of the core must be generalized, as we will see below.)

Consider one more notion. For each point (x, y) of the contract curve there is a common tangent line $\ell_{x,y}$ to Alice's and Bob's indifference curves at that point. As (x, y) runs along the contract curve, the line $\ell_{x,y}$ moves from the "bottom left" to "top right" of the Edgeworth box. Using the intermediate value theorem, then, it is easy to see that there is at least one point (x, y) for which the line $\ell_{x,y}$ passes through the initial endowment (x_0, y_0). Such an allocation (which must be in the core) is called *Walrasian* or *competitive*.

What is the meaning of the Walras condition? Let us recall the notion of a price vector $p \in V^*$, which assigns a price to every bundle of commodities. Suppose that a price vector p is given. Then Alice might argue as follows: now that she knows the price of apples and bananas, she knows the cash value (namely $p(x_0, y_0)$) of her initial endowment. She might try to solve the following problem: find that bundle that maximizes her utility, subject to the budget constraint $p(x, y) \leqslant p(x_0, y_0)$ that she may not spend more than her initial endowment. This problem has (because of strict convexity) a unique solution: let's call that solution $\zeta_A(p)$.

Of course Bob can carry out the same calculation, arriving at *his* utility-maximizing bundle $\zeta_B(p)$. For a general p, however, Alice and Bob can't both achieve this utility maximization simultaneously, because $\zeta_A(p)$ and $\zeta_B(p)$ may not add up to the initial total endowment of 1 unit apples, 1 unit bananas. The competitive allocations are precisely those for which this simultaneous maximization *is* possible. In other words, *an allocation is competitive precisely when there is a price vector for which the allocation simultaneously maximizes Alice's and Bob's utility, subject to the budget constraints provided by the initial endowment.* It is easy to see graphically why this is so: the required price vector p is just the normal to the line $\ell_{x,y}$.

Now we will carry out this discussion in a more general session (a finite set $\mathscr{A}$ of agents and a finite-dimensional vector space V of bundles). We have already defined the notion of Pareto efficiency in this context. Recall also the notation $\{w_\alpha\}$ for the initial endowment.

Definition 2.2. Let $\{a_\alpha\}$ be an allocation and let $S \subseteq \mathscr{A}$. We say that S *blocks* the allocation $\{a_\alpha\}$ if the total amount of goods belonging to S in the initial endowment can be redistributed among the members of S in a way that is Pareto-preferred to $\{a_\alpha\}$ by all the members of S. In other words, there exist $\{b_\beta\}_{\beta\in S}$ such that $\sum_{\beta\in S} b_\beta = \sum_{\beta\in S} w_\beta$ and $U_\beta(b_\beta) \geqslant U_\beta(a_\beta)$ for all $a \in S$, with strict inequality in at least one case.

Definition 2.3. A *core* allocation is one that is not blocked by any subset of $\mathscr{A}$.

Taking $S = \mathscr{A}$ we see that a core allocation is Pareto efficient. Taking S to have one point we see that a core allocation must improve each agent's utility over the initial endowment. When there are only two agents these are the only two conditions and we recover the notion of the core from the previous discussion.

Definition 2.4. An allocation $\{a_\alpha\}$ is *competitive* or *Walrasian* if there exists a price vector p such that, for each $\alpha \in \mathscr{A}$, the bundle a_α maximizes utility for agent α subject to the budget constraint $px \leqslant pw_\alpha$.

Lemma 2.5. *A competitive allocation is a core allocation (in particular, it is Pareto efficient).*

Proof. Let p be a price vector implementing the Walras definition of competitiveness, and let S be a potentially blocking subset for the competitive allocation $\{a_\alpha\}$. Let $\{b_\beta\}_{\beta\in S}$ be a blocking allocation. For each $\beta \in S$, we know that a_β maximizes utility subject to the budget constraint $\leqslant pw_\beta$. Since b_β is supposed to have greater utility than a_β, it must break the budget constraint: $pb_\beta \geqslant pw_\beta$ (at least one strict inequality). But

$$\sum_S pb_\beta = p\left(\sum_S b_\beta\right) = p\left(\sum_S w_\beta\right) = \sum_S pw_\beta,$$

and this is a contradiction. □

Theorem 2.6. *(First welfare theorem) Competitive allocations exist, and they are Pareto efficient,*

The second statement follows from the preceding lemma, of course. Before proving the theorem we make some comments about its application in economic ideology…

The theorem is usually taken to illustrate the "invisible hand" maxim that self-interested actors, guided by market forces (the price mechanism), can bring about the general good of society. Adam Smith expressed this idea in the context of international trade:

> By preferring the support of domestic to that of foreign industry, he [an entrepreneur] intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was not part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it. I have never known much good done by those who affected to trade for the public good. It is an affectation, indeed, not very common among merchants, and very few words need be employed in dissuading them from it.

