Day-to-day departure time choice under bounded rationality in the bottleneck model

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1. Introduction (1)

The bottleneck model (Vickrey, 1969)

1. Formulate the departure time choice or trip-timing decision of commuters during morning rush hour

2. Formulate the within-day dynamics of traffic on links in general networks

3. Capture the essence of spatio-temporal congestion dynamics

Develop a doubly dynamical system

1. Combine the within-day dynamics and the day-to-day dynamics

2. Describe the dynamic evolution process of departure time choice from non-equilibrium state to equilibrium
1. Introduction (2)

Two motivations for developing a doubly dynamical system

1. For a bottleneck system in a disequilibrium state, commuters adjust their departure time from day to day for reducing their travel costs and the departure rates evolve over time before reaching a new equilibrium state.

The dynamic adjustment process from non-equilibrium to equilibrium can be regarded as a process of finding an equilibrium point.

1. Better understanding various processes of forming traffic jam
2. Better utilizing various advanced traveler information systems

Designing an effective algorithm of solving the equilibrium problem.
Relevant researches: Ben-Akiva, Cyna and de Palma (1984), Ben-Akiva, de Palma and Kanaroglou (1986), Iryo (2008), Wu (2009), and Ge et al. (2015)

These dynamical systems are established based on an assumption that commuters are perfectly rational, namely, they always choose to depart from an origin at a time with the minimum (perceived) travel cost.
1. Introduction (4)

From a practical standpoint, the notion of perfect rationality is not entirely in line with realistic departure time choice (or route choice) behaviour and empirical observations (Avineri and Prashker, 2004; Morikawa et al., 2005; Guo and Liu, 2011; Zhao and Huang, 2016)

From a theoretical standpoint, the trajectories of these dynamical systems, which formulate the day-to-day adjustment process of departure rate, may not converge to perfectly rational user equilibrium under certain precondition (Iryo, 2008; Guo, Yang and Huang, 2016)

Establish a doubly dynamical system that formulates the day-to-day departure time choice of commuters under bounded rationality
1. Introduction (5)

Our works:

1. Extend the proportional swap system by Smith (1984a) and the network tatonnement process by Friesz et al. (1994) to formulate the day-to-day evolution of departure rate in the bottleneck model towards boundedly rational user equilibrium (BRUE)

2. Analyze the existence of the stationary points of the two dynamical systems and the equivalence between the stationary state and the BRUE state

3. Propose a day-to-day pricing policy to drive day-to-day departure rate in the two systems to evolve to the system optimum (SO) state
4. Prove that, under the implementation of the pricing policy, the stationary state is consistent with the SO state and also the day-to-day departure rate can converge to the SO state

Without the intervention of the pricing policy, the day-to-day departure rate may not converge to the BRUE state (Guo, Yang and Huang, 2016; Ramadurai and Ukkusuri, 2007; Ziegelmeyer et al., 2008; Daniel, Gisches and Rapoport, 2009)

The proposed congestion control scheme is still effective no matter whether the day-to-day departure rate without the intervention of the control scheme can evolve to the BRUE state
2. Overview of the bottleneck model (1)

- Every morning $N$ commuters travel from home (origin) to work (destination) along a highway with a bottleneck
- Each commuter drives a vehicle
- The capacity of the bottleneck is $s$ vehicles per unit time
- The queue follows the first-in-first-out (FIFO) rule
2. Overview of the bottleneck model (2)

- The travel time — \( T(t) = T^f + T^v(t) \)
- The free flow travel time — \( T^f = 0 \)
- The queuing time — \( T^v(t) = D(t)/s \)
- The queuing length — \( D(t) = \int_{\hat{t}}^{t} r(u)du - s(t - \hat{t}) \)
- The travel cost — \( C(t) = \begin{cases} \alpha T(t) + \beta \left( t^* - t - T(t) \right), & \text{if } t \leq \frac{\theta_c}{\gamma} \\ \alpha T(t) + \gamma \left( t + T(t) - t^* \right), & \text{if } t > \frac{\theta_c}{\gamma} \end{cases} \)
- The value of travel time — \( \alpha \)
- The value of schedule delay early — \( \beta \)
- The value of schedule delay late — \( \gamma \)

Arnott, de Palma and Lindsey (1990)
2. Overview of the bottleneck model (3)

