Departure Time and Route Choices with Bottleneck Congestion: User Equilibrium under Risk and Ambiguity

Yang Liu, Yuanyuan Li and Lu Hu

Department of Industrial Systems Engineering and Management
National University of Singapore

July 26, 2017
1. Introduction
2. General Model Setting
3. Single-Route Problem
4. Two-Route Problem
5. Conclusions
Morning Commute Problem

Bottleneck Model [Vickrey, 1969]
- Departure time choice
- Deterministic travel time

Extensions
- Route choice [e.g., Arnott et al. 1992; Liu and Nie 2011; Qian and Zhang 2013]
- User heterogeneity [e.g., Newell 1987; Daganzo 1985; Liu et al., 2015]
- Elastic demand [e.g., Arnott, de Palma and Lindsey 1993]
- **Stochasticity** [e.g., Lindsey 1995, 1996; Arnott et al. 1999; Fosgerau 2008; Siu and Lo 2009; Xiao et al. 2015]
Literature

Risk in decision-making

- Implications in route choice in transportation
  - Risk measures: Mean-variance [Markowitz 1952] and Expected utility theory
  - Stochastic routing [e.g., Nie and Wu 2009; Miller-Hooks and Mahmassani 2003]
  - User equilibrium [e.g., Yin et al. 2004; Connors et al. 2007; Chen and Zhou, 2010]

Ambiguity in decision-making

- Evidence: [Ellsberg 1961; Hsu et al. 2005]
- Ambiguity heterogeneity in traveler decision-making [Sikka 2012]
Literature

Risk in decision-making

- Implications in route choice in transportation
  - Risk measures: Mean-variance [Markowitz 1952] and Expected utility theory
  - Stochastic routing [e.g., Nie and Wu 2009; Miller-Hooks and Mahmassani 2003]
  - User equilibrium [e.g., Yin et al. 2004; Connors et al. 2007; Chen and Zhou, 2010]

Ambiguity in decision-making

- Evidence: [Ellsberg 1961; Hsu et al. 2005]
- Ambiguity heterogeneity in traveler decision-making [Sikka 2012]

A unified framework of risk and ambiguity [Qi et al. 2016]

- Combining CARA model and Hurwicz model [Hurwicz 1951]
- Ambiguity-aware CARA Travel Time
- Route choice in a static network equilibrium
Motivations

Uncertain travel time under ambiguity
- The perfect information of the travel time distribution is rarely available to the commuters.

Implications in departure time and route choices under risk and ambiguity
- Dynamic bottleneck equilibrium
  - Closed-form solutions
  - Homogeneous desired arrival time, value of time and unit schedule delay cost

Policy Insights
- Provide information system
- Design control strategies, e.g., congestion pricing
Model Setting

The commuters’ preferences
- Identical value of time ($\alpha$), unit schedule delay cost ($\beta$ for early arrival and $\gamma$ for late arrival)
- Attitudes towards risk and ambiguity of class $j$
  - Risk parameter: $\lambda_j \in (-\infty, +\infty)$
  - Ambiguity parameter: $\mu_j \in [0, 1]$

Uncertain travel time on route $i$

$$\widetilde{T}_i(t) = Q_i(t)/s_i + T_{0i} + \tilde{\varepsilon}_i$$

- The random travel delay belongs to a bounded distributional uncertainty set $\mathbb{H}$

$$\mathbb{H} = \{P|E_P(\tilde{\varepsilon}_i) = 0, P(\tilde{\varepsilon}_i \in [-\Delta_i, \Delta_i]) = 1\}$$

