Minimal Parameter Formulations of the Dynamic User Equilibrium using Macroscopic Urban Models: Freeway vs City Streets Revisited

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Motivation

- Traditional DTA networks
  - Hundreds of thousands of nodes and links
  - Hard to calibrate
  - Prohibitive computation times to equilibrium
- The nMFD opens the door for
  - Continuum-space DTA
  - Discrete-space DTA
  - Hybrid-space DTA
Reservoir/Bathtub Model for cities

\[
\left\{ \begin{align*}
n'(t) &= \lambda(t) - f(n), \\
n(0) &= n_0,
\end{align*} \right. \quad \text{(reservoir dynamics)}
\]

\text{(initial conditions)}
Continuum-space DTA

2D conservation law: $k_t(x, y, t) + \nabla \cdot f(k, x, y, t) = \lambda(x, y, t)$
Discrete-space DTA

single nMFD loading problem
nMFD estimation

- Method of cuts
  - analytical for *homogeneous* corridors

- Stochastic method of cuts (ISTTT21)
  - The (pdf of) nMFD depends on 2 parameters
    - density of traffic lights
    - mean red to green ratio
    - is symmetric ➔ Greenshield approximation
Today: (1) Single nMFD loading

\[ f(n) = V(n) n / \lambda \]

- At the core of numerical solution methods
- Parameter-free formulation
- Universal shape
- Analytical solutions
Today: (2) Freeway vs nMFD

- 1-parameter formulation
- Analytical solutions
Today: (3) Freeway vs nMFD (hybrid space)
Single nMFD loading problem

\[ f(n) = V(n)n / \ell \]

\[ \mu(1/\tau^*) \]

\[ V(n) \]

\[ u \]

\[ accumulation, n \]

\[ accumulation, n \]

\[ n_j \]

\[ n_1^* \]

\[ n_2^* \]

\[ n_j \]

\[ n_0 \]

\[ 0 \]

\[ n \]

\[ \lambda(t) \]

\[ \ell \]

\[ \tau^* \equiv \ell / u \]

\[\ell \] is the trip length,

\[\tau^* \] is the free-flow travel time

\[
\begin{align*}
    n'(t) &= \lambda(t) - f(n), & \text{(reservoir dynamics)} \\
    n(0) &= n_0, & \text{(initial conditions)}
\end{align*}
\]
Single nMFD loading in terms of occupancy

- In terms of:

\[ k(t) \equiv n(t)/n_j \quad \text{(occupancy)} \]

\[ \rho(t) \equiv \lambda(t)/\mu, \quad \text{(demand intensity)} \]

and measuring time in units of \( \tau^* \), \( \hat{t} = t/\tau^* \):

\[
\begin{align*}
\hat{k}'(\hat{t}) &= c\hat{\rho}(\hat{t}) - \hat{k}g(\hat{k}), \\
\hat{k}(0) &= \hat{k}_0,
\end{align*}
\]

- For Greenshield approximation, \( g(k) \equiv 1 - k \) and \( \rho \) is the only parameter.
Analytical solution – autonomous case

- \( k(t) = T^{-1} (T(k_0) + t) \)
  with \( T(k) = \int \frac{dk}{(c \rho - k g(k))} \).

- For Greenshield:
  \[
  k(t) = 1/2 - c_1 \tanh (T(k_0) + c_1 t)
  \]
  where \( T(k) = \frac{1}{c_1} \tanh^{-1} \left( \frac{1/2 - k}{c_1} \right) \).
Analytical solution – more general cases

- Solutions are functions of:

\[ \begin{align*}
\rho & \equiv \frac{\rho_{ss}}{\mu}, \\
\beta & \equiv \frac{t^*}{\tau^*},
\end{align*} \]

(1a) (steady-state intensity)

(1b) (time scale ratio)

- \( t^* \) = demand time scale, speed of \( q_0 \to q_1 \).
- \( \tau^* \) = supply time scale, speed of MFD \( \to \) equilibrium.