At the DUE state:

- The departure rate \( r^*(t) = \begin{cases} \frac{\beta s}{\alpha - \beta}, & \text{for } t \in [t_q, t_{q'}], \\ s - \frac{\gamma s}{\alpha + \gamma}, & \text{for } t \in (t_{q'}, t_{q''}] \end{cases} \)

\[
t_q = t^* - \left( \frac{\gamma}{\beta + \gamma} \right) \left( \frac{N}{s} \right)
\]

\[
t_{q'} = t^* + \left( \frac{\beta}{\beta + \gamma} \right) \left( \frac{N}{s} \right)
\]

- The travel cost \( C^*(t) = \left( \frac{\beta \gamma}{\beta + \gamma} \right) \left( \frac{N}{s} \right) \) for \( t \in [t_q, t_{q'}] \)
3. Systems description (1)

- The time horizon —  \([0, \bar{T}] \supseteq [t_q, t_{q'}]\)

- The length of each slice —  \(\Delta t\)

- The set of feasible departure rates —

\[
\Omega \equiv \left\{ \mathbf{r} = (r_i, i = 1, 2, \ldots, M)^T \left| \sum_{i=1}^M r_i \Delta t = N, r_i \geq 0, i = 1, 2, \ldots, M \right. \right\}
\]

- The travel time —  \(T_i(\mathbf{r}) = T^f_i + T^v_i(\mathbf{r})\)

- The free flow travel time —  \(T^f_i = 0\)

- The queuing time —  \(T^v_i(\mathbf{r}) = D_i(\mathbf{r})/s\)
3. Systems description (2)

- The length of the queue — Ramadurai et al., 2010

\[ D_i(r) = \begin{cases} 
(r_i - s)\Delta t, & \text{for } i = 1, \\
[D_{i-1}(r) + (r_i - s)\Delta t]_+, & \text{for } i = 2, 3, L, M,
\end{cases} \]

- The travel cost —

\[ C_i(r) = \begin{cases} 
\alpha T_i(r) + \beta \left( t^* - (i-1)\Delta t - T_i(r) \right), & \text{if } (i-1)\Delta t + T_i(r) \leq t^*, \\
\alpha T_i(r) + \gamma \left( (i-1)\Delta t + T_i(r) - t^* \right), & \text{if } (i-1)\Delta t + T_i(r) > t^*,
\end{cases} \]

- \( r \) is a BRUE departure rate pattern if it holds that (Mahmassani and Chang, 1987)

\[ C_i(r) \leq \mu + \varepsilon \text{ if } r_i > 0, \forall i = 1, 2, L, M. \]
3. Systems description (3)

- The decision-making travel cost (Han, Szeto and Friesz, 2015) —
  \[
  \overline{C}_i(r) = \max \left\{ C_j(r), \min_{j=1,2,L,M} \left\{ C_j(r) \right\} + \varepsilon \right\}
  \]

- The proportional swap system

  \[
  r_i^{(n+1)} = F_i \left( r^{(n)} \right) = r_i^{(n)} + \eta \Gamma_i \left( r^{(n)} \right)
  \]

  \[
  \Gamma_i \left( r^{(n)} \right) = \sum_{j=1}^{M} \left( r_j^{(n)} \left[ \overline{C}_j \left( r^{(n)} \right) - \overline{C}_i \left( r^{(n)} \right) \right]_+ - r_i^{(n)} \left[ \overline{C}_i \left( r^{(n)} \right) - \overline{C}_j \left( r^{(n)} \right) \right]_+ \right)
  \]

  for \( i = 1, 2, L, M \), \( n = 0, 1, 2, L \), \( r^{(0)} \in \Omega \). \( \eta > 0 \)
3. Systems description (4)

The tatonnement adjustment process

\[ r^{(n+1)} = F(r^{(n)}) = r^{(n)} + \lambda \Gamma(r^{(n)}) \]

\[ \Gamma(r^{(n)}) = P_{\Omega} \left( r^{(n)} - \theta \mathcal{C}(r^{(n)}) \right) - r^{(n)} \]

for \( n = 0,1,2,\ldots \) and \( r^{(0)} \in \Omega \). \( 0 < \lambda \leq 1 \), \( \theta > 0 \)

The proportional swap system and network tatonnement process are referred to as two doubly dynamical systems.

day-to-day dynamics within-day dynamics
4. Systems properties (1)

**Theorem 1.** The travel cost function $C$ is Lipschitz continuous with respect to $r$ on the set $\Omega$.  

The travel cost function $C$ is not differentiable with respect to $r$ on the set $\Omega$.  