- $\Delta_i$: the maximum variation of travel time
Definition of ACT

ACT $G_j(\widetilde{T}_i(t))$ for the commuter in group $j$ who departs from home at $t$ on route $i$, $\Omega \to \mathcal{R}$ for the commuter with parameter $\mathcal{V} = (\lambda_j, \mu_j, \mathbb{H})$ is

$$
G_j(\widetilde{T}_i(t)) = \begin{cases} 
\mu_j \sup_{P \in \mathbb{H}} \frac{1}{\lambda_j} \ln E_P(\exp(\lambda_j \cdot \widetilde{T}_i(t))) + (1 - \mu_j) \inf_{P \in \mathbb{H}} \frac{1}{\lambda_j} \ln E_P(\exp(\lambda_j \cdot \widetilde{T}_i(t))), & \lambda_j \neq 0 \\
\mu_j \sup_{P \in \mathbb{H}} E_P(\widetilde{T}_i(t)) + (1 - \mu_j) \inf_{P \in \mathbb{H}} E_P(\widetilde{T}_i(t)), & \lambda_j = 0 
\end{cases}
$$

- $G_j(\widetilde{T}_i(t))$: the perceived travel time
- $\Omega$: the state-space of the uncertain $\widetilde{T}_i(t)$
- $\mathcal{P}$: the unknown true probability distribution of $\widetilde{T}_i(t)$
- $\mathbb{H}$: a distributional uncertainty set
- Adopted from (Qi et al., 2016), ACT is a function of departure time
The Closed Form of ACT

Proposition

The ambiguity-aware CARA travel time $G_j(\tilde{T}_i(t))$ for the commuter with parameter $V = (\lambda_j, \mu_j, \mathbb{H})$ admits the closed form:

$$G_j(\tilde{T}_i(t)) = T_i(t) + f(\lambda_j, \mu_j, \Delta_i).$$

where

$$f(\lambda_j, \mu_j, \Delta_i) = \begin{cases} 
\frac{\mu_j}{\lambda_j} \ln\left(\frac{1+\exp(2\cdot\lambda_j\cdot\Delta_i)}{2}\right) - \mu_j \cdot \Delta_i, & \lambda_j > 0 \\
\frac{1-\mu_j}{\lambda_j} \ln\left(\frac{1+\exp(2\cdot\lambda_j\cdot\Delta_i)}{2}\right) - (1 - \mu_j) \cdot \Delta_i, & \lambda_j < 0 \\
0, & \lambda_j = 0
\end{cases}$$

- Interpretation of $f(\lambda_j, \mu_j, \Delta_i)$: perceived uncertainty in travel time
- $f(\lambda_j, \mu_j, \Delta_i) < 0$ when $\lambda < 0$; $f(\lambda_j, \mu_j, \Delta_i) > 0$ when $\lambda > 0$. 

Yang Liu, Yuanyuan Li and Lu Hu
National University of Singapore
Ambiguity-Aware CARA Commute Cost (ACC)

**Definition**

The ACC $F(\tilde{C}_i(t))$ for the commuter who departs from home at $t$ on route $i$, with parameter $V = (\lambda_j, \mu_j, H)$ is:

$$F_j(\tilde{C}_i(t)) = \alpha G_j(\tilde{T}_i(t)) + \begin{cases} \beta(t^* - t - G_j(\tilde{T}_i(t))), & t \leq t^* - G_j(\tilde{T}_i(t)) \\ \gamma(t + G_j(\tilde{T}_i(t)) - t^*), & t \geq t^* - G_j(\tilde{T}_i(t)) \end{cases}$$

- The commuters make decisions based on ACT
- At equilibrium, no one can reduce her own ACC commute cost by changing departure time or route choices.
Single-Route Single-Class Problem

- Homogeneous preference: \( V = (\lambda, \mu, H) \)

Queuing rate at equilibrium

\[
\frac{dF(\tilde{C}(t))}{dt} = 0 \Rightarrow \frac{dQ(t)}{dt} = \begin{cases} 
\frac{\beta}{\alpha-\beta} s, & t_s \leq t \leq t_m \\
\frac{-\gamma}{\alpha+\gamma} s, & t_m \leq t \leq t_e 
\end{cases}
\]

- ACC commute cost: \( F(\tilde{C}) = \frac{\delta N}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda, \mu, \Delta) \)

- Perceived uncertainty cost: \( \alpha \cdot f(\lambda, \mu, \Delta) \)
Both the ACC commute cost and the perceived uncertainty cost in time unit are increasing with ambiguity and risk.

