Reflection statistics of weakly disordered optical medium when its mean refractive index is different from the outside medium

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Based on the difference between an optical sample’s mean background refractive index \( n_0 \) and the outside medium \( n_{\text{out}} (\neq n_0) \), we study the reflection statistics of a one-dimensional weakly disordered optical medium with refractive index \( n(x) = n_0 + dn(x) \). Considering \( dn(x) \) as a color noise with the exponential correlation decay length \( l_c \) and \( k \) is the incident wave vector, our results show that, for weak correlation length \( kl_c < 1 \), the average reflectance \( r \) follows a form that is similar to that of the matched refractive-index case \( n_0 = n_{\text{out}} \), i.e., \( <r(dn, l_c)> \propto dn^2 l_c \). However, the standard deviation of \( r \) shows that \( \sigma[r(dn, l_c)] \propto dn l_c^{1/2} \) is different from the matched case. Applications to light scattering from biological cells/tissues are discussed. © PACS : 78.20.-e, 78.20.Bh, 78.67.-n, 87.19.xj

The statistical transport properties of one-dimensional (1D) mesoscopic disordered optical and electronics media are now well studied [1-4]. The Schrödinger equation and Maxwell’s wave equation are similar in the sense that they can be projected to the Helmholtz equation; therefore, the formalisms are the same for corresponding scalar waves in both cases [5-8]. After the Landauer formalism showed that the reflection coefficient is related to the resistance/conductance of the sample, the outer scattering components such as the reflection and transmission coefficient become important for the study of localization and conductance fluctuations in the electronics case [5, 6]. Similarly, extending the ideas from the electronics system, in previous studies of light scattering and localization properties of different optical disordered media, the fluctuation part of the refractive index is primarily considered, while the sample’s mean refractive index is the same as the outside medium [5, 7-9]. The results show that both the average reflection and the fluctuations have the same form for the mesoscopic optical sample. However, the mismatch of the refractive index between the sample and the outside medium and its effects upon the reflection statistics remain poorly understood.

In this paper, we study reflection statistics in the context of the synergistic effects between index mismatch values and the fluctuation part of the refractive index. Enhancement of the backscattering signals from the weakly
fluctuating part of the refractive index, as mediated by the refractive index mismatch, is also addressed. Finally, applications of the method for light scattering from biological cells are discussed in terms of the enhancement of the signal from refractive index nano-fluctuations for potential application to cancer detection.

Consider a 1D sample of length \( L \) with refractive index \( n(x) = n_0 + dn(x) \) \((L < x < 0)\), where the average \( n_0 = <n(x)> \) is the spatial fluctuation part and \( dn(x) \) is the refractive index of the outside medium \( n_{out} \). The ‘matched’ case can be defined as equality between the average refractive index of the sample and the outside medium, i.e., \( n_0 = n_{out} \), whereas the ‘mismatched’ case can be defined as \( n_0 \neq n_{out} \). In the literature, reflection statistics from optical media are primarily studied assuming the matched case, as defined above. Therefore, we will first briefly review the results of a one-dimensional matched case before describing the results of the mismatched case. Consider \( R(L) \) as the complex reflection amplitude of a sample of length \( L \) which is illuminated by a plane wave of wave vector \( k \). Consider the matched case \( n_0 = n_{out} = 1 \). Then a stochastic differential equation (Langevin equation) for the reflection amplitude \( R(L) \) varying with length can be derived \([5,7,9]\) as

\[
\frac{dR(L)}{dL} = 2ikR(L) + \frac{ik}{2}(2n_0dn(L))[1 + R(L)]^2. \tag{1}
\]

In a general case, the statistics of \( dn(x) \) represent color noise, i.e., \( <dn(x)> = 0 \) and \( <dn(x)dn(x')> = dn^2 \exp(|x-x'|/\xi) \), where \( \xi \) is the spatial correlation decay length of the fluctuation \( dn \). Then, using the Fokker-Planck approach, the above Langevin equation can be solved analytically, and the results describe the weakly disordered case \([5,7]\) (i.e., \( dn \ll n_0 \)). Here, the mean reflection \( \langle r = RR^* \rangle \) and its standard deviation (std) \( \sigma(r) \) both have the same value, \( \langle r \rangle = \sigma(r) = L/\xi \), where the inverse of the localization length has the form \( \xi^{-1} = 2k^2dn^2/\left[1 + (2klc)^2\right] \). This is true for a weakly disordered sample when \( \xi > L \), which also can be called as a Born approximation to the weak disorder part.