The travel cost function $C$ is not monotonic with respect to the departure rate $r$.  

**Corollary 1.** The function $F : r \mapsto r + \eta \Gamma(r)$ is Lipschitz continuous on the set $\Omega$.  

Corollary 2. The function \( \bar{F}(r) = r + \lambda \bar{\Gamma}(r) \) is Lipschitz continuous on the set \( \Omega \).

Theorem 2. The proportional swap system has at least one stationary point.

Theorem 3. The tatonnement adjustment process has at least one stationary point.
Theorem 4. A departure rate $r^{(n)}$ is a stationary departure rate of the proportional swap system if and only if it is a BRUE departure rate.

Theorem 5. A departure rate $r^{(n)}$ is a stationary departure rate of the tatonnement adjustment process if and only if it is a BRUE departure rate.
Combining Theorems 2 and 4 (or combining Theorems 3 and 5) concludes that there exists at least one BRUE point for the bottleneck model. In this way, the existence of BRUE departure rate is proved from the viewpoint of dynamical systems.
5. A day-to-day pricing policy (1)

When a departure rate \( r \) satisfies \( r_i \leq s \) for any \( i = 1, 2, L, M \), the departure rate \( r \) is called an SO departure rate (Doan, Ukkusuri and Han, 2011).

The dynamic pricing policy is governed by

\[
p_{i}^{(n+1)} = \begin{cases} 
\bar{p} - \beta \left[ t^* - (i-1)\Delta t \right]_+ - \gamma \left[ (i-1)\Delta t - t^* \right]_+, & \text{if } r_i^{(n)} > s, \\
\bar{p} - \beta \left[ t^* - (i-1)\Delta t \right]_+ - \gamma \left[ (i-1)\Delta t - t^* \right]_+, & \text{if } r_i^{(n)} \leq s,
\end{cases}
\]

for any \( i = 1, 2, L, M \) and \( n = 0, 1, 2, L \)

\[
\bar{p} = \max_{i=1,2,L,M} \left\{ \beta \left[ t^* - (i-1)\Delta t \right]_+ + \gamma \left[ (i-1)\Delta t - t^* \right]_+ \right\}
\]

\( \overline{\epsilon}^{(n)} \geq \epsilon \)
5. A day-to-day pricing policy (2)

- The generalized travel cost —

\[ \mathcal{C}_i^0(r^{(n)}) = C_i(r^{(n)}) + p_i^{(n+1)}, \quad i = 1, 2, L, M \]

- The decision-making travel cost —

\[ \bar{C}_i(r^{(n)}) = \max \left\{ \mathcal{C}_i^0(r^{(n)}), \min_{j=1,2,L,M} \left\{ \mathcal{C}_j^0(r^{(n)}) \right\} + \varepsilon \right\}, \quad i = 1, 2, L, M \]

- The proportional swap system

\[
\begin{align*}
\Gamma_i(r^{(n)}) &= \sum_{j=1}^{M} \left( r_j^{(n)} \left[ \bar{C}_j(r^{(n)}) - \bar{C}_i(r^{(n)}) \right]_+ - r_i^{(n)} \left[ \bar{C}_i(r^{(n)}) - \bar{C}_j(r^{(n)}) \right]_+ \right) \\
&\text{for } i = 1, 2, L, M, \quad n = 0, 1, 2, L, \quad \text{and } \ r^{(0)} \in \Omega
\end{align*}
\]

\[ r_i^{(n+1)} = F_i(r^{(n)}) = r_i^{(n)} + \eta \Gamma_i(r^{(n)}) \quad \eta > 0 \]
The tatonnement adjustment process

\[ r^{(n+1)} = \overline{F}(r^{(n)}) = r^{(n)} + \lambda \overline{\Gamma}(r^{(n)}) \]
\[ \overline{\Gamma}(r^{(n)}) = P_{\Omega} \left( r^{(n)} - \theta \overline{C}(r^{(n)}) \right) - r^{(n)} \]

for \( n = 0, 1, 2, \ldots \) and \( r^{(0)} \in \Omega \)