The congestion shifts to earlier for risk-averse and later for risk-seeking.

\[ F(\tilde{C}) \]

\[ f(\lambda, \mu, \Delta) \]
Model Setting for Single-Route Two-Class Problem

The commuters with parameter $V_1 = (\lambda_1, \mu_1, H)$
- The risk-averse class ($j = 1$): the perceived uncertainty $f(\lambda_1, \mu_1, \Delta)$
- The risk-seeking class ($j = 2$): the perceived uncertainty $f(\lambda_2, \mu_2, \Delta)$

The preference gap between the two classes

$$f_{12} = f(\lambda_1, \mu_1, \Delta) - f(\lambda_2, \mu_2, \Delta) > 0$$

- Analytically derive UE solution
- The UE solution depends on $f_{12}$
Proposition (User equilibrium for the single-route two-class problem)

For the single-route bottleneck model with a risk-averse class and a risk-seeking class (parameter $V_j = (\lambda_j, \mu_j, H)$, $j = 1, 2$, the UE solution depends on the gap between the costs of uncertainty in time unit $f_{12} \equiv f(\lambda_1, \mu_1, \Delta) - f(\lambda_2, \mu_2, \Delta)$

- **Case 1**: $f_{12} \in \left[\frac{\delta N_1}{\gamma s} + \frac{\delta N_2}{\beta s}, \infty\right)$
- **Case 2**: $f_{12} \in \left[\left|\frac{\delta N_1}{\gamma s} - \frac{\delta N_2}{\beta s}\right|, \frac{\delta N_1}{\gamma s} + \frac{\delta N_2}{\beta s}\right]$
- **Case 3**: $f_{12} \in \left[0, \frac{\delta N_2}{\beta s} - \frac{\delta N_1}{\gamma s}\right]$
- **Case 4**: $f_{12} \in \left[0, \frac{\delta N_1}{\gamma s} - \frac{\delta N_2}{\beta s}\right]$
Single-Route Two-Class Problem: User Equilibrium

Small preference gap $f_{12}$: $f_{12} \in [0, |\frac{\delta}{\beta} \frac{N_2}{s} - \frac{\delta}{\gamma} \frac{N_1}{s}|]$

- Two classes share the same isocost curve
- Given certain parameter setting, only one of them occurs
- We focus on the aggregated departure curve
  - Assumption: the risk-averse class depart earlier than the risk-seeking class to avoid infinite possibilities of departure order

Case 3

Case 4
Single-Route Two-Class Problem: User Equilibrium

Intermediate preference gap \( f_{12} \in \left[ \left| \frac{\delta}{\gamma} \frac{N_1}{s} - \frac{\delta}{\beta} \frac{N_2}{s} \right|, \frac{\delta}{\gamma} \frac{N_1}{s} + \frac{\delta}{\beta} \frac{N_2}{s} \right] \)

- Two peaks occur in peak-hour congestion
- The risk-averse class prefer earlier time slots

Case 2
Large preference gap $f_{12} \in \left[ \frac{\delta}{\gamma} \frac{N_1}{s} + \frac{\delta}{\beta} \frac{N_2}{s}, \infty \right)$

- Two separate peaks
- The risk-averse class prefer earlier time slots
Two classes’ departure time windows deviate with $f_{12}$.
The total expected travel time (TE) is non-increasing with $f_{12}$.
The heterogeneous preferences towards risk and ambiguity can spread the departures and alleviate average congestion.

<table>
<thead>
<tr>
<th></th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_{12}$</th>
<th>$t_{s1}$</th>
<th>$t_{m1}$</th>
<th>$t_{e1}$</th>
<th>$t_{s2}$</th>
<th>$t_{m2}$</th>
<th>$t_{e2}$</th>
<th>TE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.8</td>
<td>-0.7</td>
<td>1.5</td>
<td>5.04</td>
<td>5.69</td>
<td>7.13</td>
<td>7.21</td>
<td>7.59</td>
<td>8.46</td>
<td>5718</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.6</td>
<td>-0.5</td>
<td>1.1</td>
<td>5.08</td>
<td>5.79</td>
<td>6.68</td>
<td>6.68</td>
<td>7.01</td>
<td>8.42</td>
<td>5742</td>
</tr>
<tr>
<td>Case 3</td>
<td>0.4</td>
<td>-0.1</td>
<td>0.5</td>
<td>4.95</td>
<td>5.79</td>
<td>5.76</td>
<td>5.76</td>
<td>5.98</td>
<td>8.28</td>
<td>7233</td>
</tr>
<tr>
<td>Case 4</td>
<td>0.4</td>
<td>-0.1</td>
<td>0.5</td>
<td>4.45</td>
<td>5.48</td>
<td>7.50</td>
<td>7.50</td>
<td>7.17</td>
<td>7.78</td>
<td>7233</td>
</tr>
</tbody>
</table>
Outline

