Vulnerability Bounds for Transportation Networks under Simultaneous Multi-Link Disruptions

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Outline

• Motivation
• Mathematical Formulation
• Solution Algorithm
• Numerical Examples
• Concluding Remarks
Disasters and Transport Networks

Disasters (Natural or Manmade)

- Natural Disaster: Earthquake, Tsunami, Avalanche, Flood, Wildfire, Volcano
- Structural Collapse: Bridge, Tunnel, Overpass
- Terrorist: 9/11 Attack, London Tube, Riot
- Incident: Accidents, Road Closure, Work zone, Special events
- Regulation, Policy: Hazard Material Routes

Disruption to Transportation Network and Economic Impact

(Thailand Mega Flood in 2011)
Estimates of economic losses = 1,425 billion THB (~$ 45.7 billion)
(World Bank, 2012)

(Hurricane Sandy in 2012)
Estimates of losses due to the shutdown of business activity in the East Coast = $30 billion-$50 billion (IHS Global Insight, 2013)
Related Topics

- Reliability (supply degradation)
- Flexibility (demand fluctuation)
- Resiliency (pre & post disaster)
- Redundancy (resiliency enhancement)
- **Vulnerability**

\[ Pre \]
\[ \begin{align*}
\text{• Redundancy} \\
\text{• Robustness}
\end{align*} \]

\[ Post \]
\[ \begin{align*}
\text{• Rapidity} \\
\text{• Resourcefulness}
\end{align*} \]
Transportation Network Vulnerability

- Vulnerability is the susceptibility of a system to threats and incidents that results in operational degradation (Berdica, 2002).
- The core is to **identify the critical/vulnerable/important components** (links/nodes), whose disruptions have a significant impact on user behaviors and network performance.
- A great deal of attention has been devoted to this topic (e.g., Bell and Cassir, 2000; Berdica, 2002; Bell and Iida, 2003; Nicholson and Dantas, 2004; Sumalee and Kurauchi, 2006; Chen et al., 2007a,b; Murray and Grubesic, 2007; Kurauchi and Sumalee, 2008; Kurauchi et al., 2009; Schmocker and Lo, 2009; Nagurney and Qiang, 2010; Levinson et al., 2010, 2012; Luathep et al. 2011; Lam et al., 2012; Chen et al., 2012; Ho et al., 2013; Jansuwan and Chen, 2015; Jenelius and Mattsson, 2015, Wang et al., 2016; Bell et al., 2017).
## Vulnerability Assessment Approaches

<table>
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<th>Category</th>
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<tr>
<td><strong>Disruption scenario enumeration approach</strong></td>
<td>Reviewed by Mattsson and Jenelius (2015)</td>
<td>One link/node is removed at a time, and the impact of each link/nodal removal is evaluated and ranked according to different indictors</td>
<td>Combinatorial complexity when considering simultaneous disruption of multiple links/nodes</td>
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| Disruption scenario enumeration approach      | Bell (2000); Bell and Cassir (2002); Szeto *et al.* (2007)                | Critical links are likely to be destroyed by the demon as a consequence of the game                                                                                                                                                                       | •Only consider the worst-case scenario  
•Pessimistic evaluation                                                                                                                                                                                                                                                                                                                                                                                |
| **Game theoretic approach**                   | Nicholson and Du (1997); Chen *et al.*, (2002); Luathep *et al.* (2011); Yang *et al.* (2013) | Link criticality is the proportion of overall uncertainty of performance measure contributed by its link capacity uncertainty                                                                                                                                  | •Only valid for minor perturbations  
•Inapplicable to large perturbations in disruption scenarios                                                                                                                                                                                                                                                                                                                                                                                          |
| **Sensitivity and uncertainty analyses**      |                                                                                           |                                                                                                                                                                                                 |                                                                                                                                                                                                                                                                                                                                 |
Simultaneous Disruption of Multiple Links

- A network could be resilient to a single-link failure, but simultaneous disruptions can be very problematic, resulting in disruption propagations and widespread disruptions.
Combinatorial Complexity

- Simulation may miss some important/phantom scenarios that are unapparent due to the large scale and complex network structure.
- To consider the range of all potential disruption scenarios.

There are a total of 100 links.

- Large number of possible scenarios.
- Unknown occurrence possibility of each scenario.