Theorem 6. A departure rate \( r^{(n)} \) is a stationary departure rate of the proportional swap system if and only if it is an SO departure rate.
Theorem 7. A departure rate $r^{(n)}$ is a stationary departure rate of the tatonnement adjustment process if and only if it is an SO departure rate.

\[
\bar{T}_i\left( r^{(n)} \right) = \begin{cases} 
(\alpha - \beta)T_i\left( r^{(n)} \right), & \text{if } (i-1)\Delta t \leq t^* - T_i\left( r^{(n)} \right), \\
(\alpha + \gamma)T_i\left( r^{(n)} \right) - (\beta + \gamma)(t^* - (i-1)\Delta t), & \text{if } t^* - T_i\left( r^{(n)} \right) < (i-1)\Delta t \leq t^*, \\
(\alpha + \gamma)T_i\left( r^{(n)} \right), & \text{if } (i-1)\Delta t > t^*.
\end{cases}
\]

\[
I^{(n)} \equiv \left\{ i \mid r_i^{(n)} > s, i = 1, 2, \ldots, M \right\} \quad \text{and} \quad \bar{I}^{(n)} \equiv \left\{ i \mid r_i^{(n)} \leq s, i = 1, 2, \ldots, M \right\}
\]
5. A day-to-day pricing policy (5)

For any \( i \in I^{(n)} \), define \( H_i (r^{(n)}) \) as

\[
H_i (r^{(n)}) = \sum_{j \in I^{(n)}} \left( r_j^{(n)} \left[ \bar{T}_j (r^{(n)}) - \bar{T}_i (r^{(n)}) \right]_+ - r_i^{(n)} \left[ \bar{T}_i (r^{(n)}) - \bar{T}_j (r^{(n)}) \right]_+ \right)
\]

\( \Omega_0 \equiv \{ r \in \Omega | r_i < s + h, i = 1, 2, \ldots, M \} \) \hspace{1cm} h \; \text{is a positive constant}

**Theorem 8.** Suppose that the adjustment parameter \( \eta \) is sufficiently small. The constant \( \bar{\varepsilon}^{(n)} \) on each day satisfies

\[
\bar{\varepsilon}^{(n)} \geq \varepsilon
\]

\[
\bar{\varepsilon}^{(n)} \geq \max_{(i,j) \in I^{(n)} \times I^{(n)}} \left\{ \max \left\{ \bar{T}_j (r^{(n)}), \varepsilon \right\} - \bar{T}_i (r^{(n)}) \right\}
\]
5. A day-to-day pricing policy (6)

$$\bar{\varepsilon}^{(n)} \geq \max_{i \in I^{(n)}} \left\{ \frac{H_i \left( r^{(n)} \right) + \bar{h}}{r_i^{(n)} |\bar{I}^{(n)}|} + \frac{1}{|\bar{I}^{(n)}|} \sum_{j \in \bar{I}^{(n)}} \max \left\{ \bar{T}_j \left( r^{(n)} \right), \varepsilon \right\} - \bar{T}_i \left( r^{(n)} \right) \right\}$$

where $\bar{h}$ is a positive constant. The trajectory of the proportional swap system will enter and remain forever in the set $\mathcal{O}_0$ after some $n_1$.

For any $n$, we rearrange the sequence of these time slices in the set $\bar{I}^{(n)}$ so that

$$\theta \max \left\{ \bar{T}_{i_1} \left( r^{(n)} \right), \varepsilon \right\} - r_{i_1}^{(n)} \leq \theta \max \left\{ \bar{T}_{i_2} \left( r^{(n)} \right), \varepsilon \right\} - r_{i_2}^{(n)} \leq L \leq \theta \max \left\{ \bar{T}_{i_{|\bar{I}^{(n)}|}} \left( r^{(n)} \right), \varepsilon \right\} - r_{i_{|\bar{I}^{(n)}|}}^{(n)}$$

where $i_1, i_2, L, i_{|\bar{I}^{(n)}|}$ belong to $\bar{I}^{(n)}$. 
Let \( \bar{H} \left( r^{(n)} \right) \) denote

\[
\bar{H} \left( r^{(n)} \right) = \frac{1}{\bar{M}^{(n)}} \left( \frac{N}{\Delta t} - \sum_{k=1}^{\bar{M}^{(n)}} \left( \theta \max \{ T_{i_k} \left( r^{(n)} \right), \varepsilon \} - r_{i_k}^{(n)} \right) \right)
\]