However, mismatched weakly disordered optical samples are quite common for the optical scattering experiments that we perform. For example, biological cells and tissues have refractive indices \( n_0 \sim 1.3 - 1.5 \) and \( dn \sim 0.01 - 0.1 \) with the outside air medium \( n_{out} = 1 \). In the case of weak refractive index fluctuations, the backscattering light transport properties of a biological cell can be decomposed into a multiple-transport 1D channel or a quasi-1D multichannel problem \([10]\). It was recently shown that quasi-1D multichannel backscattering would provide sensitivity to changes in the nanoscale disordered signal relative to a three-dimensional (3D) bulk for weakly disordered media such as biological cells. Furthermore, the approach has proven potential for early pre-cancer screening by detecting changes in the nanoscale refractive index fluctuations of cells with the progress of carcinogenesis in different type of cancers \([11-14]\).
Following an approach similar to the derivation of Eq. (1), a Langevin equation (i.e., \(dR/dL\) equation) can be derived for the index mismatch case \((n_{\text{out}} \neq n_0)\). For simplicity, we will consider that the sample is kept in air, i.e., \(n_{\text{out}} = 1\) and \(n_0 > 1\). Substituting these terms \((n_{\text{out}}\) and \(n_0\) and \(dn\)) with color noise and approaching a derivation similar to Eq. (1), the Langevin equation for the mismatched case is as follows:

\[
\frac{dR(L)}{dL} = 2ikR(L) + \frac{ik}{2} [(n_0^2 - 1) + 2n_0dn(L)] \times [1 + R(L)]^2.
\]  

For the index-mismatched case, the total complex reflection \(R_t(L)\) from a weakly disordered medium such as a biological cell can be considered as a combination of: (i) a deterministic sinusoidal oscillation component \(R_s\) based on the pure background of a thin-film slab of length \(L\) without any \(dn\) terms; and (ii) an \(R\) component that contains \(dn\) terms. The Langevin equations for \(R_s\) and \(R\) can then be easily derived as follows:

\[
R_s(L) = R_s(L) + R(L),
\]  
\[
\frac{dR_s(L)}{dL} = 2ikR_s + \frac{ik}{2} (n_0^2 - 1) \times [1 + R_s]^2,
\]  
\[
\frac{dR(L)}{dL} = 2ikR + \frac{ik}{2} (2n_0dn(L)) \times [1 + R_s + R]^2
\]
\[
+ \frac{i}{2} k(n_0^2 - 1) \times [2R(1 + R_s) + R^2].
\]

In Eqs. (3a-c), the perturbative contribution by \(dn\) has many cross-terms between \(R_s\) and \(R\). We will assume that \(R\) is in the first order in \(dn\). Therefore, by doing a phase transformation in Eqs. (3b-c), we will further simplify and assimilate the \(R_s\) - \(R\) cross-terms in the equation. For this, we introduce new variables, \(Q(L)\) and \(\alpha(L)\), which are derived from \(R(L)\) by a phase transformation as follows:

\[
R_s(L) = Q_s(L) \cdot e^{2ika(L)},
\]  
\[
R(L) = Q(L) \cdot e^{2ika(L)}.
\]

This yields a new set of simplified equations for \(Q(L)\) and \(\alpha(L)\) which further simplifies to:

\[
\frac{d\alpha(L)}{dL} = 1 + \frac{(n_0^2 - 1)}{2} (1 + R_s),
\]  

\[
\frac{dQ(L)}{dL} = 2i\alpha(L) R_s + \frac{i}{2} k(n_0^2 - 1) \times [2Q(1 + R_s) + Q^2].
\]

\[
\frac{dR_s(L)}{dL} = 2ikR_s + \frac{ik}{2} (n_0^2 - 1) \times [1 + R_s]^2.
\]  

\[
\frac{dR(L)}{dL} = 2ikR + \frac{ik}{2} (2n_0dn(L)) \times [1 + R_s + R]^2
\]
\[
+ \frac{i}{2} k(n_0^2 - 1) \times [2R(1 + R_s) + R^2].
\]
\[
\frac{dQ}{dL} = \frac{i}{2} k(2n_0 dn(L))e^{-2ika} [1 + R_s + Oe^{2ika}]^2 \\
+ \frac{i}{2} k(n_0^2 - 1)(Q)^2 e^{2ika}.
\] (5b)

With the above new representation (5), the mean \( \langle r \rangle \) and the standard deviation (std) \( \sigma(r) \) of the reflectance \( r = RR^* \) can be derived. By using Eq. (3a) and performing a realization or disorder average, we obtain:

\[
\langle r(L) \rangle = \langle RR^* \rangle = \langle (Q_s + Q) \times (Q_s + Q)^* \rangle \\
= R_s + Q_s^* \langle Q \rangle + \text{c.c.} + \langle |Q|^2 \rangle,
\] (6)

where, \( r = |R|^2 \) and \( r_s = |R_s|^2 = |Q_s|^2 = \frac{(n_0^2 - 1)^2 \sin^2(n_0 kL)}{4n_0^2 + (n_0^2 - 1)^2 \sin^2(n_0 kL)} \).