\[ \beta \gg 1 \to \text{demand varies "slowly"} \]
(2) Freeway vs nMFD (UE)

\[
\begin{align*}
\tau_0(t) &= \tau_0^* + w_0(t), \quad \text{(freeway travel time)} \\
\tau_1(t) &= n(t)/f(n(t)), \quad \text{(inst. CS travel time)} \\
n'(t) &= \lambda_1(t) - f(n(t)), \quad \text{(reservoir dynamics)} \\
\lambda(t) &= \lambda_0(t) + \lambda_1(t), \quad \text{(demand conservation)}
\end{align*}
\]
User Equilibrium

- UE condition in differential form:

\[ \tau'_0(t) = \tau'_1(t) \]

It turns out that:

\[
\begin{align*}
    k'(t) &= (c\rho(t) - kg(k))/(1 - cmg'/g^2), \\
    k(0) &= k_0,
\end{align*}
\]

where:

\[
\begin{align*}
    m &= \mu_0/\mu_1, \\
    \rho(t) &\equiv (\lambda(t) - \mu_0)/\mu_1
\end{align*}
\]

(capacity ratio) \hspace{1cm} (MFD demand intensity)
Greenshield solution

- $k(t) = T^{-1} (T(n_0) + t)$, and:

$$T(k) = (1 + (2 - \rho)m/\rho^2) T^1(k) + \frac{m}{\rho^2} \left( \frac{\rho}{1 - k} + 2 \log \left( \frac{\rho - 4(1 - k)k}{(1 - k)^2} \right) \right) \quad (1)$$

where $T^1(k)$ is the corresponding $T$-function for the single NMFD problem.

- as $m \to 0$ (no freeway) then $T \to T^1$
- as $m \to \infty$ (no CS) then $T \to \infty$
Analytical solution – autonomous case
Today: (3) Freeway vs nMFD (hybrid space)
Numerical solution

**case** \( k_0 < 1/2: \)

\[
\begin{align*}
\xi_0 &= 0+, \ k_0 = 0, \ \mu_0 = 0.5Q_0, \ q = 0.95Q_0, \ u_1 = \\
&= 30 \text{, 45 \text{, 60 \text{, 80}}} \\
\end{align*}
\]

\[
\begin{align*}
k(t) \\
0 & 0.05 \ 0.10 \ 0.15 \ 0.20 \ 0.25 \ 4 \\
t, \text{hr} \\
0 & 1 \ 2 \ 3 \ 4 \\
\end{align*}
\]

\[
\begin{align*}
k(t)/k^*_1 \\
0 & 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1 \ 12 \\
t/\tau^*_{1,ss} \\
0 & 2 \ 4 \ 6 \ 8 \ 10 \ 12 \\
\end{align*}
\]

\[
\begin{align*}
\xi(t), \text{km} \\
0 & 0.5 \ 1.0 \ 1.5 \ 2.0 \ 2.5 \ 4 \\
t, \text{hr} \\
0 & 1 \ 2 \ 3 \ 4 \\
\end{align*}
\]

\[
\begin{align*}
\xi(t)/\xi_{ss} \\
0 & 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1 \ 12 \\
t/\tau^*_{1,ss} \\
0 & 2 \ 4 \ 6 \ 8 \ 10 \ 12 \\
\end{align*}
\]
Numerical solution

case $k_0 > 1/2$:
→ $k_2^*$ not a repellor anymore

City streets MFD coverage area
Q & A  Thank you!
A continuum approximation for off-ramps

- The network length can be expressed as:

\[ L(\xi) = \frac{\pi \xi^2}{2\delta}, \]  

(1)

- If trip length is proportional to the network length: \( \gamma \equiv L/\ell \) is a constant, then

\[
\begin{align*}
\lambda(t) &= (\tau_1'(t) + 1)\mu_0 + \phi \xi(t), \\
n'(t) &= \phi \xi(t) - f(n(t), \xi(t)) \\
(n(0), \xi(0)) &= (n_0, \xi_0),
\end{align*}
\]

where \( n_0 < \kappa L(\xi_0) \) to ensure well-posedness.