Number of Disrupted Links

- \( \binom{100}{2} \)
- \( \binom{100}{3} \)
- \( \binom{100}{4} \)
- \( \binom{100}{5} \)
- \( \binom{100}{6} \)
- \( \binom{100}{7} \)
- \( \binom{100}{8} \)

Number of Possible Scenarios
Research Objective

- No analytical approach of transportation network vulnerability with a systematic consideration and quantification of all possible simultaneous disruptions
- To develop an optimization approach for deriving the upper and lower bounds of network vulnerability, while circumventing the need of enumerating all possible disruption scenarios
Mathematical Formulation (1)

- **Upper-level**

\[
\max \text{ or } \min \ f_n (x, z) = \sum_{w \in W} q^w z^w
\]

s.t. \( x_a = \{0, 1\}, \ \forall a \in A \) \hspace{0.5cm} 1: Link \( a \) is disrupted

\( z^w = \{0, 1\}, \ \forall w \in W \) \hspace{0.5cm} 1: O-D \( w \) is connected

\[
\sum_{a \in A} x_a = n \hspace{0.5cm} \text{Total number of simultaneously disrupted links}
\]

\[-M \left(1 - z^w\right) + \varepsilon \leq V^w - p \leq M z^w, \ \forall w \in W \hspace{0.5cm} \text{auxiliary variables } V^w: \text{virtual maximum flow}\]

\[
\begin{align*}
z^w = 1 & \iff \varepsilon \leq V^w - p \leq M \\
z^w = 0 & \iff V^w - p \leq 0 \iff V^w \leq p
\end{align*}
\]

Binary integer linear program

Remaining network throughput or capacity after disruptions

Binary decision variables

Total number of simultaneously disrupted links

\( x_a \rightarrow z (V) = ? \)
Mathematical Formulation (2)

• Lower-level I: virtual link capacity-based maximum flow
 maximum number of distinct paths

\[
\max_x \quad V^w (\text{or } V_{rs}) \\
\text{Virtual maximum flow b/t O-D } rs \\

\text{s.t.} \\
\sum_{a \in O_i} y_a - \sum_{a \in I_i} y_a = \begin{cases} 
+V^w, & \text{if } i = r \\
-V^w, & \text{if } i = s \\
0, & \forall i \in N, i \neq r, s 
\end{cases} \\

\text{Flow conservation} \\
0 \leq y_a \leq 1, \quad \forall a \in A \\
1: \text{Link } a \text{ is selected} \\
\text{Decision variable} \\
y_a \leq 1 - x_a, \quad \forall a \in A \\
\text{Mutual exclusion b/t } x_a \text{ and } y_a \\
y_a \leq 1 - x_a \equiv \begin{cases} 
\text{if } x_a = 1 \text{ (disrupted), } y_a = 0 & \text{(not usable in any path)} \\
\text{if } x_a = 0 \text{ (connected), } y_a = 0 \text{ or } 1 & \text{(usable)}
\end{cases} \\
\text{Integral Flow Theorem: integral capacity of each link } \rightarrow \text{ integral maximal flow}
Mathematical Formulation (3)

- **Lower-level II: virtual link cost-based shortest path problem**

\[
\begin{align*}
\min_{y} & \quad u^w = \sum_{a \in A} \left( t_a + M \cdot x_a \right) y_a \quad \text{Minimum cost b/t O-D rs} \\
\text{s.t.} & \quad \sum_{a \in O_i} y_a - \sum_{a \in I_i} y_a \begin{cases} 
= 1 & \text{, if } i = r \\
= -1 & \text{, if } i = s \\
= 0 & \text{, otherwise}
\end{cases} \quad \text{Node conservation} \\
y_a \geq 0, \quad \forall a \in A \quad 1: \text{Link } a \text{ is used in SP} \quad \text{Binary decision variable}
\end{align*}
\]

\[
- Mz^w + \varepsilon \leq u^w - \bar{u}^w \leq M \left( 1 - z^w \right), \quad \forall w \in W
\]

\[
\begin{cases} 
\begin{align*}
z^w = 1 & \iff \frac{1}{M \cdot \varepsilon} \leq u^w - \theta u^w_0 \leq 0 \iff u^w \leq \theta u^w_0 & \text{exist usable path} \\
\text{unrestrictive} & \\
\end{align*}
\end{cases}
\]

\[
\begin{cases} 
\begin{align*}
z^w = 0 & \iff \varepsilon \leq u^w - \theta u^w_0 \space \underline{\notin} M \iff u^w > \theta u^w_0 & \text{no usable path} \\
\text{unrestrictive} & \\
\end{align*}
\end{cases}
\]
Modeling Flexibility

1. Flexible specifications of **vulnerability measures** (i.e., objective functions) for different modeling purposes
   - Remaining travel throughput (or network capacity) after disruptions
   - Remaining route diversity (or network redundancy) after disruptions.