In Eq. (6), we have separated the thin-film slab uniform barrier effect (that is \( r_s \) and the disorder contribution). To calculate (6), we require the disorder average of the following terms in leading order of \( dn \) as:

\[
\langle |Q|^2 \rangle, \langle (Q)^2 \rangle, \langle Q \rangle,
\]

which can be written explicitly as:

\[
\langle |Q|^2 \rangle = -\frac{i}{2} k n_0 \int_0^L dL [e^{2ika'}(1 + R_s')^2 \times < 2dnQ > - \text{c.c.}],
\] (7a)

\[
\langle (Q)^2 \rangle = i k n_0 \int_0^L dLe^{-2ika}(1 + R_s)^2 \times < 2dnQ >,
\] (7b)

\[
\langle Q \rangle = i k \int_0^L dL [n_0 (1 + R_s) < 2dnQ > + \frac{1}{2} (n_0^2 - 1)e^{-2ika} \langle (Q)^2 \rangle].
\] (7c)

The disorder averaging of the above Eqs. (7a-c) was performed using the Novikov identity [5], and the averages were obtained. For example, the disorder average of the product term \( < 2dn(L)R > \) is (following [5]):

\[
< 2dn(L)R >= \frac{i}{2} k n_0 (\frac{g}{2}) \times [1 - \frac{\partial}{\partial (L / l_c)}] e^{-2ika}(1 + R_s)^2 \\
+ O(dn^2 \cdot le^3) + O(dn^4 \cdot le^3).
\] (8)

where the value of \( g \) and \( \alpha \) can be expressed as:

\[
g = 8 \cdot dn^2 lc \quad \text{(disorder strength), and}
\] (9a)
Finally, by using Eqs. (7-9) and averaging over the disorder, we obtain the average $\langle r \rangle$ in terms of $dn$ and $l_c$ as:

$$< r > = r_s + dn^2 l_c \cdot k^2 [Q_1^2 (n_0^2 I_1 + k n_0^2 (n_0^2 - 1) I_2) + c.c.] + O(dn^4).$$  \hspace{1cm} (10)$$

where we have defined $I_1$, $I_2$, and $I_3$ in the above deterministic equations as:

$$I_1 = -2 \int_0^L dL (1 + R_s) \times \left[ 1 - \frac{\partial}{\partial (L / l_c)} \right] e^{-2ik\alpha} (1 + R_s)^2,

$$I_2 = -i \int_0^L dL e^{-2ik\alpha (L)} \int_0^L dL' e^{-2ik\alpha (L')} (1 + R_s (L'))^2 \times \left[ 1 - \frac{\partial}{\partial (L' / l_c)} \right] e^{-2ik\alpha} (1 + R_s)^2,$$

$$I_3 = \int_0^L dL e^{-2ik\alpha} (1 + R_s)^2 \times \left[ 1 - \frac{\partial}{\partial (L / l_c)} \right] e^{-2ik\alpha} (1 + R_s)^2.$$

Eq. (10) can now be rewritten in terms of a thin-slab term (without $dn$) and the fluctuation term averages. For weak disorder with short range correlation $2kl_c < L$, the quantities $I_1$, $I_2$, and $I_3$ are approximately independent of $l_c$. In this case we obtain.

$$< r > = r_s + (1/2)dn^2 l_c \cdot k^2 L \times F(k, n_0) + O(dn^4 l_c^{-2})$$

$$= r_s + \frac{L}{\xi} \times F(k, n_0, L).$$  \hspace{1cm} (12a)$$

Thus, we can write a linear relationship between the matched and mismatched cases from Eq. (12).

$$< r >_{\text{mismatched}} =< r >_{\text{slab}} +< r >_{\text{matched}} \times F(k, n_0, L).$$  \hspace{1cm} (12b)$$

The mean-square fluctuation of $r$, $\sigma^2(r)$, can be evaluated as:

$$\sigma^2(r) = < r^2 > - (< r >)^2$$

$$= < (r_s + 2 \text{Re}(Q_0^* O) + |Q_0|^2) >$$

$$- (< r_s + 2 \text{Re}(Q_0^* O) + |Q_0|^2 >)^2.$$  \hspace{1cm} (13)$$
The above equation can now be further modified by applying Eqs. (4) and (7):

$$\sigma^2(r) = Q_s^* < Q > + c.c. + 2 r_s < |Q|^2 > + O(dn^4 l_c^2).$$  \hspace{1cm} (14a)

Taking the average over the disorder, we obtain a deterministic expression of the above equation:

$$\sigma^2(r) = dn^2 l_c \cdot k^2 n_0^2 [Q_s^* I_4 + c.c. + 2 r_s I_3 + c.c.] + O(dn^4 l_c^2).$$  \hspace{1cm} (14b)