- In steady-state:

\[
\begin{align*}
\lambda_{1,ss} &= \lambda_{ss} - \mu_0, \\
\xi_{ss} &= \frac{\lambda_{1,ss}}{\phi}, \\
\ell_{ss} &= \frac{\pi \xi_{ss}^2}{4\gamma \delta}, \\
L_{ss} &= \ell_{ss} \gamma \\
\mu_{1,ss} &= \frac{1}{4} \gamma \kappa u_1, \\
\tau_{1,ss}^* &= \frac{\ell_{ss}}{u_1}.
\end{align*}
\]

(2)
Symmetry: Isosceles transformation

Flow density $u$ transformed density $\frac{1}{2} - \frac{1}{2}$ transformation $L$

FD transformation

Flow

$A'$

$\frac{1}{2}$ $-\frac{1}{2}$ transformed density

$A$

$\frac{1}{2}$ $-\frac{1}{2}$

MFD transformation

Capacity

$0.5$ $0.4$ $0.3$ $0.2$ $0.1$

$0.0$ $0.2$ $0.4$ $0.6$ $0.8$ $1.0$

$0.0$ $0.2$ $0.4$ $0.6$ $0.8$ $1.0$
Symmetry: Yokohama NMFD

- Greenshield nMFD may be good approximation
Lights (R11 and R14) density clustering (200m * 200m cells) over Lyon lights perimeter
Method of Cuts

- $\phi = \text{maximum passing rate}$
- $\nu = \text{average observer speed}$

$$q(k) = \min_{s} \{ q_s(k) \}$$

- analytical for homogeneous* corridors
- intractable for heterogeneous corridors $\Rightarrow$ numerical methods (Leclercq and Geroliminis, 2013)

*homogeneous = same segment length and signal timing
Single nMFD loading in terms of occupancy

\[ k(t) \equiv \frac{n(t)}{n_j} \quad \text{(occupancy)} \]
\[ \rho(t) \equiv \frac{\lambda(t)}{\mu}, \quad \text{(demand)} \]

\[ n'(t) \quad \frac{n_j}{n_j} = \frac{\lambda(t) - f(k)}{n_j} \]
\[ = \frac{\mu}{n_j} \rho - \frac{1}{\tau^*} g(k) k \]
\[ = \frac{1}{\tau^*} \left( c \rho(t) - k g(k) \right) \]

\[ g(k) = \frac{v(k)}{u} \quad (u = \text{free-flow speed}) \]

\[ f(k) = V(k) n / \ell \]
\[ = u g(k) k n_j / \ell \]
\[ = g(k) k n_j / \tau^* \]

\[ \mu^* = \frac{\mu}{\mu^*} = \frac{\mu \tau^*}{n_j} \]

\[ c = \frac{1}{\tau^*} \]

\[ f(n) \quad \frac{\text{outflow}}{\text{accumulation, } n} \]

\[ \frac{\text{outflow}}{\text{outflow}} \]

nMFD definition

FD + Network topology + signal timing + Route choice = NMFD
Single nMFD loading in terms of occupancy

- In terms of $k(t) \equiv n(t)/n_j$

\[
\begin{align*}
  k'(t) &= \frac{1}{\tau^*} (c\rho(t) - k g(k)), \\
  k(0) &= k_0,
\end{align*}
\]

where $c \equiv \mu \tau^*/n_j$ is the MFD “shape” parameter.

- measuring time in units of $\tau^*$, $\hat{t} = t/\tau^*$:

\[
\begin{align*}
  \hat{k}'(\hat{t}) &= c\hat{\rho}(\hat{t}) - \hat{k} g(\hat{k}), \\
  \hat{k}(0) &= \hat{k}_0,
\end{align*}
\]

- For Greenshield approximation, $g(k) \equiv 1 - k$ and $\rho$ is the only parameter!
Macro DTA

Lights density clustering (500 * 500m squares) over Lyon lights perimeter