1. Introduction

2. General Model Setting

3. Single-Route Problem

4. Two-Route Problem

5. Conclusions
Model Setting for the Two-Route Problem

- Two parallel routes: A highway (route 1) and a local arterial (route 2)
  - The maximum variation: $\Delta_1 > \Delta_2$
  - The free flow travel time: $T_{01} < T_{02}$

- Homogeneous commuters: $(\lambda, \mu, \Delta_i), i = 1, 2.$

- Flow on route $i$: $n_i$

- ACC commute cost on route $i$: $F_i(\tilde{C}) = \alpha( T_{0i} + f(\lambda, \mu, \Delta_i)) + \frac{\delta n_i}{s_i}$

- The gap of uncertainty costs between two routes:
  $$\tilde{f}_{12} = f(\lambda, \mu, \Delta_1) - f(\lambda, \mu, \Delta_2)$$
Proposition 4

For the two-route problem with homogeneous commuters, the commuter distribution \((n_1\text{ and } n_2)\) between routes 1 and route 2 at UE is:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Flow distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1 (N_1^* \geq 0) and (N \leq N_1^*)</td>
<td>(n_1 = N) (n_2 = 0)</td>
</tr>
<tr>
<td>Case 2 (N_1^* \geq 0) and (N &gt; N_1^*)</td>
<td>(n_1 = \frac{s_2}{s_1 + s_2} N_1^* + \frac{s_1}{s_1 + s_2} N) (n_2 = \frac{s_2}{s_1 + s_2} (N - N_1^*))</td>
</tr>
<tr>
<td>Case 3 (N_2^* \geq 0) and (N \leq N_2^*)</td>
<td>(n_1 = 0) (n_2 = N)</td>
</tr>
<tr>
<td>Case 4 (N_2^* \geq 0) and (N &gt; N_2^*)</td>
<td>(n_1 = \frac{s_1}{s_1 + s_2} (N - N_2^<em>)) (n_2 = \frac{s_1}{s_1 + s_2} N_2^</em> + \frac{s_2}{s_1 + s_2} N)</td>
</tr>
</tbody>
</table>

where \(N_1^* = s_1 \frac{\alpha}{\delta} (T_{02} - T_{01} - \bar{f}_{12})\) and \(N_2^* = s_2 \frac{\alpha}{\delta} (T_{01} - T_{02} + \bar{f}_{12})\).

- The impacts of uncertainty on flow distribution by comparing the interior solutions with deterministic model
  - \(\frac{s_2}{s_1 + s_2} N_1^*\) more commuters on route 1 in Case 2
  - \(\frac{s_1}{s_1 + s_2} N_2^*\) more commuters on route 2 in Case 4
Impacts of Uncertainty

**Proposition 5**

For the two-routes bottleneck model with homogeneous commuters, the route choices of commuters with parameter \( V = (\lambda, \mu, \mathbb{H}) \) between routes 1 and route 2 depend on the attitude towards risk, i.e., the flow on route 1 \((n_1)\) with respect to maximum variation has the following tendency:

<table>
<thead>
<tr>
<th>Risk attitude</th>
<th>(\Delta_1)</th>
<th>(\Delta_2)</th>
<th>Flow pattern at UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda &gt; 0)</td>
<td>↓</td>
<td>↑</td>
<td>Any one of Case 1-4</td>
</tr>
<tr>
<td>(\lambda &lt; 0)</td>
<td>↑</td>
<td>↓</td>
<td>Any one of Case 1-2</td>
</tr>
<tr>
<td>(\lambda = 0)</td>
<td>–</td>
<td>–</td>
<td>Any one of Case 1-2</td>
</tr>
</tbody>
</table>