2. Flexible modeling approaches in the **lower-level to check the O-D connectivity** under a network disruption scenario without path enumeration

3. Flexible thresholds to implicitly define route availability/usability and to capture the **travelers’ tolerance** for accepting these alternative routes when checking O-D connectivity under disruptions
Model Reformulation

• BLP is not directly solvable
• We reformulate it as a single-level MILP by substituting the lower-level with its KKT conditions along with linearization techniques of complementarity conditions and bilinear terms

\[
\begin{align*}
\tau_{a1} & \leq M \cdot (1 - y_a) \\
\tau_{a1} & \geq 0 \\
y_a & \in \{0, 1\}
\end{align*}
\]

\[
\begin{align*}
\tau_{a2} & \leq M \cdot (1 - (1 - x_a - y_a)) \\
1 - x_a - y_a & \geq 0 \\
\tau_{a2} & \geq 0 \\
y_a & \in \{0, 1\}
\end{align*}
\]

\[
\begin{align*}
\min \ u^w = \sum_{a \in A} (t_a + M \cdot x_a) y_a \\
P_a^w & \leq x_a, \quad P_a^w \geq x_a + y_a^w - 1 \\
P_a^w & \leq y_a^w, \quad P_a^w \geq 0
\end{align*}
\]
Equivalent MILP Reformulations

\begin{align*}
\text{max or min } & \sum_{w \in W} q^w z^w \\
\text{subject to :} & \\
\sum_{a} x_a &= n \\
-M \left(1 - z^w\right) + \varepsilon & \leq V^w \leq Mz^w, \ \forall w \\
x_a &= \{0, 1\}, \ \forall a \\
z^w &= \{0, 1\}, \ \forall w \\
V^w &= \sum_{\omega \in \Omega} y^w_{\omega} - \sum_{\omega \in \Omega} y^w_{s\omega} \\
\left(\delta_a^{r+} - \delta_a^{r-}\right) + \lambda^w \left(\delta_a^{r+} - \delta_a^{r-} + \delta_a^{s+} - \delta_a^{s-}\right) \\
+ & \sum_{i \in N, i \neq r, s} \mu_i^w \left(\delta_a^{i+} - \delta_a^{i-}\right) - \tau_{a1}^w + \tau_{a2}^w = 0, \ \forall a \\
\left(\sum_{\omega \in \Omega} y^w_{\omega} - \sum_{\omega \in \Omega} y^w_{s\omega}\right) + \left(\sum_{\omega \in \Omega} y^w_{s\omega} - \sum_{\omega \in \Omega} y^w_{s\omega}\right) &= 0 \\
\sum_{\omega \in \Omega} y^w_{\omega} - \sum_{\omega \in \Omega} y^w_{s\omega} &= 0, \ \forall i \in N, i \neq r, s \\
\tau_{a1}^w & \leq M \cdot \left(1 - y_{a}^w\right), \ \forall a \\
\tau_{a2}^w & \leq M \cdot \left(1 - (1 - x^w_a - y_a^w)\right), \ \forall a \\
1 - x^w_a - y_a^w & \geq 0, \ \tau_{a1}^w \geq 0, \ \tau_{a2}^w \geq 0, \ y_a^w \in \{0, 1\}, \ \forall a
\end{align*}

\begin{align*}
\text{max or min } & \sum_{w \in W} q^w z^w \\
\text{subject to :} & \\
\sum_{a} x_a &= n \\
-Mz^w + \varepsilon & \leq u^w - \bar{u}^w \leq M \left(1 - z^w\right), \ \forall w \\
x_a &= \{0, 1\}, \ \forall a \\
z^w &= \{0, 1\}, \ \forall w \\
u^w &= \sum_{a} t_a y_a^w + Mp_a^w \\
p_i^w & \leq x_a, \ p_s^w \leq y_s^w, \ p_r^w \geq x_a + y_a^w - 1, \ p_s^w \geq 0, \ \forall a \\
\left(t_a + Mx_a\right) + \mu_r^w \left(\delta_a^{r+} - \delta_a^{r-}\right) + \mu_s^w \left(\delta_a^{s+} - \delta_a^{s-}\right) \\
+ & \sum_{i \in N, i \neq r, s} \mu_i^w \left(\delta_a^{i+} - \delta_a^{i-}\right) - \tau_{a1}^w = 0, \ \forall a \\
\left(\sum_{\omega \in \Omega} y^w_{\omega} - \sum_{\omega \in \Omega} y^w_{s\omega}\right) &= 1, \ \text{if } i = r \\
\sum_{\omega \in \Omega} y^w_{\omega} - \sum_{\omega \in \Omega} y^w_{s\omega} &= -1, \ \text{if } i = s \\
& = 0, \ \text{otherwise} \\
\tau_{a1}^w & \leq M \cdot \left(1 - y_{a}^w\right), \ \tau_{a2}^w \geq 0, \ y_a^w \in \{0, 1\}, \ \forall a
\end{align*}

Existing algorithms in commercial software packages
Global optimal solution: exact upper and lower bounds with an implicit consideration of all possible combinations of multiple disruptions
Numerical Examples

• **Example 1**: verify the correctness of the optimal upper and lower bounds by comparing with the complete enumeration approach, and explore the implication of the vulnerability envelope.