The integer \( \bar{M}^{(n)} \) satisfies \( \bar{M}^{(n)} \leq |I^{(n)}| \) to guarantee

\[
r_{i_j}^{(n)} - \theta \max \left\{ \bar{T}_{i_j} \left( r^{(n)} \right), \varepsilon \right\} + \bar{H} \left( r^{(n)} \right) \geq 0
\]

for \( j = 1, 2, L, \bar{M}^{(n)} \)

\[
r_{i_j}^{(n)} - \theta \max \left\{ \bar{T}_{i_j} \left( r^{(n)} \right), \varepsilon \right\} + \bar{H} \left( r^{(n)} \right) < 0
\]

for \( j = \bar{M}^{(n)} + 1, \bar{M}^{(n)} + 2, L, |I^{(n)}| \)
5. A day-to-day pricing policy (8)

Theorem 9. Assume that the adjustment parameter $\lambda$ is sufficiently small. The constant $\bar{\varepsilon}^{(n)}$ on each day satisfies

$$\bar{\varepsilon}^{(n)} \geq \varepsilon$$

$$\bar{\varepsilon}^{(n)} \geq \max_{i \in I^{(n)}} \left\{ \frac{r_i^{(n)}}{\theta} - \bar{T}_i \left( r^{(n)} \right) \right\} + \frac{\bar{H} \left( r^{(n)} \right)}{\theta}$$

The trajectory of the tatonnement adjustment process will enter and remain forever in the set $\emptyset$ after some $n_1$. 
Define the potential on day $n$ as

$$V\left( r^{(n)} \right) = \sum_{i=1}^{M} \sum_{j=1}^{M} r_i^{(n)} \left[ \overline{C}_i \left( r^{(n)} \right) - \overline{C}_j \left( r^{(n)} \right) \right]^2$$

| $N = 6000$ | $s = 3000$ | $\alpha = 10$ | $\beta = 5$ | $\gamma = 15$ |
| [0, $\bar{T}$] = [0, 2] | $M = 40$ | $t^* = 1.5$ |
6. Numerical examples (2)

Figure 1. (a) The potential of the proportional swap system when the adjustment parameter $\eta = 10^{-4}$ and the boundedly rational threshold $\varepsilon = 0, 1, 2, \text{ and } 3$. (b) The potential of the tatonnement adjustment process when the adjustment parameter $\lambda = 0.2$, the sensitivity parameter $\theta = 50.0$, and the boundedly rational threshold $\varepsilon = 1, 2, 3, \text{ and } 4$.

Without the toll charge
6. Numerical examples (3)

Figure 2. (a) The potential of the proportional swap system when the adjustment parameter \( \eta = 10^{-3} \) and the boundedly rational threshold \( \varepsilon = 0, 1, 2, \) and 3. (b) The potential of the tatonnement adjustment process when the adjustment parameter \( \lambda = 0.5 \), the sensitivity parameter \( \theta = 100.0 \), and the boundedly rational threshold \( \varepsilon = 1, 2, 3, \) and 4. With the toll charge

\[
V\left( r^{(n)} \right) = \sum_{i=1}^{M} \left[ r_{i}^{(n)} - S \right]_+^2
\]
7. Conclusions (1)

1. Extend the proportional swap system by Smith (1984a) and the network tatonnement process by Friesz et al. (1994) to formulate the day-to-day evolution of departure time choice under bounded rationality in the bottleneck model.

2. Show the existence of the stationary points of the two systems and the equivalence between the stationary state and the BRUE state.

3. Propose a dynamic pricing policy, implemented in the day-to-day evolution process, to change the evolutionary trend of the day-to-day departure rate for achieving the SO target.
Appendix. References (1)

Appendix. References (2)

- Han, K., Szeto, W.Y., Friesz, T.L., 2015. Formulation, existence, and computation of boundedly rational dynamic user equilibrium with fixed or endogenous user tolerance. Transportation Research Part B 79, 16–49.
Appendix. References (3)

Appendix. References (4)

Discussion & Question?

Thank you!