- For the risk-averse commuters: the larger \(\Delta_1\) (\(\Delta_2\)) will make the highway less (more) attractive.
- For the risk-seeking commuters: the larger \(\Delta_1\) (\(\Delta_2\)) will make the highway more (less) attractive.
- For the risk-neutral commuters, the uncertainties have no impact on the route choices.
Numerical Examples

The impacts of uncertainty on route choice at UE

The flow distribution with $\Delta_1$

The flow distribution with $\Delta_2$

The monotonicity of flow on the highway

- Risk-averse class: decreases with $\Delta_1$ and increases with $\Delta_2$
- Risk-seeking class: increases with $\Delta_1$ and decreases with $\Delta_2$
Numerical Examples

The monotonicity of system performance at UE

Total expected travel time given $\Delta_1$

System cost at UE given $\Delta_1$

Policy insights:

- Risk-averse commuters: reducing uncertainty on the highway will improve the two system performance measures simultaneously.
- Risk-seeking commuters: the trade-off between TE and TC should be considered.
Numerical Examples

The sensitivity of price of anarchy (PoA) given risk $\lambda$ and ambiguity $\mu$

The PoA for risk-averse class

- The upper bound and lower bound in this example can be observed
- For risk-averse commuters, the more pessimistic preference towards ambiguity results in higher efficiency loss.

The PoA for risk-seeking class

- For risk-seeking commuters, the more optimistic preference towards ambiguity results in higher efficiency loss.
1 Introduction

2 General Model Setting

3 Single-Route Problem

4 Two-Route Problem

5 Conclusions
Conclusions

Single-route single-class problem

- Both the perceived uncertainty in time unit $f(\lambda, \mu, \Delta)$ and the ACC commute cost are monotonically increasing with ambiguity and risk.
- Compared with the risk-neutral class,
  - The risk-averse (risk-seeking) class will overestimate (underestimate) the commute cost.
  - The departure time window is shifted earlier (later) for the risk-averse (risk-seeking) class.

Single-route two-classes problem

- Four congestion patterns at UE based on the preference gap between two classes
- A mixture of heterogeneous preferences towards risk and ambiguity may spread the departure time choices and relieve the average peak hour congestion
Conclusions

Two-routes problem

- Decreasing maximum time variation on the highway, e.g., by providing traffic information,
  - will increase its flow when commuters are risk-averse, but will decrease its flow when commuters are risk-seeking.
  - will reduce total expected travel time and perceived system cost for risk-averse commuters
Thanks!

-The end-
<table>
<thead>
<tr>
<th>Risk attitude</th>
<th>Reference</th>
<th>Uncertainty source</th>
<th>Probability distribution</th>
<th>Risk measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutral</td>
<td>de Palma et al. (1983)</td>
<td>Utility</td>
<td>Gumbel</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Arnott et al. (1991)</td>
<td>Capacity</td>
<td>Two possible levels</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Daniel (1995)</td>
<td>Arrival number</td>
<td>Poisson</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Noland and Small (1995)</td>
<td>Travel time</td>
<td>Uniform, Exponential</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Arnott et al. (1999)</td>
<td>Capacity and demand</td>
<td>General</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Fosgerau (2010)</td>
<td>Capacity and demand</td>
<td>General</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Fosgerau and Small (2013)</td>
<td>Capacity</td>
<td>Two possible levels</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Coulombel and de Palma (2014)</td>
<td>Travel time</td>
<td>Uniform</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Xiao et al. (2014)</td>
<td>Capacity</td>
<td>Uniform</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Xiao et al. (2015)</td>
<td>Capacity</td>
<td>Uniform</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Xiao et al. (2017)</td>
<td>Travel time</td>
<td>Uniform</td>
<td>None</td>
</tr>
<tr>
<td>Aversion</td>
<td>Yin et al. (2004)</td>
<td>Travel time</td>
<td>Normal</td>
<td>Concave utility function</td>
</tr>
<tr>
<td></td>
<td>Li et al. (2008)</td>
<td>Capacity</td>
<td>Uniform</td>
<td>Mean-standard deviation</td>
</tr>
<tr>
<td></td>
<td>Li et al. (2009)</td>
<td>Capacity</td>
<td>General, Normal</td>
<td>Mean-standard deviation</td>
</tr>
<tr>
<td></td>
<td>Siu and Lo (2009)</td>
<td>Travel time</td>
<td>Uniform</td>
<td>Punctuality reliability</td>
</tr>
<tr>
<td></td>
<td>Yao et al. (2010)</td>
<td>Toll</td>
<td>Uniform, Log-normally</td>
<td>Mean-standard deviation</td>
</tr>
<tr>
<td></td>
<td>Yao et al. (2012)</td>
<td>Toll</td>
<td>Uniform, GBM</td>
<td>Mean-standard deviation</td>
</tr>
<tr>
<td></td>
<td>Siu and Lo (2013)</td>
<td>Travel time</td>
<td>Uniform</td>
<td>Punctuality reliability</td>
</tr>
<tr>
<td></td>
<td>Siu and Lo (2014)</td>
<td>Travel time</td>
<td>Uniform</td>
<td>Punctuality reliability</td>
</tr>
</tbody>
</table>
At UE