But there are two caveats that need to be entered:

(a) As noted above, one cannot identify Pareto efficiency straightforwardly with "the general good" or "the interest of society". In fact, the theory is built on incommensurable individual preferences, and there is no room for a concept of "the general good".

(b) In the theorem above (and the standard generalizations in General Equilibrium Theory), the agents have a rather passive role with respect to prices. It is simply assumed that "the market" will arrive somehow at a competitive price vector. But how can this come about? Walras proposed a model where the markets are governed by an Auctioneer who repeatedly calls out prices and learns from the agents which goods they would be prepared to trade at the specified prices. When, and only when, the Auctioneer has arrived at a competitive price vector, are the agents permitted to trade, and then only at the specified price. Obviously this is a long way from our intuitive picture of the "free market".

Proof. To show that a competitive allocation exists, we will use the Brouwer fixed point theorem. Let p be a (positive) price vector; we may as well normalize it so that $\sum p_j = 1$ (this doesn't affect anything — multiplying all prices by a constant leaves the utility-maximizing allocations unchanged). Thus p lies in the simplex $\Delta = \{p_j : p_j \geqslant 0,\ \sum p_j = 1\}$, which is topologically a disc.

For given p and an agent α, let $\zeta_\alpha(p) \in V$ be the bundle that maximizes α's utility subject to the budget constraint pw_α given by the initial allocation. It is uniquely defined and depends continuously on p (by the strict convexity assumption). Let

$$e(p) = \sum_{\alpha} (\zeta_\alpha(p) - w_\alpha) \in V$$

be the *excess demand function*: we are seeking a p such that $e(p) = 0$. Note that, by construction,

$$pe(p) = \sum_{\alpha} p\,(\zeta_\alpha(p) - w_\alpha) = 0.$$

Now define a map $T\colon \Delta \to \Delta$ as follows: define $\delta_j = \max\{e_j(p), 0\}$ and then set

$$T(p)_j = (p_j + \delta_j)/Q, \quad \text{where } Q = (1 + \sum \delta_i).$$

By Brouwer's theorem, T has a fixed point. At that fixed point, we have $p_j Q = p_j + \delta_j$ for all j, and so

$$\delta_j = p_j \sum \delta_i.$$

We are going to show that all the δ_i equal zero. If not, then by the last displayed equation $\delta_j > 0$ (and therefore $e_j > 0$) whenever $p_j > 0$. It follows that $\sum p_j e_j > 0$, a contradiction.

Thus we have shown that for the p selected by Brouwer's fixed-point theorem, $e_j(p) \leqslant 0$ for all j. Since $\sum p_j e_j(p) = 0$ we deduce that in fact $e_j(p) = 0$ for all j and we are done.

Strictly speaking we need to rule out cases where some $p_j = 0$. However, this corresponds to the case where good j is free. At price 0 we may expect that an infinite amount of j will be demanded; thus $e_j > 0$ and this case cannot in fact occur. □

Lecture 3

The second fundamental theorem

We recall the set-up of the previous lecture. We have our usual vector space V of bundles of commodities, and a set $\mathscr{A}$ of *agents*. An *allocation* is an element of the vector space $V^{\mathscr{A}}$, i.e., a list of bundles, one for each agent. We consider only allocations lying in the positive cone. An initial allocation $\{w_\alpha\}$ called the *endowment* is given and we restrict attention to allocations $\{a_\alpha\}$ that may be obtained from the initial endowment by exchange, i.e. such that $\sum a_\alpha = \sum w_\alpha$. Such an allocation is called *competitive* for a price vector p if

$$a_\alpha = \zeta_\alpha(p, pw_\alpha);$$

that is, if each agent's bundle is utility-maximizing subject to the budget constraint given by the dollar value of the initial endowment. A competitive allocation is Pareto efficient, and the *first fundamental theorem* of welfare economics says that competitive allocations always exist. We proved this last time using the Brouwer fixed point theorem.

To state the second fundamental theorem we need a more general notion.

Definition 3.1. A *competitive allocation with transfers* is an allocation a_α for which there exists a price vector p and a list $\{t_\alpha\}$ of real numbers (the *transfers*) such that $\sum_\alpha t_\alpha = 0$), such that

$$a_\alpha = \zeta_\alpha(p, pw_\alpha - t_\alpha).$$

The interpretation of this condition is that government redistributes money among the agents in accordance with the vector $\{t_\alpha\}$ of transfers: once this is done, the "market takes over" and produces a competitive allocation.