• **Example 2**: demonstrate the flexibility of the proposed framework in terms of allowing different models in the lower-level subprogram, and implicitly defining route availability without path enumeration.

• **Example 3**: compare the vulnerability envelope between individual link disruption and pairwise link disruption.
Example Network

- For simplicity, all O-D pairs have the same demand of 1 unit
- Examples 1 and 2 consider **14 O-D pairs**: (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3), (5, 6), (6, 5)
- Example 3 considers all **30** (i.e., 6×5) O-D pairs
Example 1

Maximum flow-based model, $p=0$
Shortest path-based model, $\theta=M$

Objective Value

Number of Disrupted Links ($n$)

Largest Range = 13

Largest Slope of Upper Bound

Largest Slope of Lower Bound

Completely Disconnected
Example 1

Complete enumeration: $C_{16}^4 = 1820$

Combinatorial complexity: solve 25,480 (1820 scenarios $\times$ 14 O-D pairs) SP
Example 1

Maximum flow-based model, $p=0$
Shortest path-based model, $\theta=M$

Number of Disrupted Links ($n$)

Objective Value

Largest Range = 13

Largest Slope of Upper Bound

Largest Slope of Lower Bound

Completely Disconnected

$U(n=10)$

$L(n=2)$

$L(n=4)$

$L(n=12)$
Example 2  
Shortest path-based model: $\theta=1.5, 2, 3$
Example 2

Maximum flow-based model: $p=0, 1$

**Objective Value**

- **Best Case ($p=0$)**
- **Worst Case ($p=0$)**
- **Best Case ($p=1$)**
- **Worst Case ($p=1$)**

**Largest Range**

- Largest Range = 12 ($p=1$)
- Largest Range = 13 ($p=0$)

**Link 5**

![Graph showing the relationship between the number of disrupted links and objective value for different cases.](image)
Example 3  Individual vs. pairwise link disruption

Number of Disrupted Links ($n$)

Objective Value

- Best Case-Individual Disruption
- Worst Case-Individual Disruption
- Best Case-Pairwise Disruption
- Worst Case-Pairwise Disruption

Additional Disruptions on Links 9 & 10

Additional Disruptions on Links 5 & 6 or 13 & 14
Concluding Remarks

- Developed an optimization framework for deriving both the upper and lower bounds of network vulnerability envelope under simultaneous disruptions of multiple links, without the need of enumerating/evaluating all possible multi-disruption scenarios (but it is capable of considering all scenarios)
- Formulated it as a bi-level program
- Reformulated as a single-level MILP (w/t path enumeration)
- Demonstrated the validity, capability, and flexibility
- The vulnerability envelope could be used as a network assessment tool to more cost-effectively plan for system protection against disruptions, and prioritize improvements to minimize disruption risks with limited resources.
Main Information Summary

- **Small range & Large lower bound**
  - Stable across various possible scenarios

- **LARGE range & SMALL lower bound**
  - Vulnerable to target disruptions
  - Resilient to most random failures

- **SMALL range & SMALL lower bound**
  - Susceptible to most possible scenarios
Future Research

- Traveler’s rerouting effect on the remaining network travel throughput
- Node-based disruptions and interdependent infrastructures
- More efficient algorithms with a better utilization of the problem/model structure (e.g., use parallel computing due to the O-D pair specific lower-level subprogram)
  - Add $z_{js} - z_{is} \leq x_{ij}, \forall (i, j) \in A$
  - Add bounds of stage $n-1$ as a constraint of stage $n$
  - ......
Thank You!
**Example 1**

Best case at $n=10$

Worst case at $n=2$
Example 1

Worst case at $n=4$

Worst case at $n=12$
Formulation with Path Enumeration

\[
\begin{align*}
\text{min or max } & \quad f_n(x, y, z) = \sum_{w \in W} q_w^w z^w \\
\text{s.t.} & \quad \begin{cases}
    y_k^w & \leq \sum_{a \in A} x_a \delta_{ak}^w, \quad \forall k \in K^w, \ w \in W \\
    y_k^w & \geq x_a, \quad \forall a \in \Gamma_k^w, k \in K^w, \ w \in W \\
    z^w & \leq \sum_{k \in K^w} (1 - y_k^w), \quad \forall w \in W \\
    z^w & \geq 1 - y_k^w, \quad \forall k \in K^w, \ w \in W \\
    \sum_{a \in A} x_a & = n \quad x_a = \{0, 1\}, \ \forall a \in A \\
    y_k^w & = \{0, 1\}, \ \forall k \in K^w, \ w \in W \quad z^w = \{0, 1\}, \ \forall w \in W
\end{cases}
\end{align*}
\]