- The critical times

\[
\begin{align*}
  ts &= t^* - T_0 - f(\lambda, \mu, \Delta) - \frac{\delta}{\beta} \frac{N}{s} \\
  tm &= t^* - T_0 - f(\lambda, \mu, \Delta) - \frac{\delta}{\alpha} \frac{N}{s} \\
  te &= t^* - T_0 - f(\lambda, \mu, \Delta) + \frac{\delta}{\gamma} \frac{N}{s},
\end{align*}
\]

- Total expected travel time at UE:

\[
TE = \frac{1}{2s} \frac{\delta}{\alpha} N^2 + N \cdot T_0
\]
Single-route Single-class problem

At SO

- The critical times

\[
\begin{align*}
t_s &= t^* - T_0 - f(\lambda, \mu, \Delta) - \frac{\delta}{\beta} \frac{N}{s} \\
t_m &= t^* - T_0 - f(\lambda, \mu, \Delta) \\
t_e &= t^* - T_0 - f(\lambda, \mu, \Delta) + \frac{\delta}{\gamma} \frac{N}{s}
\end{align*}
\]

- The minimum system cost at SO:

\[
TC^* = \frac{\delta N^2}{2s} + \alpha \cdot T_0 N + \alpha \cdot f(\lambda, \mu, \Delta) N
\]

- In the deterministic bottleneck model, the system cost (excluding the free flow travel time $\alpha T_0 N$) is reduced by half at SO

- In this model, the system cost is reduced by less than a half for risk-averse commuters and more than a half for risk-seeking commuters at SO.
Case 1: \( f_{12} \in \left[ \frac{\delta N_1}{\gamma s} + \frac{\delta N_2}{\beta s}, \infty \right) \)

- The ACC commute cost
  \[
  F(\tilde{C}_j) = \delta \frac{N_j}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda_j, \mu_j, \Delta),
  \]

- The queue length is given by
  \[
  Q(t) = \begin{cases} 
  \frac{\beta}{\alpha - \beta} s(t - t_{s_j}), & t_{s_j} \leq t \leq t_{m_j} \\
  \frac{-\gamma}{\alpha + \gamma} s(t - t_{m_j}) + Q(t_{m_j}), & t_{m_j} \leq t \leq t_{e_j},
  \end{cases}
  \]

  with
  \[
  \begin{aligned}
  t_{s_j} &= t^* - T_0 - f(\lambda_j, \mu_j, \Delta) - \frac{\delta N_j}{\beta s}, \\
  t_{m_j} &= t_{s_j} + \frac{1}{\theta} \frac{\delta N_j}{\beta s}, \\
  t_{e_j} &= t_{m_j} + \frac{1}{\rho} \frac{\delta N_j}{\gamma s},
  \end{aligned}
  \]