Therefore, the standard deviation of $r$, $\sigma(r)$, can be written as indicated below. For the mismatched case, it should be noted that the mean-square fluctuation of $r$ has a barrier or mismatch-induced leading term $dn^2 l_c$. In the matched case, the leading term is two orders less, that is, $dn^4 l_c^2$. Therefore, we obtain the expression for the standard deviation $\sigma(r)$ as follows:

$$\sigma(r) = dn L^{1/2} \cdot k \left[ Q_s^* I_4 + c.c. + 2 r_s I_3 + c.c. \right]^{1/2} + O(dn^2),$$  \hspace{1cm} (15)

where we have defined:

$$I_3 = \int_0^L dL e^{-2ika} (1 + R_s^*)^2 [1 - \frac{\partial}{\partial(L/l_c)}] e^{-2ika} (1 + R_s)^2,$$  \hspace{1cm} (16a)

$$I_4 = -2 \int_0^L dL e^{-2ika} (1 + R_s)^2 [1 - \frac{\partial}{\partial(L/l_c)}] e^{-2ika} (1 + R_s)^2.$$  \hspace{1cm} (16b)

Eq. (15) can be further written as (following Ref.[5]):

$$\sigma(r) = dn L^{1/2} \cdot k L^{1/2} [G(n_0, L, k)].$$  \hspace{1cm} (17)

where $G$ is a function without the $dn$ term. It can be noted that, in the matched case ($n_0=1$), the value of $G$ is 0 since $Q_s$ and $r_s$ have the multiplicative factor ($n_0^2 - 1$)=0 as $n_0=1$. In this case, in $\sigma(r)$, first order term in $dn$ vanishes and second order term, that is $dn^2$ is the leading term, similar to matched case.
Theoretical Eqs. (12a) and (15) were plotted for $<r(\dn)>$ and $\sigma(r)$ in Figures 1(a) and (b), respectively, for a sample length $L=2\mu$; similarly, we plot $<r(\dn)>$ and $\sigma(r)$ in Figures 1(c) and (d), respectively, for $L=5\mu$. The other parameters remain the same: wavelength = 500nm and $l_c=20$nm.

We also performed direct stochastic simulation of Eq. (2) and then performed realization averages numerically (shown by the dotted line). It can be seen that the theoretical results and the corresponding numerical results agree well for all of the parameters, thereby validating our calculations. For the case $2klc >1$, a detailed calculation will be reported in a separate paper.

In conclusion, our results show that the average reflection, for the mismatched case with weak disorder and short range correlation of the refractive index, is linearly dependent upon its matched case; that is, the average reflection $<r(\dn,l_c)>_{mismatched}$ is still proportional to $\dn^2l_c$ when other parameters are constant. However, the value of the standard deviation of the reflection has a different form (Eq. (12)). As seen from Eq. (14(b)), the index-mismatch parameter $(n_0^2 - 1)$ contributes to the standard deviation, and this changes the leading term of the mean-square fluctuations from $(\dn^2l_c)^2$ to $\dn^2l_c$. Therefore, the RMS fluctuations, or std($r$) for the mismatched case, are proportional to $\dn l_c^{1/2}$. The relative fluctuation $\sigma(r)/<r>$ decreases ($<1$) with the increase of the mismatch parameter $(n_0^2 - 1)$. However, the relative $\sigma(r)$ value for the mismatched case, compared to the matched case, increases. The phenomenon of increasing the average and the fluctuations can be useful for enhancing a weakly reflective signal from the weak refractive index fluctuation ($\dn$) hidden in a strong uniform refractive index background ($n_0$). This clearly shows that the backscattering signal from a disordered sample can be enhanced by increasing the mean refractive index. For example, one can dip biological cells in a nonreactive liquid having a higher refractive index to enhance the reflection signal from the fluctuating part.

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Figure 1: Plots of analytical calculations (line) and numerical simulations (circle). Matching the analytical calculation in Eq. (10) with the pure numerical integration of Eq. (2) for $<r>$ with lengths (a) $L=2\mu$ and (c) $L=5\mu$. Matching of the plot of the analytical calculation Eq. (15) and numerical simulations of Eq. (3) for the standard deviation of $r$, $\sigma(r)$ for the lengths (b) $L=2\mu$ and (d) $L=5\mu$. Refractive index of outside medium is taken as air $n_{out}=1$. The mean refractive indexes of the samples are $n_0 = 1, 1.1, 1.2, 1.3, 1.4$, and $1.5$. Refractive index fluctuations were varied such that $dn = 0–0.03$. Correlation length $l_c$ of $dn$ is 20nm.