Theorem 3.2. *Any Pareto efficient allocation is a competitive allocation with suitable transfers.*

The traditional interpretation of this is as follows. Any socially desirable outcome must be Pareto efficient (otherwise, it could be improved at no cost to anyone, and why would you not do that)? So the outcomes that a benevolent government might wish to bring about should include only the Pareto efficient ones. According to the second welfare theorem, then, any such objective can be achieved just by suitable redistributive taxation followed by the operation of the free market. Thus, "social welfare" questions are reduced simply to moving around money which agents then freely use (no strings attached). There is no need for the government to actively steer people towards the choices it regards as desirable ("interfering in the market").

The non-uniqueness of equilibria interferes somewhat with this happy picture, as does the level of governmental omniscience that would be required in practice to compute the transfers needed to achieve some particular objective. Other reasons for scepticism will be discussed later.

Proof. Let $W = V^{\mathscr{A}}$ be the vector space of allocations. Let U be the subspace of W defined by $U = \{(x_\alpha) : \sum x_\alpha = 0\}$ (this is the vector space of *exchanges*) and let $U^- = \{(x_\alpha) : \sum x_\alpha \leqslant 0\}$, where the inequality refers to the ordering of V (*exchanges with wastage*). Let $w = (w_\alpha)$ be the initial endowment and let $a = (a_\alpha)$ be the Pareto efficient allocation given in the statement of the theorem; by assumption, $a - w \in U$, that is, a and w are equivalent under exchange.

Define subsets A and B of W as follows:

$$A = w + U^- = a + U^-, \qquad B = \{(x_\alpha) : U_\alpha(x_\alpha) > U_\alpha(a_\alpha)\}.$$

Here U_α is the utility function for agent α, so that B is the collection of all allocations that are (strongly) Pareto preferred to a. The hypothesis of efficiency says that no allocation accessible by exchange from a is preferred to a; that is, $A \cap B = \emptyset$.

Since A and B are convex and B is open, the Hahn-Banach theorem gives a linear functional $\phi \in W^*$ and a constant $\lambda \in \mathbb{R}$ such that

$$\phi(x) \leqslant \lambda < \phi(y)$$

for all $x \in A$ and $y \in B$. Since A contains a coset of the subspace U, the linear functional ϕ is bounded on U (by $2|\lambda|$) and is therefore zero there (because U is a subspace). Now the map $\sigma\colon (x_\alpha) \mapsto \sum x_\alpha$ induces an isomorphism of W/U onto V, and by the first isomorphism theorem there is $p \in V^*$ such that $\phi = p\sigma$.

We show that p is a price vector (i.e., it is positive). Let v be any bundle (an element of the positive cone of V). Then for sufficiently large $r > 0$, the element $(-rv, -rv, \ldots)$ must belong to A. Thus $-Nrp(v)$ is bounded above as $r \to \infty$. This can happen only if $p(v) \geqslant 0$.

Next we show that $a_\alpha = \zeta_\alpha(p, pa_\alpha)$ for each α — in other words, given the price vector p, each agent α finds his/her bundle a_α optimal for its price. Suppose not. Then, for at least one agent, say β, there exists a new allocation b_α such that

$$b_\alpha = a_\alpha \ (\alpha \neq \beta), \quad p(b_\beta) = p(a_\beta), \quad U_\beta(b_\beta) > U_\beta(a_\beta).$$

In particular, $\phi(b) = \sum p(b_\alpha) = \phi(a)$. Since the U's are assumed continuous and increasing, one can redistribute a small amount of one good from agent β to the other agents to arrive at a new allocation c for which

$$\phi(c) = \phi(b) = \phi(a), \quad U_\alpha(c_\alpha) > U_\alpha(a_\alpha)\, \forall \alpha \in \mathscr{A}.$$

But now $c \in B$, $a \in A$, and $\phi(c) = \phi(a)$; this is a contradiction.

Finally define the transfers $t_\alpha = p(w_\alpha) - p(a_\alpha) \in \mathbb{R}$. The sum of the transfers is indeed zero because $w - a \in U \subseteq \ker \phi$. By construction, then,

$$a_\alpha = \zeta_\alpha(p, pw_\alpha - t_\alpha)$$

so a is indeed a competitive allocation with transfers. □

We went on to talk about the idea of Kaldor-Hicks efficiency, and cost-benefit analysis. We went on to discuss the limits of the model, especially the individualistic nature of the model for welfare (my welfare depends on my "stuff" and nothing else).

Lecture 4

Production