- The total expected travel time for all commuters is
  \[
  TE = \frac{1}{2s} \frac{\delta}{\alpha} \sum_j N_j^2 + N \cdot T_0
  \]
Case 2: \( f_{12} \in \left[ \left| \frac{\delta N_1}{\gamma} - \frac{\delta N_2}{\beta} \right|, \frac{\delta N_1}{\gamma} + \frac{\delta N_2}{\beta} \right] \)

- The ACC commute cost

\[
F(\tilde{C}_1) = \frac{1}{2} \left( \beta s + \frac{\delta N}{s} - \beta f_{12} \right) + \alpha \cdot T_0 + \alpha \cdot f(\lambda_1, \mu_1, \Delta)
\]

\[
F(\tilde{C}_2) = \frac{1}{2} \left( \gamma s + \frac{\delta N}{s} - \gamma f_{12} \right) + \alpha \cdot T_0 + \alpha \cdot f(\lambda_2, \mu_2, \Delta)
\]

- the queue length is given by

\[
Q(t) = \begin{cases} 
\frac{\beta}{\alpha - \beta} s(t - t_{s1}), & t_{s1} \leq t \leq t_{m1} \\
\frac{-\gamma}{\alpha + \gamma} s(t - t_{m1}) + Q(t_{m1}), & t_{m1} \leq t \leq t_{e1} \\
\frac{\beta}{\alpha - \beta} s(t - t_{s2}) + Q(t_{e1}), & t_{s2} \leq t \leq t_{m2} \\
\frac{-\gamma}{\alpha + \gamma} s(t - t_{m2}) + Q(t_{m2}), & t_{m2} \leq t \leq t_{e2} 
\end{cases}
\]

where

\[
\begin{align*}
t_{s1} &= t^* - T_0 - \frac{1}{2} \left[ \frac{N_1}{s} + \frac{\delta}{\beta} N s + \sum_j f(\lambda_j, \mu_j, \Delta) \right] \\
t_{m1} &= t_{s1} + \frac{1}{2 \theta} \left( \frac{N_1}{s} + \frac{\delta}{\beta} N s - f_{12} \right) \\
t_{e1} &= t_{m1} + \frac{1}{2 \rho} \left( \frac{N_1}{s} - \frac{\delta}{\beta} N s + f_{12} \right) \\
t_{s2} &= t_{e1} \\
t_{m2} &= t_{s2} + \frac{1}{2 \theta} \left( -\frac{N_1}{s} + \frac{\delta}{\beta} N s + f_{12} \right) \\
t_{e2} &= t_{m2} + \frac{1}{2 \rho} \left( \frac{N_2}{s} + \frac{\delta}{\gamma} N s - f_{12} \right)
\end{align*}
\]
Single-route Two-classes Problem: UE

Case 3: \( f_{12} \in [0, \frac{\delta N_2}{\beta s} - \frac{\delta N_1}{\gamma s}] \) and \( \frac{N_2}{\beta} \geq \frac{N_1}{\gamma} \)

- The ACC commute cost

\[
\begin{align*}
F(\tilde{C}_1) &= \delta \frac{N}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda_1, \mu_1, \Delta) - \beta f_{12} \\
F(\tilde{C}_2) &= \delta \frac{N}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda_2, \mu_2, \Delta)
\end{align*}
\]

- The queue length is given by

\[
Q(t) = \begin{cases} \\
\frac{\beta}{\alpha-\beta} s(t - t_{s1}), & t_{s1} \leq t \leq t_{e1} \\
\frac{\beta}{\alpha-\beta} s(t - t_{e1}) + Q(t_{e1}), & t_{e1} \leq t \leq t_{m2} \\
\frac{\gamma}{\alpha + \gamma} s(t - t_{m2}) + Q(t_{m2}), & t_{m2} \leq t \leq t_{e2}
\end{cases}
\]

where

\[
\begin{align*}
t_{s1} &= t^* - T_0 - f(\lambda_2, \mu_2, \Delta) - \frac{\delta N}{\beta s} \\
t_{e1} &= t_{s1} + \frac{\alpha-\beta}{\alpha} \frac{N_1}{s} \\
t_{s2} &= t_{e1} \\
t_{m1} &= t^* - T_0 - f(\lambda_1, \mu_1, \Delta) + \frac{\beta}{\alpha} f_{12} - \frac{\delta N}{\alpha s} \\
t_{m2} &= t^* - T_0 - f(\lambda_2, \mu_2, \Delta) - \frac{\delta N}{\alpha s} \\
t_{e2} &= t^* - T_0 - f(\lambda_2, \mu_2, \Delta) + \frac{\delta N}{\gamma s},
\end{align*}
\]
Case 4: $f_{12} \in [0, \frac{\delta N_1}{\gamma s} - \frac{\delta N_2}{\beta s}]$ and $\frac{N_2}{\beta} \geq \frac{N_1}{\gamma}$

- The ACC commute cost

$$F(\tilde{C}_1) = \delta \frac{N}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda_1, \mu_1, \Delta)$$

$$F(\tilde{C}_2) = \delta \frac{N}{s} + \alpha \cdot T_0 + \alpha \cdot f(\lambda_2, \mu_2, \Delta) - \gamma f_{12}$$

- The queue length is given by

$$Q(t) = \begin{cases} 
\frac{\beta}{\alpha-\beta} s(t - t_{s1}), & t_{s1} \leq t \leq t_{m1} \\
-\frac{\gamma}{\alpha+\gamma} s(t - t_{m1}) + Q(t_{m1}), & t_{m1} \leq t \leq t_{e1} \\
-\frac{\gamma}{\alpha+\gamma} s(t - t_{e1}) + Q(t_{e1}), & t_{e1} \leq t \leq t_{e2} 
\end{cases}$$

where

$$t_{s1} = t^* - T_0 - f(\lambda_1, \mu_1, \Delta) - \frac{\delta N}{\beta s}$$

$$t_{m1} = t^* - T_0 - f(\lambda_1, \mu_1, \Delta) - \frac{\delta N}{\beta s}$$

$$t_{m2} = t^* - T_0 - f(\lambda_2, \mu_2, \Delta) + \frac{\gamma}{\alpha} f_{12} - \frac{\delta N}{\alpha s}$$

$$t_{e1} = t^* - T_0 - f(\lambda_1, \mu_1, \Delta) + \frac{\delta N}{\gamma s} - \frac{\alpha+\gamma}{\alpha s} N_2$$

$$t_{s2} = t_{e1}$$

$$t_{e2} = t^* - T_0 - f(\lambda_1, \mu_1, \Delta) + \frac{\delta N}{\gamma s}.$$
### Findings and Contributions

**Single-route model with homogeneous preference**

- **The impact of risk preference**

<table>
<thead>
<tr>
<th></th>
<th>Risk-averse</th>
<th>Risk-neutral</th>
<th>Risk-seeking</th>
</tr>
</thead>
<tbody>
<tr>
<td>The movement of</td>
<td>←</td>
<td>-</td>
<td>→</td>
</tr>
<tr>
<td>departure time</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>window</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **The impact of ambiguity**

<table>
<thead>
<tr>
<th></th>
<th>Pessimistic</th>
<th>Neutral</th>
<th>Optimistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude of the</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>movement</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perceived uncertainty cost</td>
<td>↑</td>
<td>-</td>
<td>↓</td>
</tr>
</tbody>
</table>

**Single-route model with heterogeneous preferences**

- Stagger the departure time choice
- Relieve traffic congestion
Findings and Contributions

Two-route model
The impacts of uncertainty on highway

<table>
<thead>
<tr>
<th>The flow on highway</th>
<th>The expected travel time (TE)</th>
<th>Total ACC commute cost (TC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-averse</td>
<td>↘</td>
<td>↗</td>
</tr>
<tr>
<td>Risk-seeking</td>
<td>↗</td>
<td>↗</td>
</tr>
</tbody>
</table>

Policy insights

- Reduce uncertainty on highway to relieve traffic congestion if the commuters are risk-averse
- Make the trade-off between TE and TC if the commuters are risk-seeking